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THE GEGENBAUER-LIKE AND JACOBI-LIKE POLYNOMIALS: AN OVERVIEW IN THE LIGHT OF SOLTIS CONSIDERATIONS, SOME APPLICATIONS

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Abstract: The exposition is based on the excellent article by Soltis [1]. Some applications that these polynomials find in synthesis of antennas and approximation of "U and V– shaped transfer functions" are indicated. Dynamics of the Lienard differential system using Gegenbauer-like and Jacobi-like polynomials is also considered. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard differential system, Gegenbauer-like and Jacobi-like polynomials, antenna factor

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1. The Gegenbauer-like polynomials

For many applications, orthogonality of a polynomial family is not necessary.

By extending the parameter range of Gegenbauer polynomials useful new functions, with applications, are shown in [1].

Gegenbauer polynomials, for N odd can be described by

$$C_N^{(\alpha)}(x) = C_{2n+1}^{(\alpha)}(x) = S_N x a_0(x)$$

where $a_0(x)$ is recursively found from

$$a_{m-1}(x) = 1 - b_m x^2 a_m(x)$$

and

$$b_m = \frac{2(n-m+1)(\alpha+n+m)}{m(2m+1)!};\ N = 2n+1;\ (m = n, n-1, \ldots, 2, 1;\ a_n(x) = 1)$$

and S_n is a scaling constant.

The standard scaling is

$$S_{NS} = \frac{2(-1)^n(\alpha)_{n+1}}{n!}$$

where $(\alpha)_{n+1} = \alpha(\alpha+1)\cdots(\alpha+n)$.

Traditionally the parameter α satisfies $(-\frac{1}{2} \leq \alpha \leq \infty)$.

Note that α need not be an integer and $\alpha = 0$ provides the ubiquitous Chebyshev polynomial.

In [1] the author present a new class for N odd and select $\alpha = -\frac{N}{2}$.

For example, for $N = 9$, we have

$$a_0(x) = 1 - b_1x^2 + b_1b_2x^4 - b_1b_2b_3x^6 + b_1b_2b_3b_4x^8.$$

In explicit form, using $S_N = -S_{NS}$, i.e. $S_9 = S_{9S}$ we have (see Fig. 1)

$$C_9^{(-4.5)}(x) = 0.273437x^9 - 1.40625x^7 + 2.95312x^5 - 3.28125x^3 + 2.46094x.$$

Consider the factor

$$R(\alpha,\theta) = \sum_{k=1}^{5} A_k \cos((2k-1)) = \sum_{k=1}^{5} C_{2k-1}^{(0)}(\frac{x}{x_0} = C_9^{(-4.5)}(x)$$

where $u = \frac{\pi d}{\lambda}$,$\theta$ is polar angle and x_0 is a design parameter for

a) $\alpha = -4.5$, $x_0 = 1.3$;

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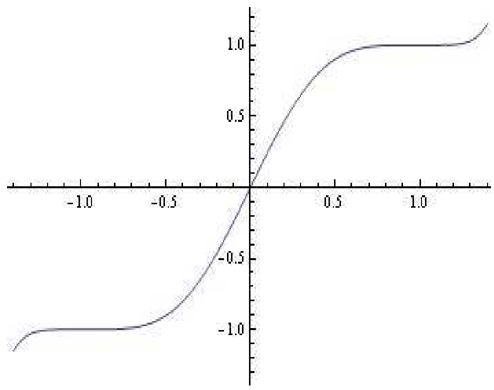

Figure 1: The polynomial $C_9^{(-4.5)}(x)$

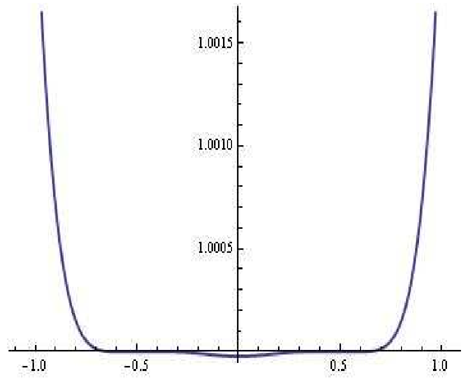

Figure 2: The case a.)

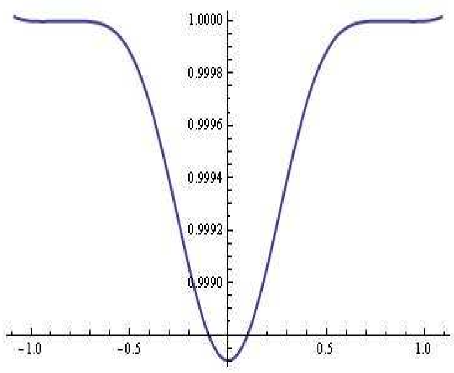

Figure 3: The case b).

