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# INVESTIGATIONS ON A HYPOTHETICAL PIECEWISE SMOOTH LOG–LOGISTIC GROWTH FUNCTION. SOME APPLICATIONS. III

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**Abstract:** In this article a hypothetical piecewise smooth Log–Logistic sigmoidal model $F(f_1(t), f_2(t))$ is defined. A new class of growth function generated by reaction networks and based on "correcting amendments of fractional linear function–type" is proposed. Some numerical examples, using *CAS MATHEMATICA* illustrating our results are given.

**AMS Subject Classification:** 41A46
**Key Words:** hypothetical piecewise smooth log–logistic sigmoidal growth function, Heaviside step–function, cut function, Hausdorff distance, reaction network

## 1. Introduction

In many applications sigmoid functions have a very steep slope around a "centre", that is they are "close" to a step function having a step (jump) at that centre. Therefore the question of estimating the "degree of closeness" between a sigmoid function and a corresponding step function arises. Such estimates can find various applications, including a possibility to substitute a sigmoid function by a step function (or vice versa) whenever appropriate. Apparently, in order to formulate the above problem in a formal way we need a measure for closeness between the sigmoid and the step function. In our case we have chosen the Hausdorff metric.

Following the ideas given in [12], in this article a hypothetical piecewise smooth Log–Logistic sigmoidal model $F(f_1(t), f_2(t))$ is defined. These sigmoid functions are

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used in biological applications, income and lifetime analysis, financial mathematics, fuzzy set theory, impulsive analysis etc. Thus the log–logistic distribution is used in fields such as biostatistics, population dynamic, medical research [3], insurance and economics [6]. The log–logistic distribution is also known as the Fisk distribution [4]. The transmuted log-logistic distribution has been used in the analysis of extreme values [2]. Shaw et al. [7], Gupta et. al. [5] study a new model which generalizes the log–logistic function [1].

**Definition 1.** The c.d.f. $f_1(t)$ of the special log–logistic distribution is defined for $t \geq 0$ and $c > 0$ by:

$$f_1(t) = \frac{1}{1 + \left(\frac{t-a}{-a}\right)^{-c}}, \quad f_1(0) = \frac{1}{2}. \tag{1}$$

The Hausdorff distance $d$ between the Heaviside step function $h_0(t)$ and the sigmoid log–logistic function $f_1(t)$ satisfies the relation

$$f_1(d) = \frac{1}{1 + \left(\frac{d-a}{-a}\right)^{-c}} = 1 - d, \tag{2}$$

or

$$\ln \frac{1-d}{d} - c \ln \left(1 - \frac{d}{a}\right) = 0. \tag{3}$$

The following proposition gives upper and lower bounds for $d$

**Theorem A.** [8]. The Hausdorff distance [13] $d = d(a, c)$ between the Heaviside step function $h_0$ and the sigmoidal log–logistic function $f_1(t)$ can be expressed in terms of the parameters $a < 0$ and $c > 0$ for any real $-\frac{c}{a} \geq 2$ as follows:

$$\frac{1}{1 - \frac{c}{a}} < d < \frac{\ln \left(1 - \frac{c}{a}\right)}{1 - \frac{c}{a}}. \tag{4}$$

## 2. Main Results

The reader can formulate a hypothetical piecewise smooth sigmoidal model using the log–logistic model:

