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A TECHNIQUE FOR SIMULATING THE DYNAMICS OF SOME EXTENDED NONLINEAR MODELS

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Abstract: This paper proposes a technique for simulating the dynamics of some extended nonlinear models, such as "extended generalized Lotka–Volterra model with K species ", "Relaxation oscillator model" and others with the introduction of corrective functions of polynomial type.

We offer a software tool for simulating the dynamics of the new extended families.

We hope that the proposed module, implemented in *CAS Mathematica*, will support the work of researchers working in this scientific field.

AMS Subject Classification: 65L07 34A34

Key Words: extended generalized Lotka–Volterra model with K species , "Relaxation oscillator model", "polynomial intervention factors"

1. Introduction

In [7]–[8] the authors considered a modification of the Lotka–Volterra model with 2 species and extended competition model.

The input functions are "polynomial intervention factors" (monotonically increasing, or monotonically decreasing in the considered interval).

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The dynamics and non-trivial equilibrium are also studied. A software tool for simulating the dynamics of a new extended family of the Lotka–Volterra model is also given.

The considered methodological aspects can be successfully applied to study the dynamics of some nonlinear models.

1. The **generalized Lotka–Volterra predator–prey system** where there are k prey species and k predators, which prey on all the prey species but with different severity (see, for example [1]) is given by:

$$\begin{cases} \dfrac{dN_i(t)}{dt} = N_i(t)\left(a_i - \sum_{j=1}^{k} b_{ij}P_j(t)\right) \\ \\ \dfrac{dP_i(t)}{dt} = P_i(t)\left(\sum_{j=1}^{k} c_{ij}N_j(t) - d_i\right) \\ \\ \qquad i = 1, 2, \ldots k. \end{cases} \tag{1}$$

A detailed analysis of the complexity and stability of the Lotka-Volterra predator-prey model the reader can find in the monograph [1].

2. Consider the following nonlinear model (from the field of Enzyme Kinetics):

$$\begin{cases} \dfrac{dx}{dt} = y(t) - ax^3(t) + bx^2(t) \\ \\ \dfrac{dy}{dt} = -c - dx^2(t) - y(t) \end{cases} \tag{2}$$

with $a, b, c, d > 0$.

For other results, see [6].

3. The following ”Relaxation oscillator” is given by van der Pol [4] (see. also Murray [1]):

$$\begin{cases} \dfrac{dx}{dt} = \frac{1}{c}(f(x) - y) \\ \\ \dfrac{dy}{dt} = x \end{cases} \tag{3}$$

with $c < 1$ and $f(x) = x - \frac{1}{3}x^3$.

In this article we offer a technique for simulating the dynamics of some extended nonlinear models with ”polynomial intervention factors”.

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2. Main Results. Simulations

In this Section we first consider the following **"extended generalized Lotka–Volterra model with K species and intervention polynomial factors"** of the type:

$$\begin{cases} \dfrac{dN_i(t)}{dt} = N_i(t)\left(a(t) - \sum_{j=1}^{k} b_{ij}P_j(t)\right) \\ \dfrac{dP_i(t)}{dt} = P_i(t)\left(\sum_{j=1}^{k} c_{ij}N_j(t) - d(t)\right) \\ i = 1, 2, \dots k \end{cases} \tag{4}$$

where

$$a(t) = \sum_{i=0}^{n} a_i t^i$$

$$d(t) = \sum_{i=0}^{n} d_i t^i.$$

The system (4) can be written as

$$\begin{cases} \dfrac{d\mathbf{N}}{dt} = \mathbf{N}^T \left(a(t) - B\mathbf{P}\right) \\ \dfrac{d\mathbf{P}}{dt} = \mathbf{P}^T \left(C\mathbf{N} - d(t)\right) \end{cases} \tag{5}$$

where the superscript T denotes the transpose.