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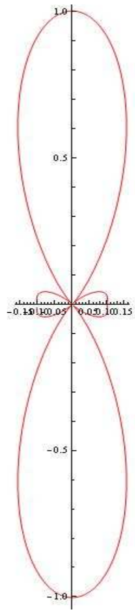

Figure 4: Dolph-Chebyshev array

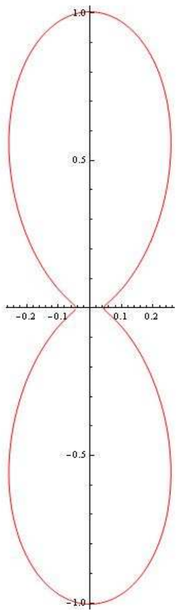

Figure 5: Binomial array

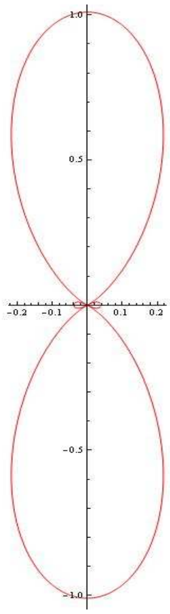

Figure 6: Soltis array

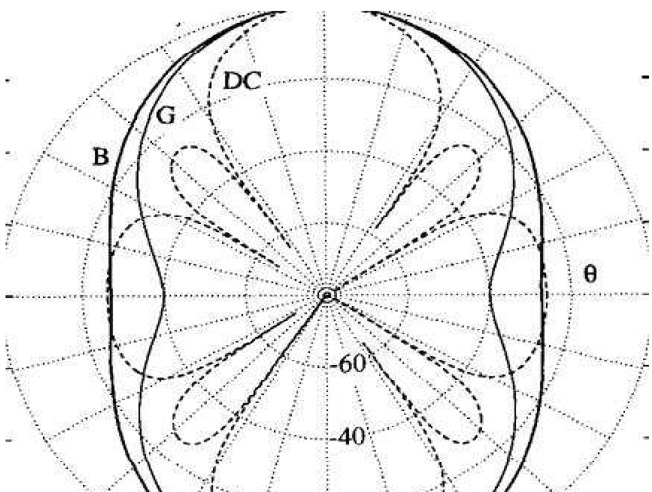

Figure 7: Comparisons between B–binomial, G- present theory by Soltis [1] and DC-Dolph-Chebyshev.

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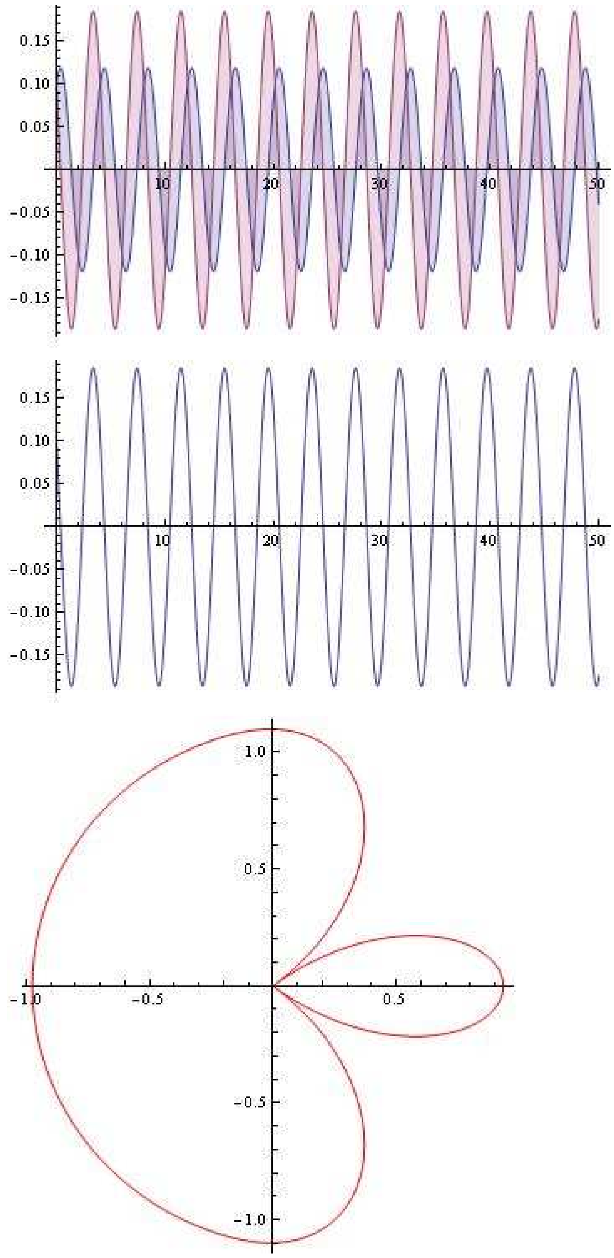

Figure 8: The simulations for $x_0 = 0.1$, $y_0 = 0.1$, $b = -0.05$, $c = -0.01$: a) solutions of the system b) y-component of the solution; b) the normalized factor.

