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LIENARD SYSTEM WITH "CORRECTING FACTORS" OF THE TYPE OF INTERPOLATING POLYNOMIALS OF SOME BASIC FUNCTIONS

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Abstract: In this article we consider a new extended Lienard differential system with "correcting factors" of the type of interpolating polynomials of some basic functions as $sin(x)$, $tanh(x)$, $sinh(x)$. The hypothetical oscillator model with Bernoulli's polynomial–correction is also presented.

The models are considered in the light of Melnikov's approach.

Numerical examples, illustrating our results using *CAS MATHEMATICA* are given.

Key Words: Lienard system, Melnikov's polynomial, "correcting factors" of the type of interpolating polynomials, number and type of limit cycles, Bernoulli's polynomial

1. Basic facts. Main Results. Simulations

1. Taylor's series is widely used in applied mathematics and mathematical analysis. The development of $\sin x$ function is well known

$$\sin\, x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} := H_n(x).$$

Fig. 1 shows sin x and its Taylor approximations by polynomials H_3, H_5, H_7. The

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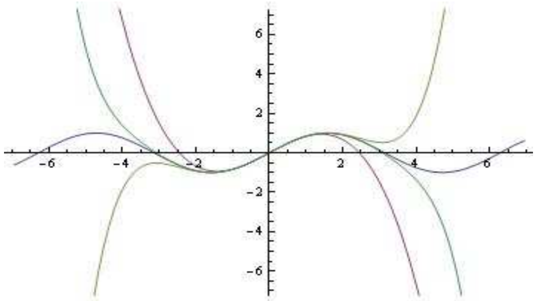

Figure 1: Approximation of $sin\, x$ by H_3, H_5 and H_7.

corresponding Lienard's system to the classical Van der Pol oscillator model

$$x'' + \epsilon(1 - x^2)x' + x = 0 \tag{1}$$

is of the form

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon f(x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{2}$$

where $\epsilon > 0$ and

$$f(x) = x - \frac{x^3}{3} \tag{3}$$

Here we consider the following model for $n = 3, 5, 7, 9, \ldots$ and $\epsilon > 0$

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon H_n(x) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{4}$$

The system (4) leads to the following Van der Pol–type oscillators:

a) for $n = 3$

$$x'' + \epsilon \left(1 - \frac{x^2}{2}\right) x' + x = 0 \tag{5}$$

b) for $n = 5$

$$x'' + \epsilon \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right) x' + x = 0 \tag{6}$$

and so on for each odd n.

The solutions of the differential equations (1), (5) and (6) are shown on Fig. 2. The simulation for $\epsilon = 0.001$ and $n = 7$, with the model (4) for $x_0 = 0.6$, $y_0 = 0.4$