Example 1. A simulation for user-selected coefficients b_{ij}, c_{ij}:

$$b_{ij} = \{\{0.1, 0.2, 0.3\}, \{0.4, 0.2, 0.1\}, \{0.3, 0.5, 0.2\}\},$$

$$c_{ij} = \{\{0.3, 0.5, 0.6\}, \{0.1, 0.6, 0.2\}, \{0.3, 0.1, 0.7\}\}$$

and vectors **a**, **b**:

$$a = \{0.5, 0.1, 0.2\},$$

$$d = \{0.3, 0.1, 0.4\}$$

and initial approximations: $N_i(0) = P_i(0) = 1;\ i = 1, 2, 3$ for the model (1) is shown in Fig. 1.

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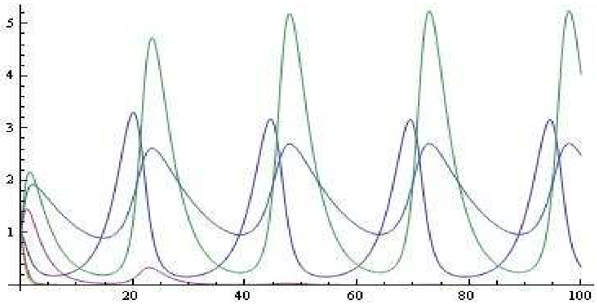

Figure 1: The solutions of the generalized Lotka-Volterra predator-prey system (1) (Example 1).

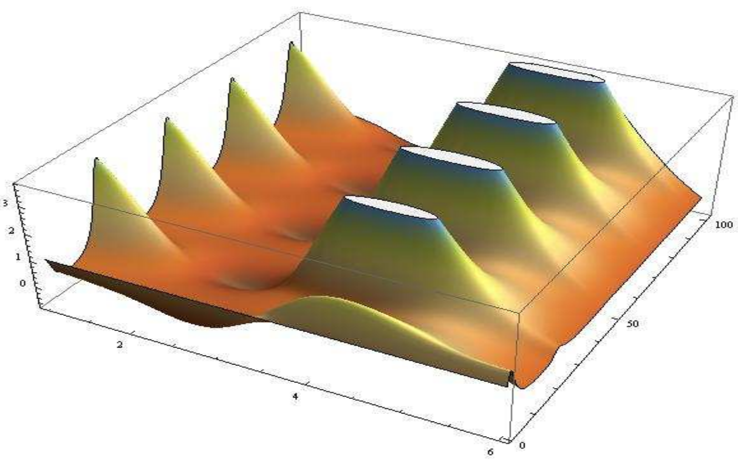

Figure 2: ListPlot3D[...] for generalized Lotka-Volterra predator-prey system (1) (Example 1).

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For ListPlot3D[...] see Fig. 2.

We propose a software module (intellectual properties) within the programming environment *CAS Mathematica* for the analysis of the considered model (5) (Figures 3–5).

Analyzing the simulations using model (4) (resp. (5) (see Figures 5–6), we conclude that the new extended model is very sensitive to the coefficients of the input polynomials.

This makes it attractive for conducting computer simulations.

Example 2. A simulation for user-selected matrices B and C:

$$B = \begin{pmatrix} 0.1 & 0.2 & 0.3 \\ 0.4 & 0.2 & 0.1 \\ 0.3 & 0.5 & 0.2 \end{pmatrix}$$

$$C = \begin{pmatrix} 0.3 & 0.5 & 0.6 \\ 0.1 & 0.6 & 0.2 \\ 0.3 & 0.1 & 0.7 \end{pmatrix}$$

and "polynomial intervention factors":

$$a(t) = 0.5 - 0.001t - 0.0001t^2$$

$$d(t) = 0.3 - 0.001t - 0.00005t^2$$

and initial approximations: $N_i(0) = P_i(0) = 1;\ i = 1, 2, 3$ for the model (5) is shown in Fig. 5.

For ListPlot3D[...] see Fig. 6.

2. Consider the following **"extended model of type (2)"**:

$$\begin{cases} \dfrac{dx}{dt} = y(t) - a(t)x^3(t) + bx^2(t) \\ \\ \dfrac{dy}{dt} = -c - d(t)x^2(t) - y(t) \end{cases} \tag{6}$$

with $a(t) = \sum_{i=0}^{n} a_i t^i;\ d(t) = \sum_{i=0}^{n} d_i t^i$.