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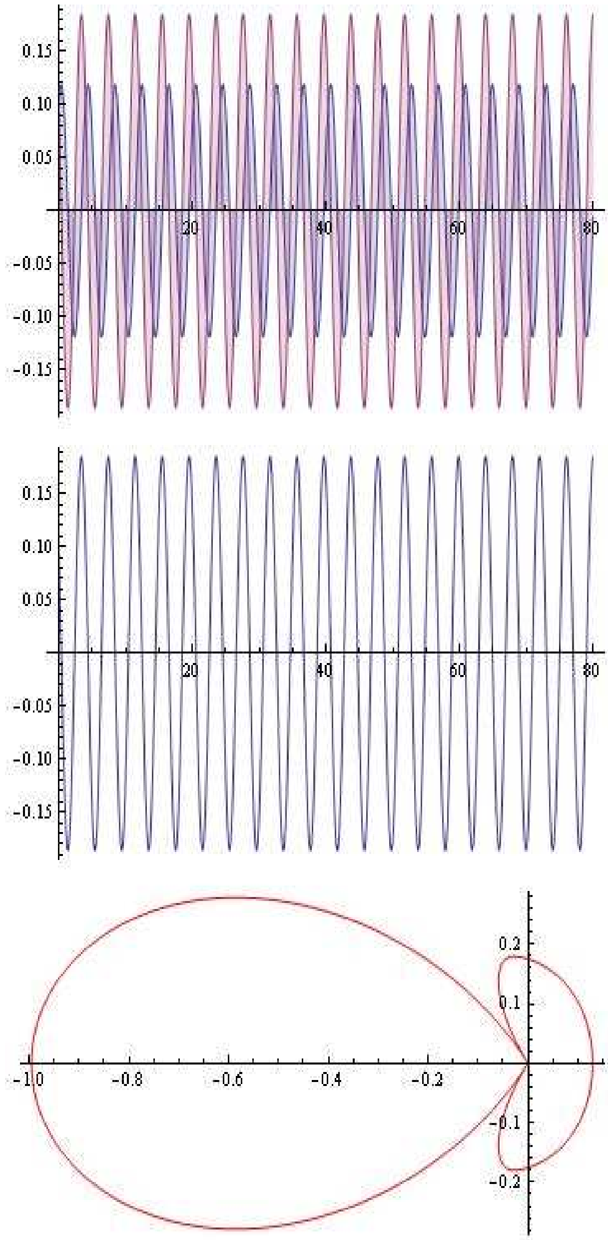

Figure 9: The simulations for $x_0 = 0.1$, $y_0 = 0.1$, $b = 0.1$, $c = 0.01$: a) solutions of the system b) y-component of the solution; c) the normalized factor.

A_4	A_3	A_2	A_1
1.357	1.374	2.496	2.798

Table 1: Normalized array coefficients (Dolph-Chebyshev)

A_4	A_3	A_2	A_1
9	36	84	126

Table 2: Normalized array coefficients (Binomial)

b) $\alpha = -4.5$, $x_0 = 1.15$.

We note that the model can be used successfully in modeling and approximating of "U and V– shaped transfer functions" (see Fig. 2–Fig. 3).

Array factor power pattern (in dB) of 10–element array, $d = \frac{\lambda}{4}$ is depicted in Fig. 7.

Consider the system

$$\begin{cases} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & -C_9^{(-4.5)}(x) + 0.1(x - \frac{1}{3}x^3)y \end{cases}$$

Define the normalized factor

$$\frac{y(b\cos\theta + c)}{m}$$

where θ is the azimuthal angle and c is the phase difference.

Tke simulations on the system for

a) $x_0 = 0.1$, $y_0 = 0.1$, $b = -0.05$, $c = -0.01$,

b) $x_0 = 0.1$, $y_0 = 0.1$, $b = 0.1$, $c = 0.011$

are depicted in Fig. 8–Fig. 9.

2. A look at the Jacobi-like polynomials

Proceeding in a similar fashion, interesting results for Jacobi–like polynomials are also found in [1]

$$P_n^{(\alpha,\beta)}(x) = \frac{1}{n}a_0(x)$$

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A_4	A_3	A_2	A_1
5.4497	15.1585	27.9146	32.2551

Table 3: Normalized array coefficients (Soltis [1])