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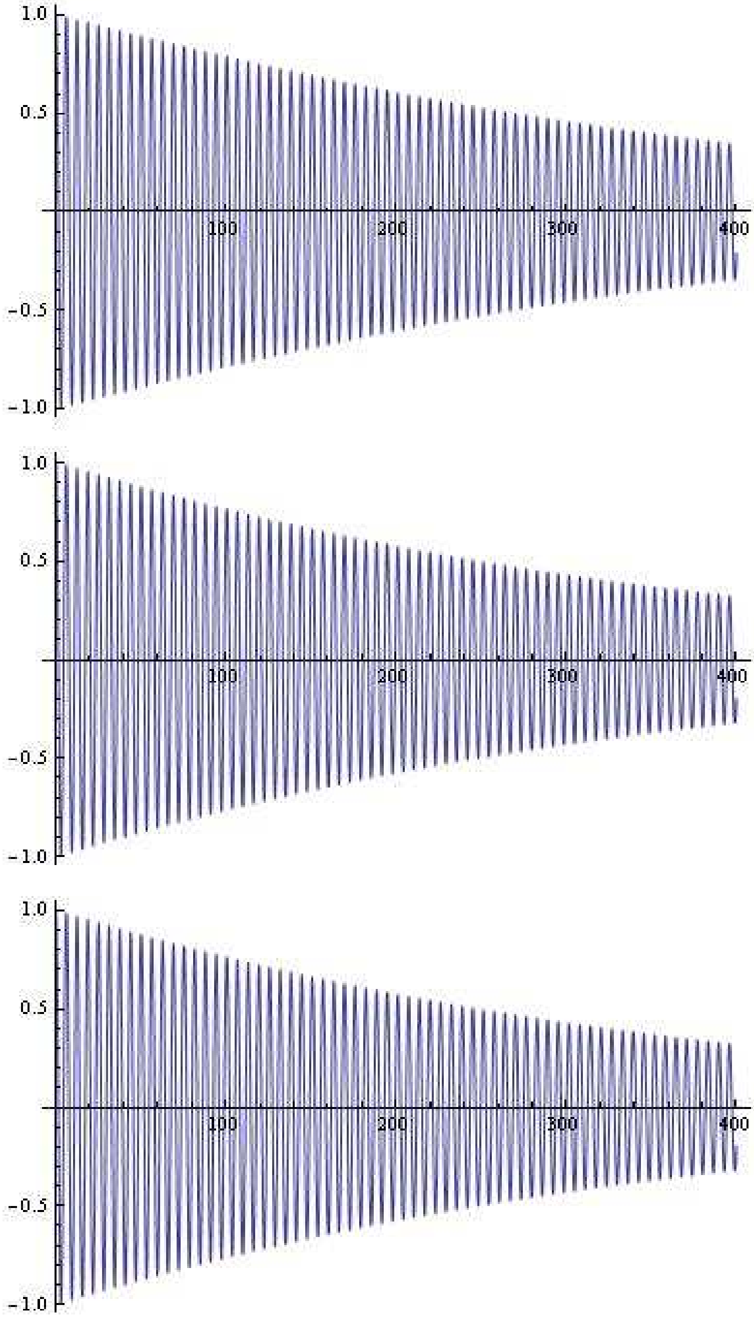

Figure 2: The solutions of the differential equations (1), (5) and (6)

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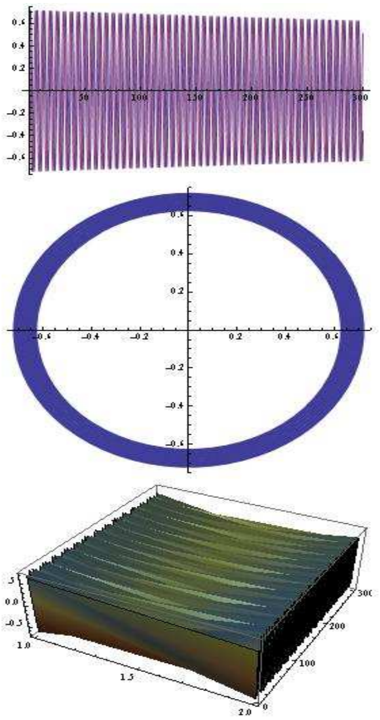

Figure 3: The solutions of the differential system (4). The portrait of the planar system.

is shown in Fig. 3. For $n = 3$ consider the model

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon(\mu x - \frac{x^3}{6}) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{7}$$

in the light of Melnikov's approach. For the Melnikov polynomial in r^2 (see Fig. 4) we have:

$$P(r^2, 1) = \frac{\mu}{2} - \frac{1}{16} r^2. \tag{8}$$

and the Lienard system has for $\mu = 1$ a limit cycle 2.82843.

2. The development of $\tanh x$ function (in $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$) is well known

$$\tanh x = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} x^{2n-1} := H_n^*(x)$$

where B_n is the nth Bernoulli number.

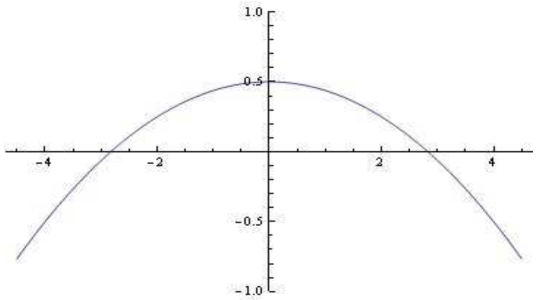

Figure 4: The Melnikov polynomial $P(r^2, 1)$ for $n = 3$ and $\mu = 1$. The root is: 2.82843.