$$f_1(t) = \frac{1}{1 + \left(\frac{t-a}{-a}\right)^{-c}}.$$

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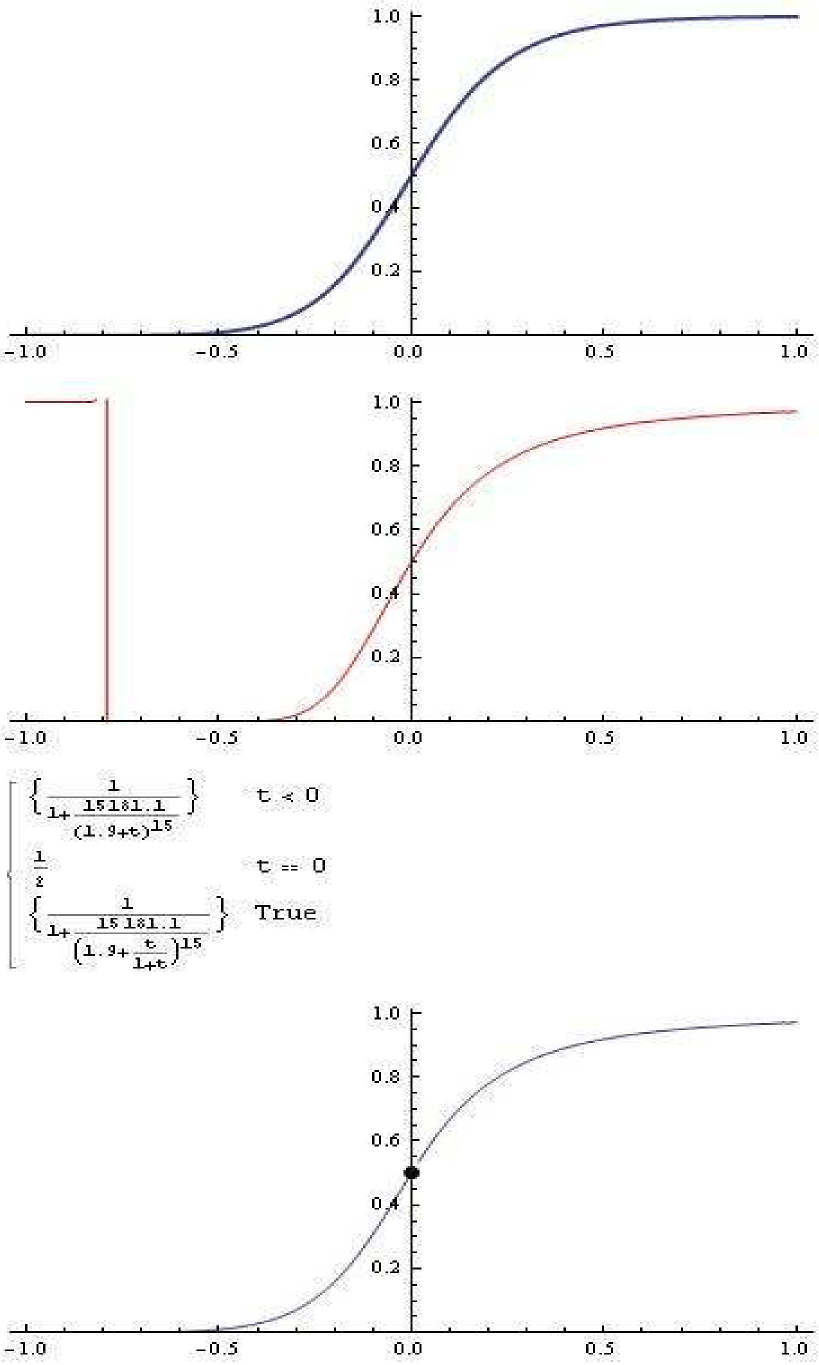


Figure 1: The functions $f_1(t)$, $f_2(t)$ and $F(f_1(t), f_2(t))$ for $a = -1.9$, $c = 15$.

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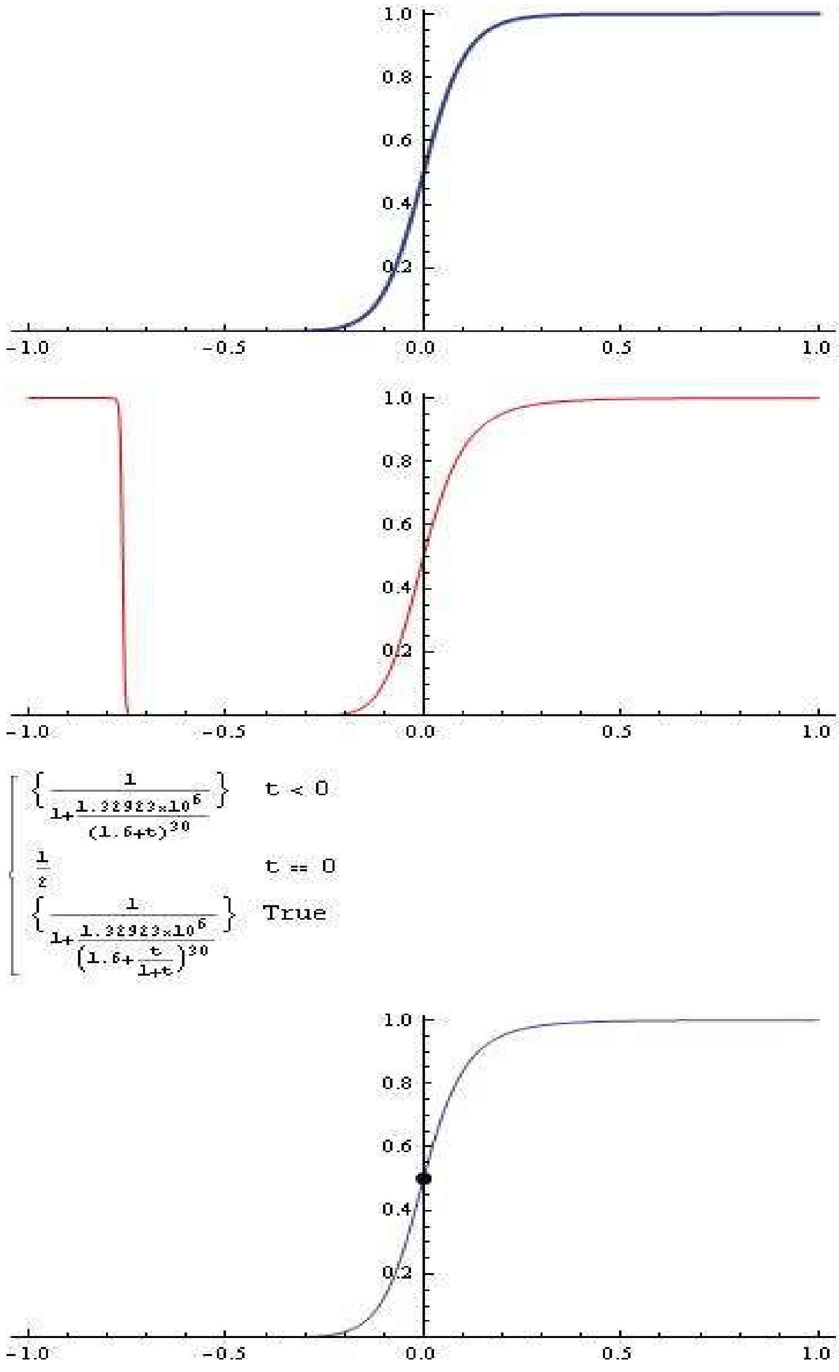