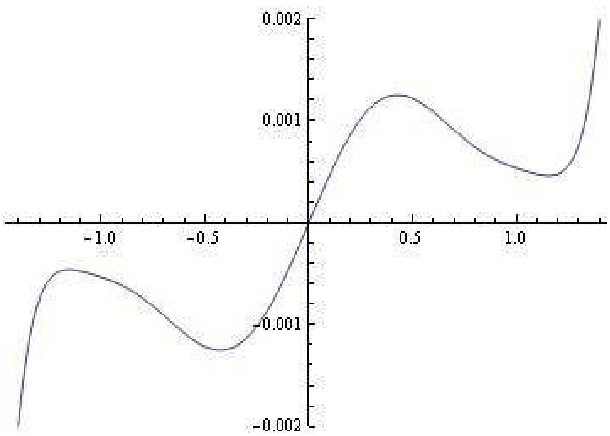

Figure 10: The polynomial $P_9^{(-4.5,-4.5)}(x)$

for $\alpha = \beta = -\frac{n}{2}$ and $a_0(x)$ is recursively found from

$$a_{m-1}(x) = 1 - b_m a_m(x)(1 - x)$$

and

$$b_m = \frac{(n-m+1)(\alpha+\beta+n+m)}{2m(\alpha+m)}.$$

For example we have [1] (see Fig. 10)

$$P_9^{(-4.5,-4.5)}(x) = 0.00195312x^9 - 0.00878906x^7 + 0.0153809x^5 - \\ -0.0128174x^3 + 0.00480652x.$$

The model can be used successfully in modeling and approximating of ”U and V– shaped transfer functions’.’

Example 1. Let $\alpha = -4.5$, $\beta = -4.5$,$x_0 = 1.81$.

The simulation is depicted in Fig. 11.

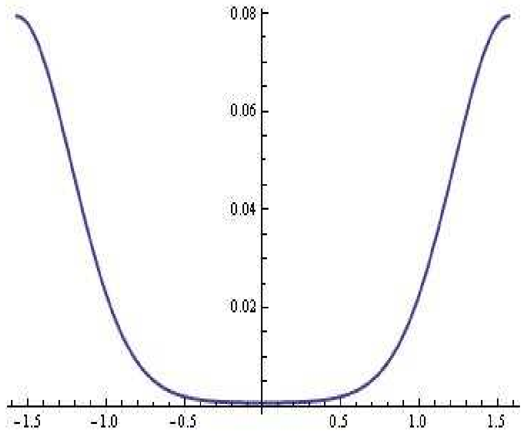

Figure 11: Example 1.)

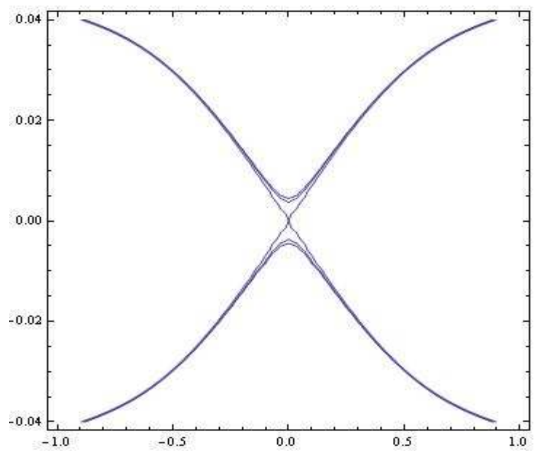

Figure 12: The level curves

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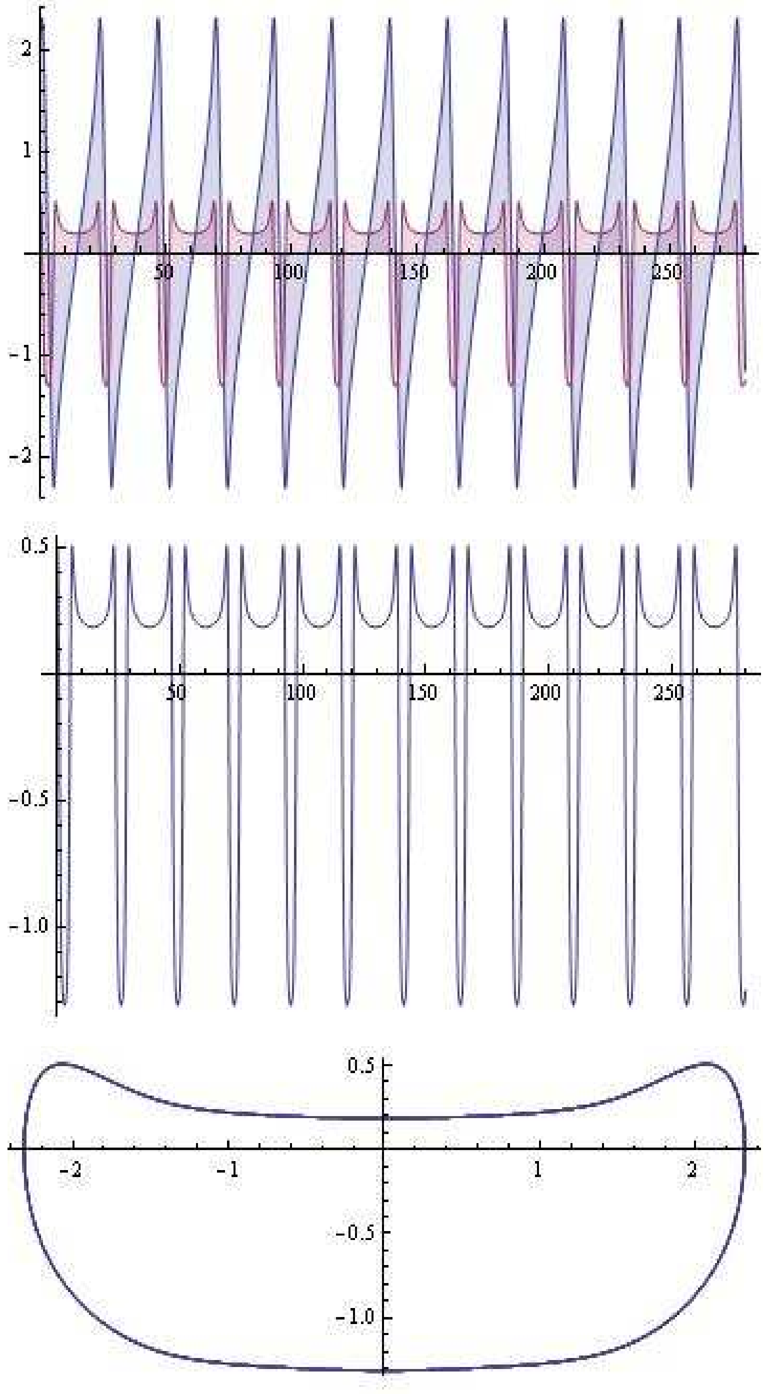