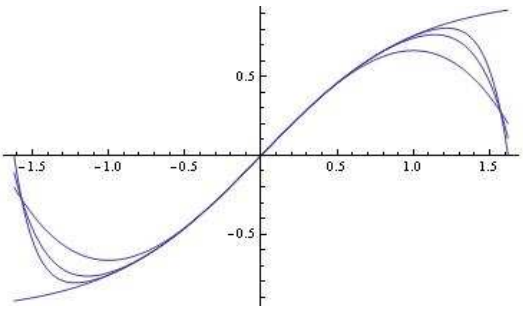

Figure 5: Approximation of $tanh\, x$ by T_3^*, T_7^*, T_{11}^*.

Fig. 5 shows tanh x and its Taylor approximations by polynomials

$$T_3^*(x) = x - \frac{x^3}{3}$$

$$T_7^*(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315}$$

$$T_{11}^*(x) = x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} - \frac{1382x^{11}}{155925}$$

We will explicitly note that the model

$$\begin{cases} \frac{dx}{dt} &= y - \epsilon(T_3^*(x)) \\ \frac{dy}{dt} &= -x \end{cases} \tag{9}$$

coincides with the classical oscillator model of Van der Pol.

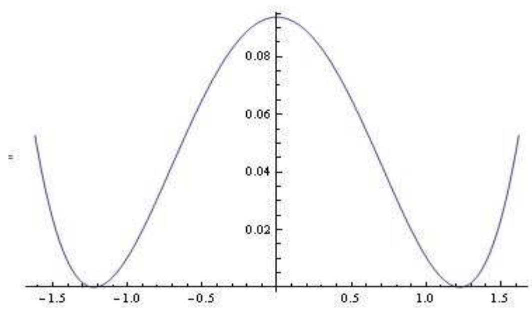

Figure 6: The Melnikov polynomial $P(r^2, 2)$ for $n = 5$ and $\mu = \frac{3}{16}$. The root is: 1.22474... with multiplicity two .

Consider the model

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(\mu x - \frac{x^3}{3} + \frac{2x^5}{15}) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{10}$$

where $\mu > 0$, $\epsilon > 0$ in the light of Melnikov's considerations. For the Melnikov polynomial in r^2 (see Fig. 6) we have:

$$P(r^2, 2) = \frac{\mu}{2} - \frac{1}{8}r^2 + \frac{1}{24}r^4. \tag{11}$$

It is shown that for $\mu = \frac{3}{16}$ the differential model has a limit cycle $\sqrt{\frac{3}{2}} \approx 1.22474$ with multiplicity two.

3. The development of $arsinh\, x$ function (in $x \in (-1, 1)$) is well known

$$arsinh\, x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n)!}{4^n (n!)^2 (2n+1)} x^{2n+1} := A_n(x).$$

Fig. 7 shows $arsinh\, x$ and its approximations by polynomials

$$A_3(x) = x - \frac{x^3}{6}$$

$$A_7(x) = x - \frac{x^3}{6} + \frac{3x^5}{40} - \frac{5x^7}{112}$$

Consider the model (in the case $n = 5$)

$$\begin{cases} \dfrac{dx}{dt} &= y - \epsilon(\mu x - \frac{x^3}{6} + \frac{3x^5}{40}) \\ \dfrac{dy}{dt} &= -x \end{cases} \tag{12}$$

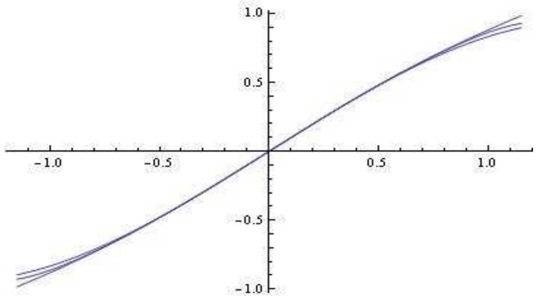

Figure 7: Approximation of $arsinh\, x$ by A_3, A_7.