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```
Print["Extended generalised Lotka-Volterra model for interacting
population with K species and intervention polynomial factors:"];

i1 = Input["Input i: "];
Print["i = ", i1];
j1 = Input["Input j: "];
Print["j = ", j1];

b = Input[Array[b## &, {i1, j1}]]
MatrixForm[b]

c = Input[Array[c## &, {i1, j1}]]
MatrixForm[c]

"Input polynomial factor a[t]j: "
Print["a[t]==0.5-0.001*t-0.0001*t^2"];

"Input polynomial factor d[t]j: "
Print["d[t]==0.3-0.001*t-0.00005*t^2"];

Print["The extended generalised Lotka-Volterra Lotka-Voltera
predator-prey system of differential equations: "];

Print["N[[1]]'[t] ==N[[1]][t](a[t]- Sum_{j=1}^{i1} b[[1,j]]*P[[j]][t])"];
Print["N[[2]]'[t] ==N[[2]][t](a[t]- Sum_{j=1}^{i1} b[[2,j]]*P[[j]][t])"];
Print["N[[3]]'[t] ==N[[3]][t](a[t]- Sum_{j=1}^{i1} b[[3,j]]*P[[j]][t])"];
Print["P[[1]]'[t] ==P[[1]][t](Sum_{j=1}^{i1} c[[1,j]]*N[[j]][t]-d[t])"];

Print["P[[2]]'[t] ==P[[2]][t](Sum_{j=1}^{i1} c[[2,j]]*N[[j]][t]-d[t])"];
Print["P[[3]]'[t] ==P[[3]][t](Sum_{j=1}^{i1} c[[3,j]]*N[[j]][t]-d[t])"];
```

Figure 3: The module in *CAS Mathematica*

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```
N10 = Input["Input initial condition - N1[0]"];
Print["Initial condition N10 = ", N10];
N20 = Input["Input initial condition - N2[0]"];
Print["Initial condition N20 = ", N20];
N30 = Input["Input initial condition - N3[0]"];
Print["Initial condition N30 = ", N30];
P10 = Input["Input initial condition - P1[0]"];
Print["Initial condition P10 = ", P10];
P20 = Input["Input initial condition - P2[0]"];
Print["Initial condition P20 = ", P20];
P30 = Input["Input initial condition - P3[0]"];
Print["Initial condition P30 = ", P30];

t0 = Input["Input t0, for which we shall investigate model for
interacting population"];
Print["t0 = ", t0];
t1 = Input["Input t1, for which we shall investigate model for
interacting population"];
Print["t1 = ", t1];

Print["Graphics of the periodic solutions of the system of differential
equations as functions of the time t"];

NDSolve[{N1'[t] == N1[t] (0.5 - 0.001*t - 0.0001*t^2 - b[[1,1]]*P1[t] - b[[1,2]]*P2[t]
        - b[[1,3]]*P3[t]),
   N2'[t] == N2[t] (0.5 - 0.001*t - 0.0001*t^2 - b[[2,1]]*P1[t] - b[[2,2]]*P2[t]
        - b[[2,3]]*P3[t]),
   N3'[t] == N3[t] (0.5 - 0.001*t - 0.0001*t^2 - b[[3,1]]*P1[t] - b[[3,2]]*P2[t]
        - b[[3,3]]*P3[t]),
   P1'[t] == P1[t] (c[[1,1]]*N1[t] + c[[1,2]]*N2[t] + c[[1,3]]*N3[t] - (0.3 - 0.001*t
          - 0.00005*t^2)),
   P2'[t] == P2[t] (c[[2,1]]*N1[t] + c[[2,2]]*N2[t] + c[[2,3]]*N3[t] - (0.3 - 0.001*t
          - 0.00005*t^2)),
   P3'[t] == P3[t] (c[[3,1]]*N1[t] + c[[3,2]]*N2[t] + c[[3,3]]*N3[t] - (0.3 - 0.001*t
          - 0.00005*t^2)),
   N1[0] == N10, N2[0] == N20, N3[0] == N30, P1[0] == P10, P2[0] == P20, P3[0] == P30},
  {N1, N2, N3, P1, P2, P3}, {t, t0, t1}];

Plot[Evaluate[{N1[t], N2[t], N3[t], P1[t], P2[t], P3[t]}
   /. First[%]], {t, t0, t1}]
```