Figure 13: The simulations for $x_0 = 2$, $y_0 = 0.5$: a) solutions of the system b) y-component of the solution; c) the portrait.

Consider the system

$$\begin{cases} \dfrac{dx}{dt} &= y \\ \dfrac{dy}{dt} &= -P_9^{(-4.5,-4.5)}(x) + 0.1(x - x^3 + x^5 - \frac{1}{7}x^7)y \end{cases}$$

The level curves $L_{h_i} = \{H(x, y) = h_i\}$ where $H(x, y)$ is the Hamiltonian of the system are depicted on Fig. 12.

The simulations on the system for $x_0 = 2$, $y_0 = 0.5$ are depicted in Fig. 13.

A new method to design a cosecant squared radiation pattern in a shaped reflector antenna is proposed in [2].

Surface of the proposed antenna is expanded by a set of modified Jacobi polynomials.

Concluding remarks

As far as the Dolph-Chebyshev technique for synthesis of filters is well known, we note that by analogy we can define the following hypothetical transmitting functions based on "Gegenbauer-like filter prototype":

$$|A|_1(\omega) = \frac{1}{1 + a^2\left(C_n^{(-\alpha)}\left(\dfrac{2\omega}{2 - a_1 a}\right)\right)^2},$$

$$|A|_2(\omega) = \sqrt{\frac{1}{1 + a^2\left(C_n^{(-\alpha)}\left(\dfrac{2 - a_1 a}{2\omega}\right)\right)^2}},$$

Example 1. The simulations for

a) $n = 9$, $\alpha = 4.5$, $a = 0.18$, $a_1 = 2.6469$

b) $n = 9$, $\alpha = 4.5$, $a = 0.24$, $a_1 = 2$

are depicted in Fig. 14–Fig. 15.

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References

[1] J. J. Soltis, New Gegenbauer–like and Jacobi–like polynomials with applications, *Journal of the Franklin Institute*, 33 (3), 1993, 635–639.