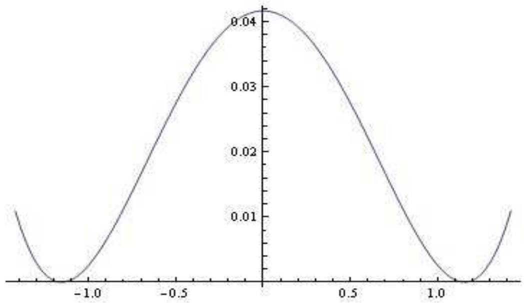

Figure 8: The Melnikov polynomial $P(r^2, 2)$ for $n = 5$ and $\mu = \frac{1}{12}$. The root is: 1.1547... with multiplicity two .

where $\mu > 0$, $\epsilon > 0$ in the light of Melnikov's considerations.

For the Melnikov polynomial in r^2 (see Fig. 8) we have:

$$P(r^2, 2) = \frac{\mu}{2} - \frac{1}{16}r^2 + \frac{3}{128}r^4. \tag{13}$$

It is shown that for $\mu = \frac{1}{12}$ the differential model has a limit cycle $\frac{2}{\sqrt{3}} \approx 1.1547$ with multiplicity two. The simulation for $\epsilon = 0.001$, $\mu = \frac{1}{12}$, $n = 5$, with the model (12) for $x_0 = 0.5$, $y_0 = 0.5$ is shown in Fig. 9.

Consider a Lienard system of type

$$\begin{cases} \dfrac{dx}{dt} & = & y \\ \dfrac{dy}{dt} & = & g(x) + \epsilon f(x) y \end{cases} \tag{14}$$

The solution of the system (14) for $\epsilon = 0.0001$, $g(x) = A_3(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$ (see oscillator model considered in [11]) is visualized on Fig. 10. The solution of

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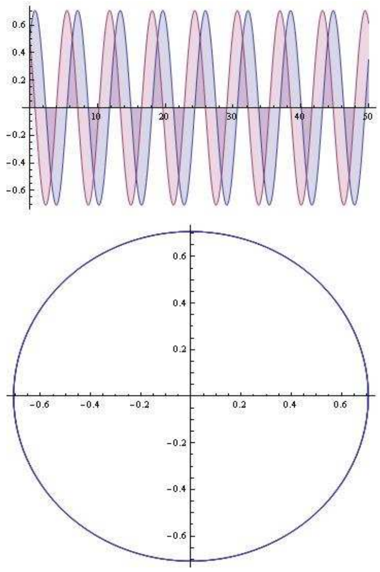

Figure 9: The solutions of the differential system (12). The portrait of the planar system.

the system (14) for $\epsilon = 0.0001$, $g(x) = A_5(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$ is depicted on Fig. 11. We will explicitly note that the $y(t)$-components of the differential systems discussed above can be used successfully in modeling and approximating functions and point sets in the field of signal theory, filters and other characteristics.

4. In this Section we consider the following model of the type:

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon B_n(x) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{15}$$

where $\epsilon > 0$ and $B_n(x)$ for $n = 3, 5, 7, 9, ...$ is the Bernoulli's polynomial.

For example we have (see Fig. 12)

$$\begin{aligned} B_3(x) &= x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x \\ B_5(x) &= x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x \\ B_7(x) &= x^7 - \tfrac{7}{2}x^6 + \tfrac{7}{2}x^5 - \tfrac{7}{6}x^3 + \tfrac{1}{6}x \\ B_9(x) &= x^9 - \tfrac{9}{2}x^8 + 6x^7 - \tfrac{21}{5}x^5 + 2x^3 - \tfrac{3}{10}x \end{aligned}$$

