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SOME ASSOCIATED POLYNOMIALS AND THEIR q–ANALOGUES. POSSIBLE APPLICATIONS

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Abstract: In this article we consider the associated Gegenbauer polynomials generated by relation $V_{n+1}(x;a,c) = \frac{2(n+a+c)}{n+c+1} x V_n(x;a,c) - \frac{n+2a+c-1}{n+c+1} V_{n-1}(x;a,c)$; $V_0(x;a,c) = 1$; $V_{-1}(x;a,c) = 0$ and continuous q–Gegenbauer polynomials generated by $(1-q^{n+1})C_{n+1}(x,a,q) = 2x(1-aq^n)C_n(x,a,q) - (1-a^2q^{n-1})C_{n-1}(x,a,q)$; $C_0(x,a,q) = 1$; $C_{-1}(x,a,q) = 0$ with possible applications in two directions – the study the dynamics of differential systems and the generation of special classes of radiation diagrams. Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard differential system, extended and continuous q–Gegenbauer polynomials, antenna factor

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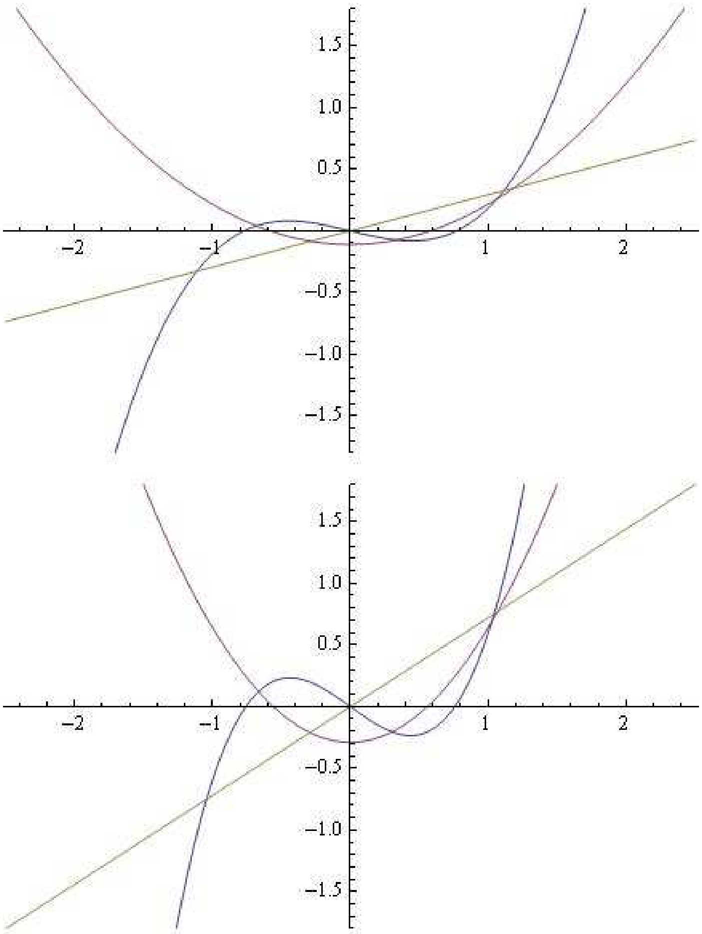

Figure 1: The associated Gegenbauer polynomials $V_1(x; a, c), V_2(x; a, c)$ and $V_3(x; a, cq)$; a) $a = 0.07,\ c = 0.09$; b) $a = 0.2,\ c = 0.25$.

1. Extended families of Gegenbauer polynomials

The associated Gegenbauer polynomials satisfy the recurrence relation

$$V_{n+1}(x; a, c) = \frac{2(n + a + c)}{n + c + 1} x V_n(x; a, c) - \frac{n + 2a + c - 1}{n + c + 1} V_{n-1}(x; a, c)$$

with

$$V_0(x; a, c) = 1; \quad V_{-1}(x; a, c) = 0.$$

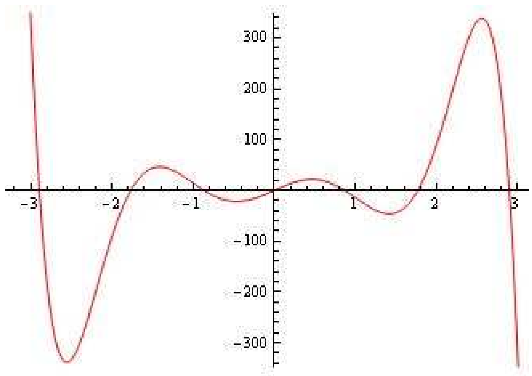

Figure 2: The function $F(x)$.