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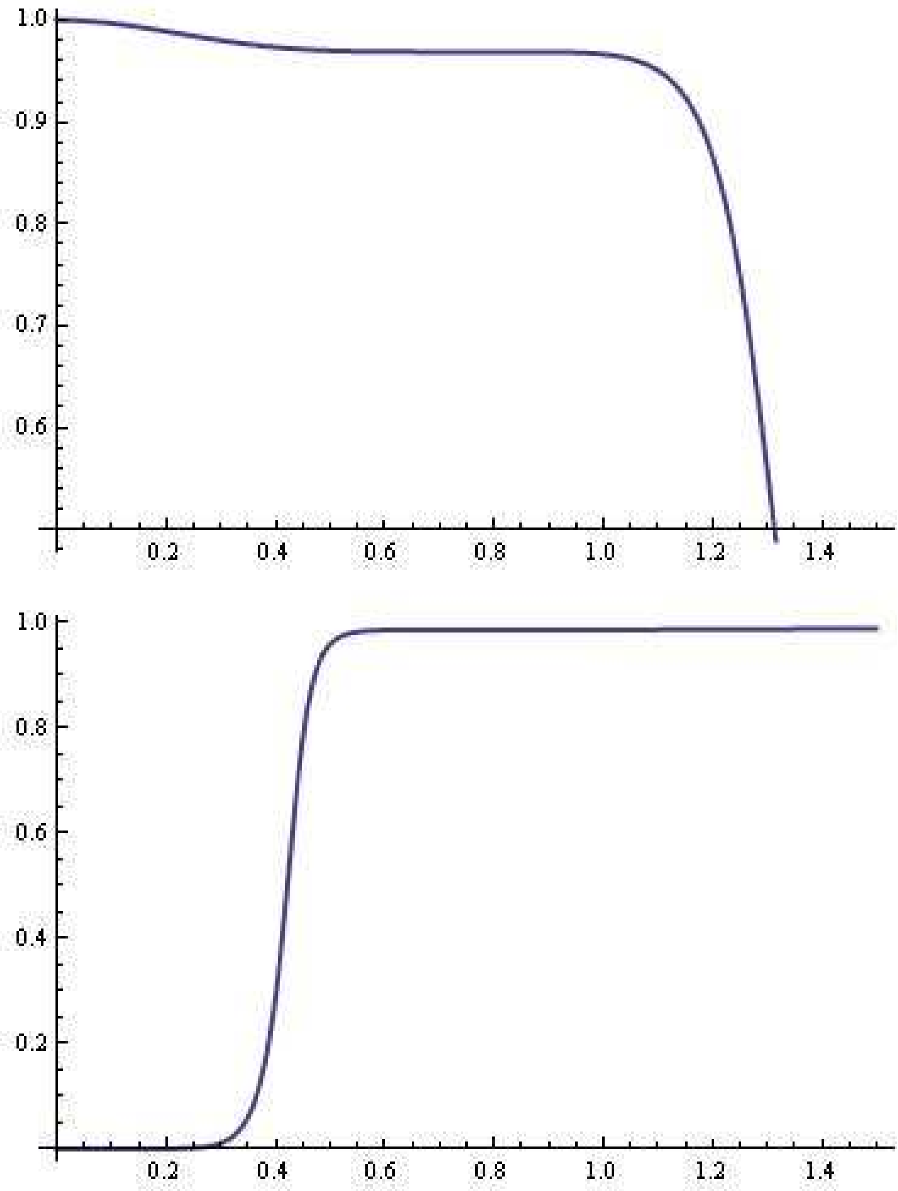

Figure 14: Example 1 a).

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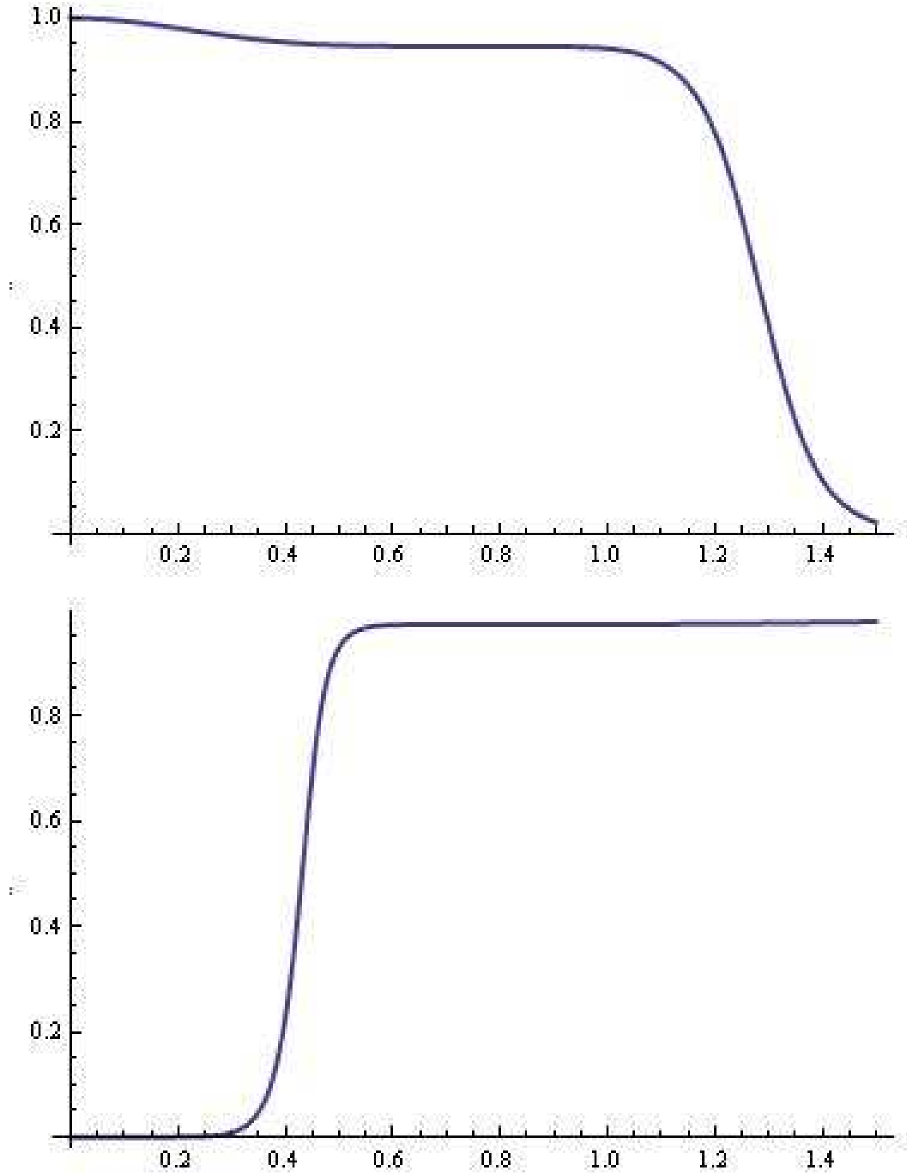

Figure 15: Example 1 b).