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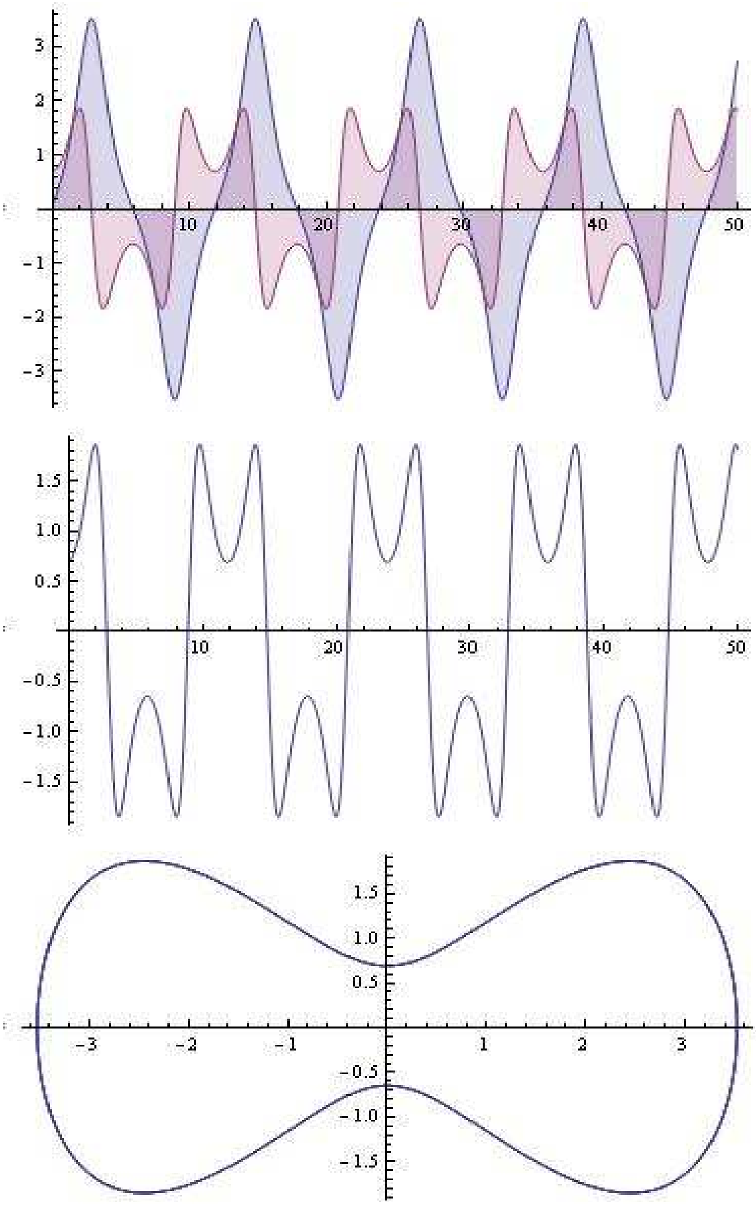

Figure 10: a) The solutions of the planar system (14) for $\epsilon = 0.001$, $\epsilon = 0.0001$, $g(x) = A_3(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$; b) the y-component of the solution; c) the portrait of the planar system.

The solutions of the system

$$\begin{cases} \dfrac{dx}{dt} = y - \epsilon(B_5(x)) \\ \dfrac{dy}{dt} = -x \end{cases} \tag{16}$$