We have (see Fig. 1)

$$\begin{cases} V_1(x;a,c) &= \frac{2(a+c)}{c+1}x \\ V_2(x;a,c) &= \frac{4(a+c)(a+c+1)}{(c+1)(c+2)}x^2 - \frac{2a+c}{c+2} \\ V_3(x;a,c) &= \frac{8(a+c)(a+c+1)(a+c+2)}{(c+1)(c+2)(c+3)}x^3 - \frac{2(2c^3+6c^2+5a^2c+6ac^2+14ac+6a^2+6a+4c)}{(c+1)(c+2)(c+3)}x \end{cases}$$

Consider the system

$$\begin{cases} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -V_3(x,a,c) + \epsilon F(x)y \end{cases}$$

where the function $F(x)$ is of the form (see Fig. 2)

$$F(x) = 72x - \frac{382}{3}x^3 + \frac{224}{5}x^5 - \frac{128}{35}x^7$$

The simulation on the system for $a = 0.07$, $c = 0.09$ $x_0 = 0.09$, $y_0 = 0$, $\epsilon = 0.0001$ is depicted on Fig. 3 (a), b), c))

Let us define the normalized antenna factor as follows

$$X(\theta) = \frac{|y(b\cos\theta + d)|}{N}.$$

The emitting chart ($b = -0.6$, $d = -0.55$) is depicted in Fig. 3).

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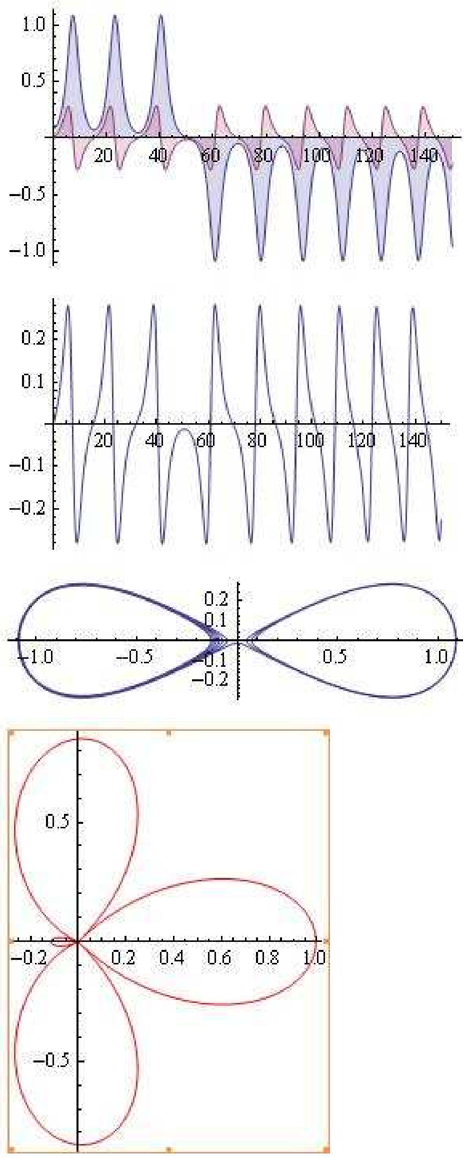

Figure 3: The simulations for $a = 0.07$, $c = 0.09$ $x_0 = 0.09$, $y_0 = 0$, $\epsilon = 0.0001$: a) the solutions of the system; b) y-component of the solution; c) the portrait; d) The antenna factor ($b = -0.6$, $d = -0.55$).

1.1. Continuous q–Gegenbauer polynomials

The continuous q–Gegenbauers polynomials satisfy the three–term recurrence relation

$$(1 - q^{n+1})C_{n+1}(x, a, q) = 2x(1 - aq^n)C_n(x, a, q) - (1 - a^2q^{n-1})C_{n-1}(x, a, q)$$

with

$$C_0(x,a,q)=1;\quad C_{-1}(x,a,q)=0.$$

We have (see Fig. 4)

$$\begin{cases} C_1(x,a,q) &= \frac{2(1-a)}{1-q}x \\ C_2(x,a,q) &= \frac{4(1-a)(1-aq)}{(1-q)(1-q^2)}x^2-\frac{1-a^2}{1-q^2} \\ C_3(x,a,q) &= \frac{8(1-a)(1-aq)(1-aq^2)}{(1-q)(1-q^2)(1-q^3)}x^3-\frac{2(1-a)(1-a^2q)(2+a+q)}{(1-q^2)(1-q^3)}x \end{cases}$$