[2] M. Samadi Taheri, A. Mallahzadeh, A. Foudazi, Shaped beam synthesis for shaped reflector antenna using PSO algorithm, Proc. 6-th European Conference: Antennas and Propagation (EUCAP), 2012.

[3] George E. Andrews, Richard Askey, Ranjan Roy, Special Functions, Cambridge Univ. Press, 1999.

[4] R. Askey, J. Wilson, Some basic hypergeometric orthogonal polynomials, Memoirs of the AMS 319, 1985.

[5] G. Gaspar, M. Rahman, Basic hypergeometric series, Second edition, Cambridge Univ. Press, 2003.

[6] R. Koekoek, R. Swarttouw, The Askey–scheme of hypergeometric orthogonal polynomials and its q–analogue

[7] J. Cigler, Continuous q–Hermite polynomials: an elementary approach, arXiv 1307.0357V2 [math HO] 24 Feb 2014.

[8] R. Koekoek, R. Askey, R. Swarttouw, Hypergeometric orthogonal polynomials and their q–analogues, Springer Monographs in Mathematics, Berlin, New York, Springer–Verlag, 2010.

[9] J. Cigler, J. Zang, A curious q–analogue of Hermit polynomials, *J. Comb. Th A*, 118 (2011), 9–20.

[10] V. K. Melnikov, On the stability of a center for time–periodic perturbation, *Tr. Mosk. Mat. Obs.*, **12** (1963).

[11] T. Blows, L. Perko, *SIAM Rev.*, **36**, 341 (1994).

[12] L. Perko, *Differential Equations and Dynamical Systems*, Springer–Verlag, New York (1991).

[13] Gavrilov L., I. D. Iliev, Perturbations of quadratic Hamiltonian two–saddle cycles, *Ann. Inst. H. Poincare (C) Non Linear Analysis*, 32, 2, 2015, 307–324.

[14] Iliev I.D., Chengzhi Li, Jiang Yu, Bifurcations of limit cycles in a reversible quadratic system with a center, a saddle and two nodes, *Commun. Pure Appl. Analysis*, 9, 3, American Institute of Mathematical Sciences (AIMS), 2010, 583–610.

[15] Horozov E., I.D. Iliev, On the number of limit cycles in perturbations of quadratic Hamiltonian systems, *Proc. Lond. Math. Soc.,* (3), 69, 1, Oxford University Press, 1994, 198–224.

[16] Iliev I.D., L. M. Perko, Higher order bifurcations of limit cycles, *J. Differential Equations,* 154, 2, Academic Press; Elsevier, 1999, 339–363.

[17] Iliev I.D., The number of limit cycles due to polynomial perturbations of the harmonic oscillator, *Math. Proc. Cambridge Philos. Soc.,* 127, 2, Cambridge University Press, 1999, 317–322.

[18] Iliev I.D., Higher–order Melnikov functions for degenerate cubic Hamiltonians. *Adv. Differential Equations*, 1, 4, Khayyam Publishing Company, Inc., 1996, 689–708.

[19] Gavrilov L., I. D. Iliev, The limit cycles in a generalized Rayleigh-Lienard oscillator, arXiv:2203.02829v1 [math.DS] 5 Mar 2023.

[20] N. Kyurkchiev, A. Andreev, *Approximation and Antenna and Filters synthesis. Some Moduli in Programming Environment MATHEMATICA*, LAP LAMBERT Academic Publishing, Saarbrucken, 150 pp., (2014), ISBN: 978-3-659-53322-8.

[21] N. Kyurkchiev, Some Intrinsic Properties of Tadmor-Tanner Functions: Related Problems and Possible Applications, *Mathematics* (2020), **8**, 1963.

[22] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Simulations on the Lienard polynomial system with Dickson-type polynomial corrections. The level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 31–44.

[23] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, A note on the extended Lienard system with Dickson polynomials of the third kind as corrections. The level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 73–87.

[24] V. Kyurkchiev, N. Kyurkchiev, On an extended relaxation oscillator model: number of limit cycles, simulations. I, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[25] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, A technique for simulating the dynamics of some extended relaxation oscillator models. II, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[26] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Another extended polynomial Lienard systems: simulations and applications. III, *International Electronic Journal of Pure and Applied Mathematics*, **16**, No. 1 (2022), 55–65.