Figure 4: The module in *CAS Mathematica* (continued)

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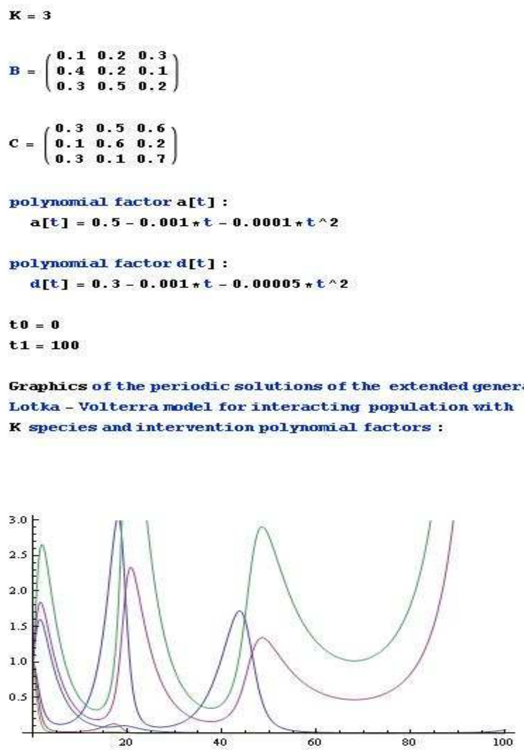

Figure 5: The solutions of the system of differential equations (5) (Example 2).

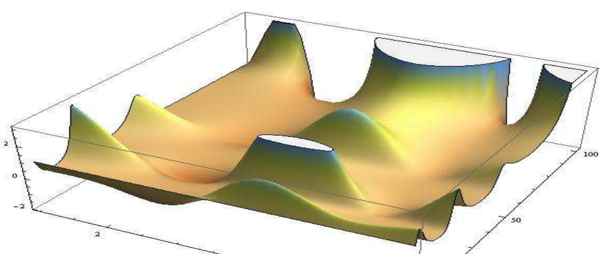

Figure 6: ListPlot3D[...] for extended generalized Lotka-Volterra predator-prey system (5) (Example 2).

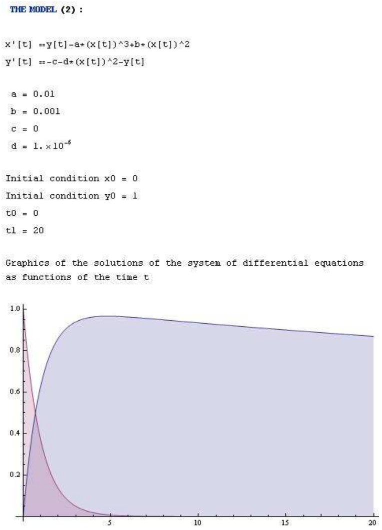

Figure 7: The solutions of the system of differential equations (2) (Example 3).

Example 3. A simulation for user-selected coefficients

$$a = 0.01;\ \ b = 0.001;\ \ c = 0;\ \ d = 0.000001$$

and initial approximations: $x(0) = 0;\ y(0) = 1$ for the model (2) is shown in Fig. 7.

Example 4. A simulation for user-selected coefficients $b = 0.001;\ \ c = 0$ and polynomials

$$a(t) = 0.01 + 0.006t - 0.00015t^2;\ \ d(t) = 0.000001 + 0.002t - 0.000001t^2$$

and initial approximations: $x(0) = 0;\ y(0) = 1$ for the model (6) is shown in Fig. 8.

3. Consider the following **"extended model of type (3)"**:

$$\begin{cases} \dfrac{dx}{dt} & = & \frac{1}{c(t)}(f(x) - y) \\ \\ \dfrac{dy}{dt} & = & x \end{cases} \tag{7}$$

with $f(x) = x - \frac{1}{3}x^3$ and $c(t) = \sum_{i=0}^{n} c_i t^i$.