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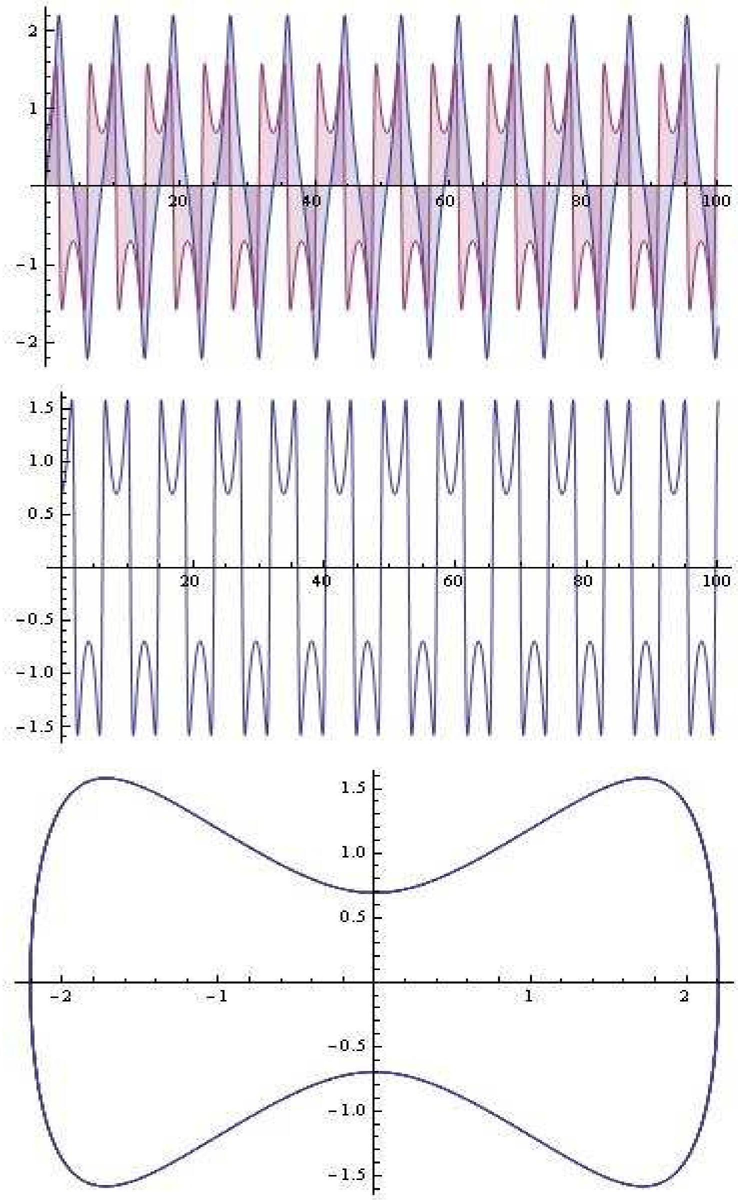

Figure 11: a) The solutions of the system (14) for $\epsilon = 0.001$, $\epsilon = 0.0001$, $g(x) = A_5(x)$, $f(x) = x - x^3 + x^5 - \frac{1}{7}x^7$; b) the y-component of the solution; c) the portrait of the planar system.

for $\epsilon = 0.001; x_0 = 0.7, y_0 = 0.3$ is depicted on Fig. 13.

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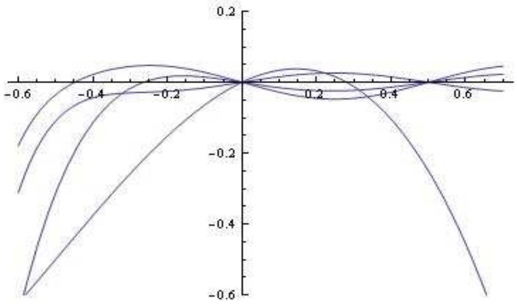

Figure 12: The polynomials $B_n(x)$ for $n = 3, 5, 7, 9$.

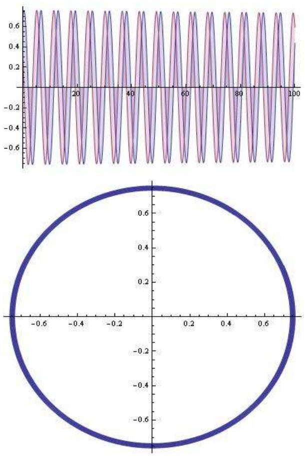

Figure 13: The solutions of the differential system (16). The portrait of the planar system.

For the model

$$\begin{cases} \dfrac{dx}{dt} & = & y - \epsilon(B_7(x)) \\ \dfrac{dy}{dt} & = & -x \end{cases} \tag{17}$$

for $\epsilon = 0.001; x_0 = 0.9, y_0 = 0.1$ see Fig. 14.

For other results see [11]–[21].

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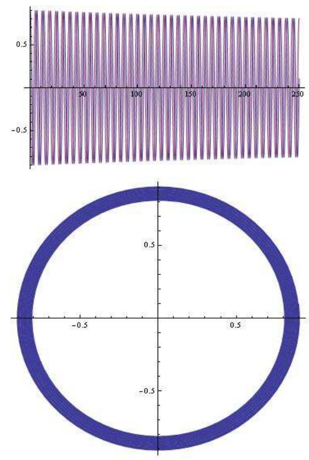

Figure 14: The solutions of the differential system (17). The portrait of the planar system.

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