[27] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Investigations on some polynomial Lienard-type systems: number of limit cycles, simulations, *International Journal of Differential Equations and Applications*, **21**, No. 1 (2022), 117–126.

[28] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, *On some extended oscillator models: a technique for simulating and studying their dynamics*, Plovdiv, Plovdiv University Press (2022); ISBN 978-619-7663-13-6.

[29] N. Kyurkchiev, A. Iliev, On the hypothetical oscillator model with second kind Chebyshev's polynomial–correction: number and type of limit cycles, simulations and possible applications, *Algorithms* 15, 12, 2022, ISSN:1999-4893

[30] V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, Lienard System with First Kind Chebyshev's Polynomial–correction in the Light of Melnikov's Approach. Simulations and Possible Applications, International Scientific Conference IMEA'2022, 23-25 November 2022, Pamporovo, Bulgaria, ISBN: 978-619-7663-33-4, Plovdiv University Press, 105–112.

[31] E. Angelova, V. Arnaudova, T. Terzieva, A. Malinova, Investigations on a Differential System with Correction of Zernike-type Radial Polynomials. Simulations, International Scientific Conference IMEA'2022, 23-25 November 2022, Pamporovo, Bulgaria, ISBN: 978-619-7663-33-4, Plovdiv University Press, 87–92.

[32] N. Kyurkchiev, The effects on the dynamics of Lienard equation with Morse–type corrections: level curves, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022).

[33] A. Malinova, T. Terzieva, O. Rahneva, E. Angelova, Legendre polynomials as "correction factors" in the Lienard differential system. Simulations, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[34] A. Malinova, T. Terzieva, O. Rahneva, E. Angelova, A. Golev, Associated Legendre polynomials as corrections in the classical Lienard system, *Communications in Applied Analysis*, **26**, No. 1 (2022).

[35] A. Golev, V. Arnaudova, Lienard system with "correcting factors" of the type of interpolating polynomials of some basic functions, *International Electronic Journal of Pure and Applied Mathematics*, **16**, No 1 (2022), 67–80.

[36] A. Golev, V. Arnaudova, Higher-Kind Chebyshev Polynomials and Their Applications to Dynamic Model Studies, *International Journal of Differential Equations and Applications*, 21, 2 (2022), 101–108.

[37] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Gegenbauer polynomials as correction in the Lienard planar system: Melnikov's approach, *International Journal of Differential Equations and Applications*, **21**, No. 2 (2022), 45–57.

[38] V. Kyurkchiev, N. Kyurkchiev, A. Iliev, A. Rahnev, *On the Lienard System with Some "Corrections of Polynomial-type": Number of Limit Cycles, Simulations and Possible Applications. Part II.*, Plovdiv, Plovdiv University Press (2022), ISBN: 978-619-7663-41-9.

[39] Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., Dynamics of the Lienard Polynomial System Using Dickson Polynomials of the (M + 1)-th Kind. The Level Curves, *International Journal of Differential Equations and Applications*, 21, 2, 2022, 109–120.

[40] Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., On a class of orthogonal polynomials as corrections in Lienard differential system. Applications, *Algorithms*, (2023). (accepted)

[41] Vasileva, M., Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., Associated Hermite polynomials. Some applications, *International Journal of Differential Equations and Applications*, **22**, No. 1 (2023), 1–17.

[42] A. Golev, V. Arnaudova, Associated Gegenbauer polynomials. Applications, *International Journal of Differential Equations and Applications*, **22**, No. 1 (2023), 19–28.

[43] V. Kyurkchiev, A.Iliev, A.Rahnev, N. Kyurkchiev, Chebyshev polynomials of the fifth kind and their application to study the dynamics of Lienard-type equations, *Communications in Applied Analysis*, 26, 1, 2022, 111–123.

[44] M. Vasileva, A. Iliev, A. Rahnev, N. Kyurkchiev, Applications of Some Associated Orthogonal Polynomials, *Communications in Applied Analysis*, **27**, No. 1 (2023), 27–39.

[45] M. Vasileva, V. Kyurkchiev, A. Iliev, A. Rahnev, N. Kyurkchiev, A Look at the Associated Lommel Polynomials. Applications, *Communications in Applied Analysis*, **27**, No. 1 (2023), 41–52.

[46] A. Golev, V. Arnaudova, One possible application of the q–Lommel polynomials associated with the Jackson q-Bessel function, *International Electronic Journal of Pure and Applied Mathematics* (2023).

[47] Vasileva, M., Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., Dynamics of the Lienard system using continuous and bivariate q–Hermite polynomials. Some applications, *International Journal of Differential Equations and Applications* (2023).

[48] Vasileva, M., Kyurkchiev, V., Iliev, A., Rahnev, A., Kyurkchiev, N., Some associated polynomials and their q–analogues. Possible applications, *International Electronic Journal of Pure and Applied Mathematics* (2023).

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