Example 5. A simulation for user-selected coefficient $c = 0.01$ for the model (3) is shown in Fig. 9.

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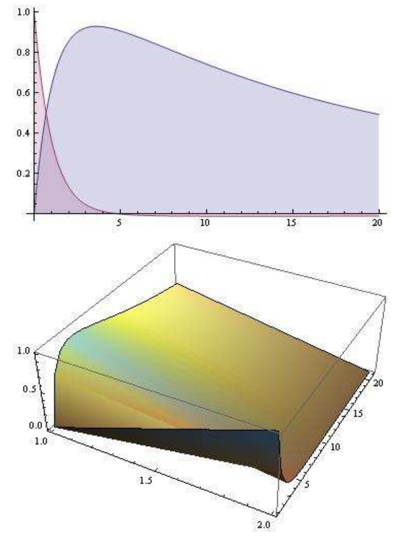

Figure 8: The solutions of the system of differential equations (6) (Example 4).

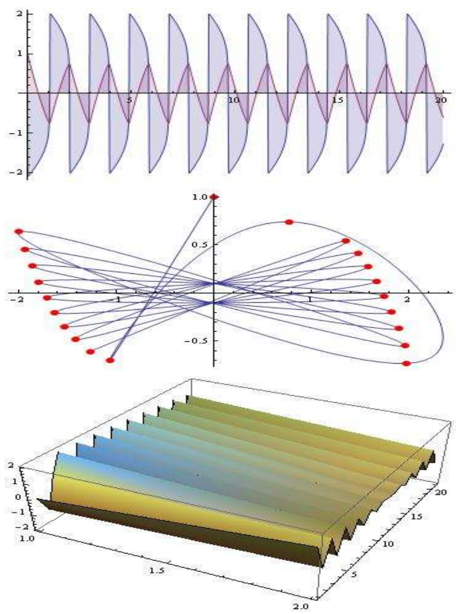

Figure 9: The solutions of the system of differential equations (3) (Example 5).

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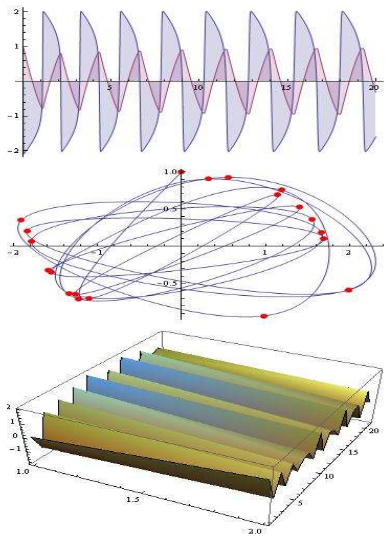

Figure 10: The solutions of the system of differential equations (7) (Example 6).

Example 6. A simulation for user-selected polynomial $c(t) = 0.01 + 0.006t - 0.00015t^2$ for the model (7) is shown in Fig. 10

Example 7. A simulation for user-selected polynomial $c(t) = 0.001 + 0.0012t + 0.001t^2 - 0.0000045t^3$ for the model (7) is shown in Fig. 11

For other results, see [5].

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3. Concluding Remarks

The module proposed in the article is only one element of the planned general upgrade of the computer algebraic system in connection with the new methodological considerations in studying the dynamics of SIR/SIRD/SEIR/GSEIR models (see for example [9]).

Acknowledgment

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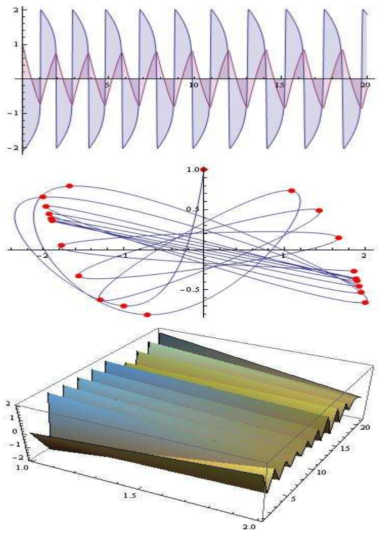

Figure 11: The solutions of the system of differential equations (7) (Example 7).

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