Figure 2: The functions $f_1(t)$, $f_2(t)$ and $F(f_1(t), f_2(t))$ for $a = -1.6$, $c = 30$.

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In the light of the discussions in this paper, the researcher can achieve saturation level

$$B_1 = \lim_{t\to+\infty} \frac{1}{1+\left(\frac{\frac{t}{1+t}-a}{-a}\right)^{-c}}$$

if he uses, for example, the function

$$f_2(t) = \frac{1}{1+\left(\frac{\frac{t}{1+t}-a}{-a}\right)^{-c}}.$$

It is easy to see that the hypothetical piecewise smooth log–logistic growth model is of the form:

$$F(t) := \begin{cases} \dfrac{1}{1+\left(\frac{t-a}{-a}\right)^{-c}} := f_1(t),\ \ t<0 \\ \dfrac{1}{2},\ \ t=0 \\ \dfrac{1}{1+\left(\frac{\frac{t}{1+t}-a}{-a}\right)^{-c}} := f_2(t),\ \ t>0. \end{cases} \tag{5}$$

Evidently,

$$f_1'(0) = f_2'(0).$$

The hypothetical piecewise smooth log–logistic model $F(f_1(t), f_2(t))$ is depicted on Fig. 1–2.

In addition, the reader can consider the interesting problem of approximating the Heaviside step function

$$h_0(t) = \begin{cases} 0, & \text{if} \quad t<0, \\ [0, B_1], & \text{if} \quad t=0, \\ B_1, & \text{if} \quad t>0, \end{cases}$$

with the new class of growth functions $F(f_1(t), f_2(t))$ with respect to the Hausdorff distance. In this regard, it is sufficient to use the methodology given in [14].

### 2.1. Applications

Consider the following data from the field of Computer Viruses Propagation

$$dataVirus := \{\{0.05, 0.63\}, \{0.1, 0.78\}, \{0.2, 0.86\}, \{0.3, 0.92\},$$

$$\{0.4, 0.95\}, \{0.5, 0.987\}, \{0.8, 0.99\}, \{0.9, 0.99\}, \{1, 0.99\}\}$$

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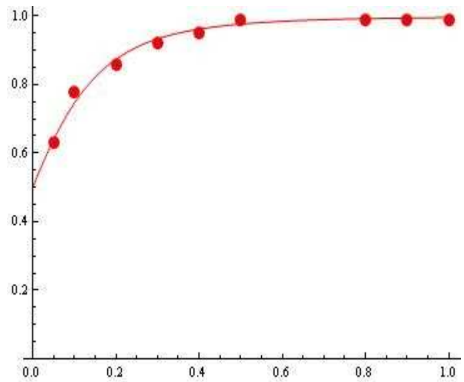


Figure 3: The fitted model $f_2(t)$ for $a = -1.81$ and $c = 21.956$ ("dataVirus").