The simulations on the system

$$\begin{cases} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -C_3(x,a,q)+\epsilon F(x)y \end{cases}$$

for

a) $a=0.07$, $q=0.09$, $x_0=0.02$, $y_0=0$, $\epsilon=0.0001$, $b=-0.42$, $d=-0.55$,

$$F(x)=\frac{107}{20160}x-\frac{7351}{151200}x^3+\frac{2131}{15120}x^5-\frac{1}{8}x^7;$$

b) $a=0.07$, $q=0.09$, $x_0=0.02$, $y_0=0$, $\epsilon=0.0001$, $b=-1.7$, $d=-0.1$,

$$F(x)=x-x^3+x^5-\frac{1}{7}x^7$$

are depicted in Fig. 5–Fig. 6.

Let us mentioned that this idea could be applied successfully for approximation of the emitting chart of antenna pattern by using some orthogonal polynomials and associated polynomials considered in this paper.

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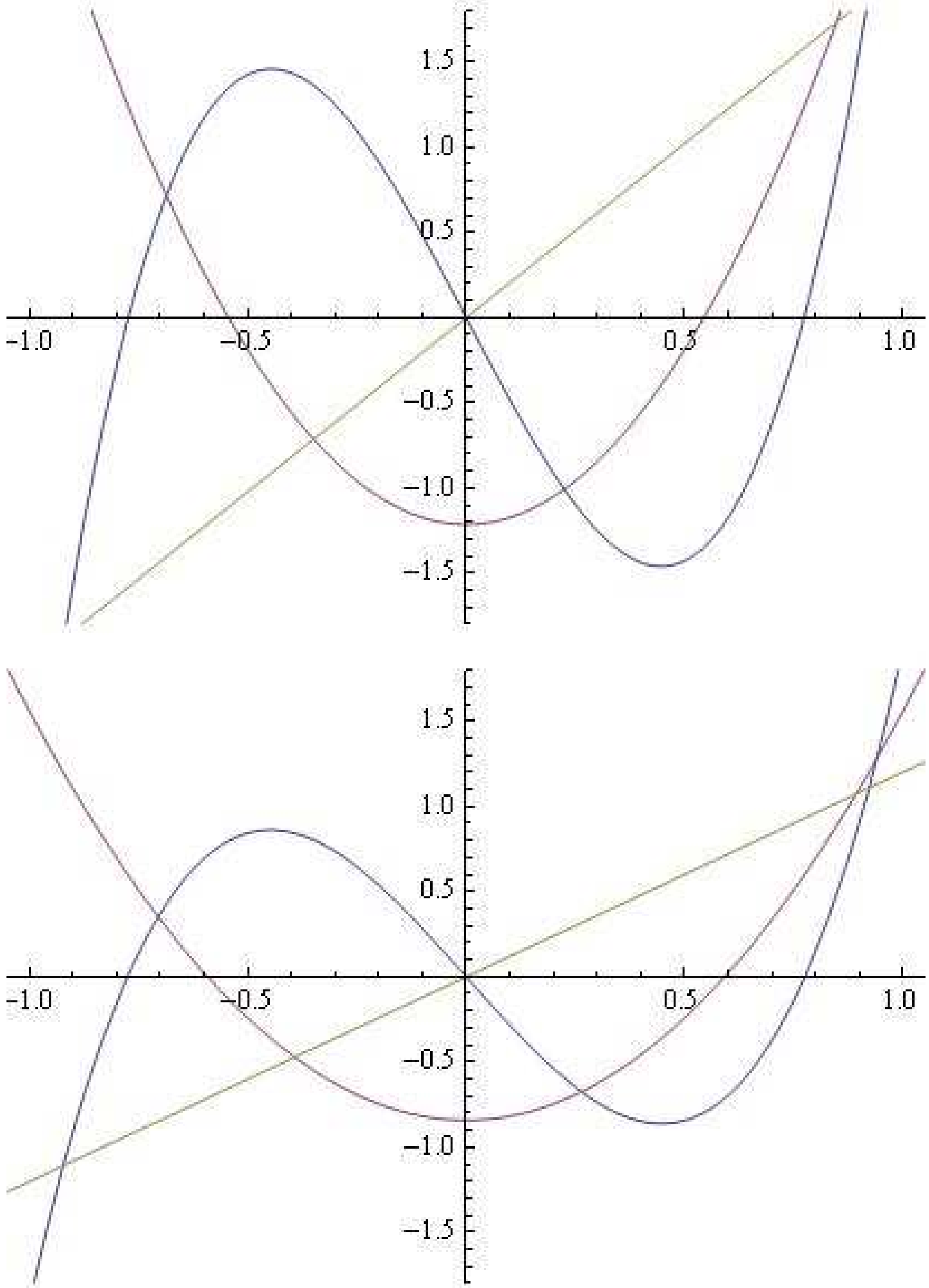

Figure 4: The q–Gegenbauer polynomials $C_1(x, a, q), C_2(x, a, q)$ and $C_3(x, a, q)$; a) $a = 0.07$, $q = 0.09$; b) $a = 0.4$, $q = 0.001$.