For the "dataVirus" the fitted model

$$f_2(t) = \frac{1}{1 + \left( \frac{\frac{t}{1+t} - a}{-a} \right)^{-c}}$$

for $a = -1.81$ and $c = 21.956$ is depicted on Fig. 3.

**Remarks.**

**1.** A possible explanation for the good approximation of the mentioned dataset is due to the fact that the magnitude of the asymptotic saturation for the proposed model is $B_1 \approx 0.999$, which corresponds to the nature of the data.

**2.** This is also our suggestion to researchers - knowing the approximate level of saturation, to use some of the proposed piecewise smooth log–logistic sigmoidal growth functions.

### 2.2. New class of growth function generated by reaction networks and based on "correcting amendments of fractional linear function–type"

Consider the logistic growth–decay pair generated by the following reaction network (in canonical form) involving two reacting species $Y, X$:

$$Y + X \xrightarrow{\gamma(t)} 2X \tag{6}$$

wherein $\gamma(t)$ is the "rate function".

We discuss the usage of the framework of chemical reaction networks for the construction of new dynamical model. The process of construction of reaction-network-based models via mass action kinetics is introduced and illustrated.

Reaction network (6) induces the following differential system

$$\begin{cases} \dfrac{dy(t)}{dt} = -\gamma(t)y(t)x(t) \\ \dfrac{dx(t)}{dt} = \gamma(t)y(t)x(t) \end{cases} \tag{7}$$

with $y(0) = y_0$; $x(0) = x_0$.

The above example illustrate the process of translation of a chemical reaction networks into systems of ordinary differential equations.

Let

$$\gamma(t) = \frac{c}{(1+t)^2((1-a)t-a)}.$$

Hence, the new model can be written for the growth function in the form:

$$\begin{aligned} & x'(t) = \frac{c}{(1+t)^2((1-a)t-a)} x(t)(1-x(t)) \\ & x(0) = \tfrac{1}{2}. \end{aligned} \tag{8}$$

Some computational examples using *CAS Mathematica* are given in Figures 4–6. Obviously, the function $x(t)$ coincides with the second component $f_2(t)$ of the defined and studied in detail in this article hypothetical piecewise smooth log–logistic growth function $F(f_1(t), f_2(t))$ (see, Fig. 6). The new model has been applied to simulate biological growth data sets coming from various fields of science.

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```
a = Input["a"];(* -1 *)
Print["a = ", a];
c = Input["c"];(* 15 *)
Print["c = ", c];
x0 = Input["Input initial condition - x[0]"]; (* 0.5 *)
Print["Initial condition x0 = ", x0];
y0 = Input["Input initial condition - y[0]"]; (* 0.5 *)
Print["Initial condition y0 = ", y0];
t0 = Input["Input t0"];
Print["t0 = ", t0];
t1 = Input["Input t1"];
Print["t1 = ", t1];
Print["The solutions of the system of differential equations"];
NDSolve[{x'[t] == c / ((1 + t)^2 * ((1 - a) * t - a)) * x[t] * (1 - x[t]), x[0] == x0,
   y'[t] == -c / ((1 + t)^2 * ((1 - a) * t - a)) * y[t] * (1 - y[t]), y[0] == y0}, {x, y}, {t, t0, t1}];
Plot[Evaluate[{x[t], y[t]} /. First[%]], {t, t0, t1}, AxesOrigin → {0, 0}]
```

```
a = -1
c = 15
Initial condition x0 = 0.5
Initial condition y0 = 0.5
t0 = 0
t1 = 1
The solutions of the system of differential equations
```

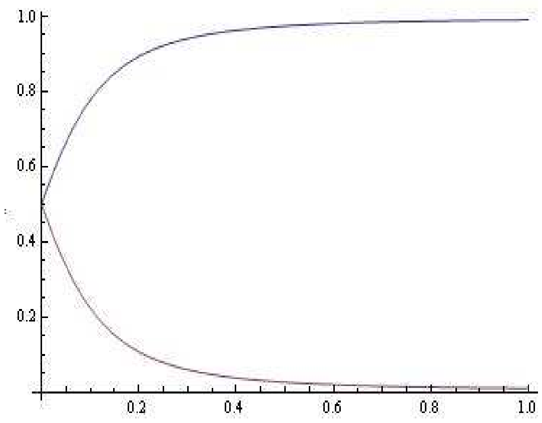


Figure 4: Module in the software environment *CAS Mathematica* for solving and visualizing the solution of system of the differential equations (7) (for $a = -1$, $c = 15$).

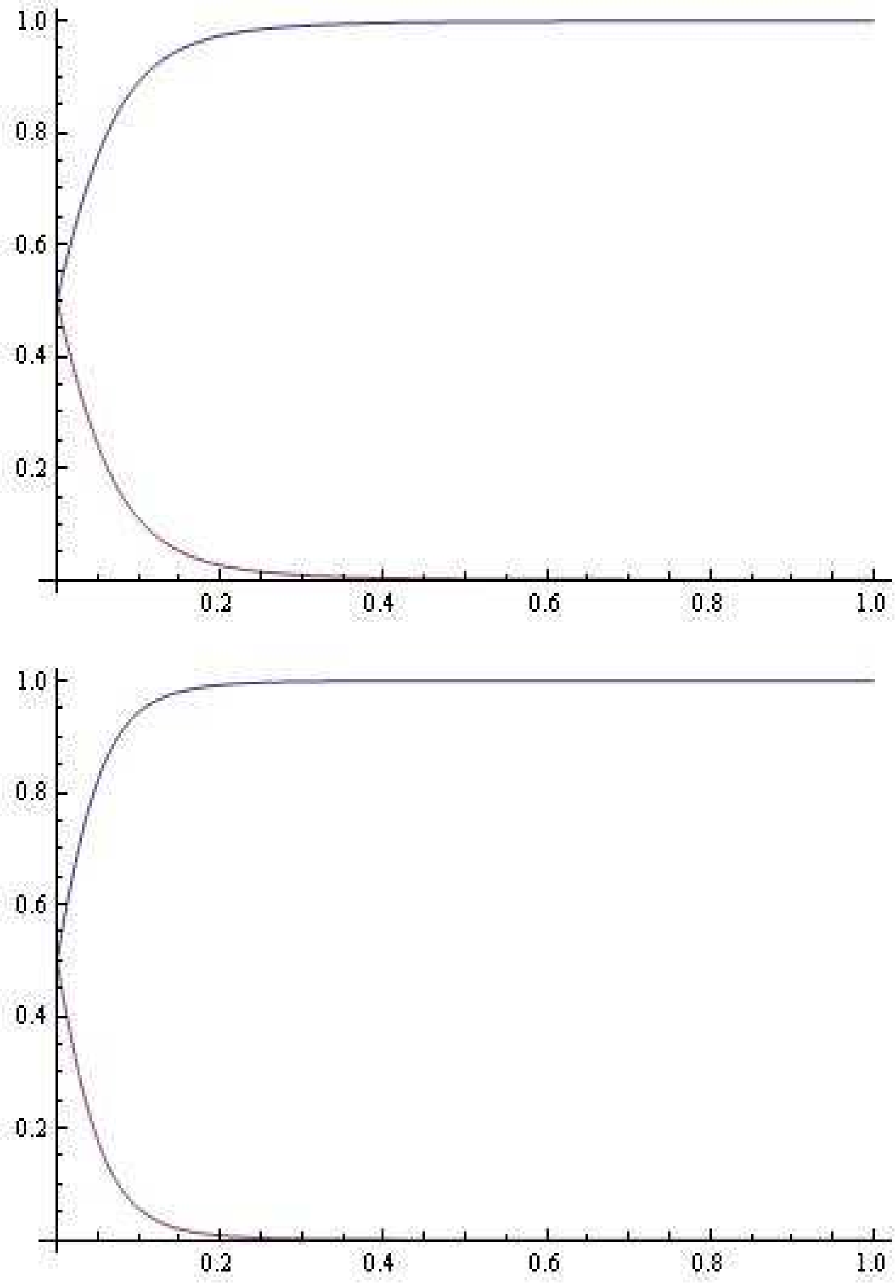


Figure 5: The solution of system of the differential equations: a) for $a = -1.2$, $c = 30$; b) for $a = -1.5$, $c = 50$ .

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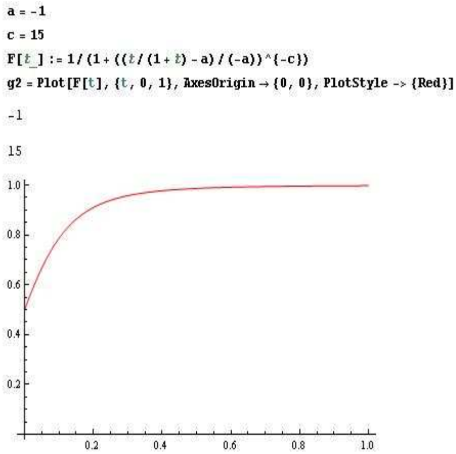


Figure 6: The component $f_2(t)$ of the function $F(f_1(t), f_2(t))$ for $a = -1$, $c = 15$.


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