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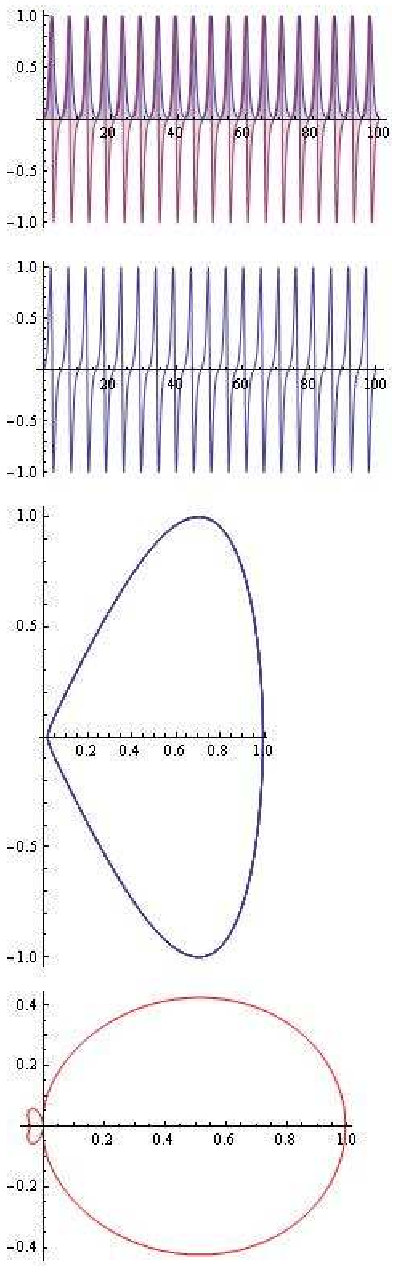

Figure 5: The simulations: the case a).

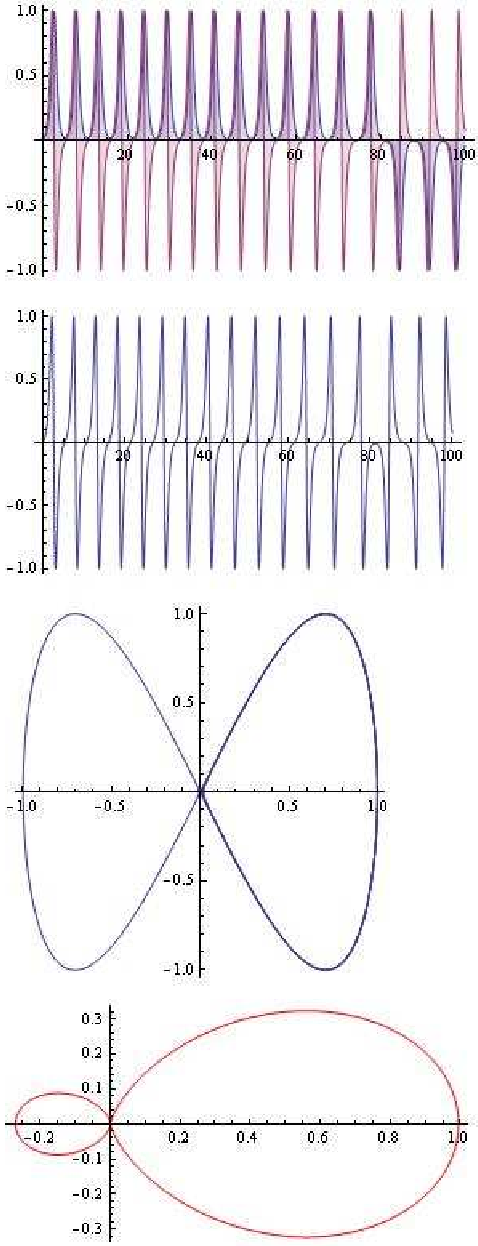

Figure 6: The simulations: the case b).

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