A NEW EXTENSION OF
THE LOMAX–GOMPertz–MAKEHAm FAMILIES OF C.D.F.

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Abstract: In \([1]\) the authors propose the following cdf of the new Lomax–Gompertz–Makeham (LGM) distribution:

\[
M(t) = 1 - b^n \left( b + \theta t + \frac{\alpha}{\beta} \left( e^{\beta t} - 1 \right) \right)^{-a},
\]

where \((\alpha, \beta, \theta, a, b) \in \mathbb{R}^+\).

Also of interest to the specialists is the task of approximating the Heaviside function

\[
h_{t_0}(t) = \begin{cases} 
0, & \text{if } t < t_0, \\
[0, 1], & \text{if } t = t_0, \\
1, & \text{if } t > t_0
\end{cases}
\]

where \(t_0\) is the median with the new cumulative function in the Hausdorff sense.

We define a new family of recurrence generated cdf of the transmuted–transmuted generalized Lomax–Gompertz–Makeham (TTLGM) distribution

\[
M_{i+1}(t) = M_i(t)(\mu_{i+1} + 1 - \mu_{i+1}M_i(t)),
\]

\(i = 0, 1, 2, \ldots, \mu_i \in [0, 1); \ M_0(t) = M(t)\).

Numerical examples, illustrating our results are presented using programming environment CAS Mathematica.

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1. Introduction and Preliminaries

Definition 1. In [1] the authors consider an extension of the Gompertz–Makeham distribution using the Lomax generator of probability distributions. The cdf of the new Lomax–Gompertz–Makeham (LGM) distribution is defined by:

\[ M(t) = 1 - b^a \left( b + \theta t + \frac{\alpha}{\beta} \left( e^{\beta t} - 1 \right) \right)^{-\alpha}, \tag{1} \]

where \((\alpha, \beta, \theta, a, b) \in \mathbb{R}^+, t > 0.\)

Definition 2. The shifted Heaviside step function is defined by

\[ h_{t_0}(t) = \begin{cases} 
0, & \text{if } t < t_0, \\
[0, 1], & \text{if } t = t_0, \\
1, & \text{if } t > t_0 
\end{cases} \tag{2} \]

Definition 3. [2] The Hausdorff distance (the H–distance) \(\rho(f, g)\) between two interval functions \(f, g\) on \(\Omega \subseteq \mathbb{R}\), is the distance between their completed graphs \(F(f)\) and \(F(g)\) considered as closed subsets of \(\Omega \times \mathbb{R}\). More precisely,

\[
\rho(f, g) = \max \left\{ \sup_{A \in F(f)} \inf_{B \in F(g)} ||A - B||, \sup_{B \in F(g)} \inf_{A \in F(f)} ||A - B|| \right\},
\]

wherein \(||.||\) is any norm in \(\mathbb{R}^2\), e. g. the maximum norm \(||(t, x)|| = \max\{|t|, |x|\}\); hence the distance between the points \(A = (t_A, x_A), B = (t_B, x_B)\) in \(\mathbb{R}^2\) is \(||A - B|| = \max(|t_A - t_B|, |x_A - x_B|)\).

Definition 4. We define a new family of recurrence generated cdf of the transmuted–transmuted generalized Lomax–Gompertz–Makeham (TTLGM) distribution:

\[ M_{i+1}(t) = M_i(t)(\mu_{i+1} + 1 - \mu_{i+1}M_i(t)), \]
\[ i = 0, 1, 2, \ldots, \mu_i \in [0, 1); M_0(t) = M(t). \tag{3} \]

For some generalized family of distributions, see [4]–[8].

In this note we study the Hausdorff approximation of the Heaviside function \(h_{t_0}(t)\) by the family \(M_i(t)\).
2. Main Results

2.1. A note on the new Lomax–Gompertz–Makeham (LGM) cdf (1)

The investigation of the characteristic “supersaturation” of the model (1) to the horizontal asymptote is important.

Sensitive analysis for the “saturation in the Hausdorff sense”.

Let $t_0$ is the value for which $M(t_0) = \frac{1}{2}$.

The one–sided Hausdorff distance $d$ between the function $h_{t_0}(t)$ and the (cdf) $M(t)$ (1) satisfies the relation

$$M(t_0 + d) = 1 - d. \quad (4)$$

For given $\alpha, \beta, \theta, a, b$ and $t_0$, the nonlinear equation (4) has unique positive root $-d$.

The model (1) for $a = 2.5, \theta = 1.5, \alpha = 0.6, \beta = 0.005, b = 0.2$ and $t_0 = 0.0304287$ is visualized on Fig. 1.

From the nonlinear equation (4) we have: $d = 0.107085$.

The model (1) for $a = 1.3, \theta = 0.00001, \alpha = 1.9, \beta = 1.9, b = 0.04$ and $t_0 = 0.0146235$ is visualized on Fig. 2.

From the nonlinear equation (4) we have: $d = 0.0889044$. 
2.2. A note on the new family of recurrence generated cdf of the transmuted–transmuted generalized Lomax–Gompertz–Makeham (TTLGM) distribution (3)

Sensitive analysis for the "saturation in the Hausdorff sense".

For $i = 0$ from (3) we have:

$M_1(t) = M(t)(\mu_1 + 1 - \mu_1 M(t))$ (5)

Let $t_1$ is the value for which $M_1(t_1) = \frac{1}{2}$.

The one-sided Hausdorff distance $d_1$ between the function $h_{t_1}(t)$ and the (cdf) $M_1(t)$ (5) satisfies the relation

$M_1(t_1 + d_1) = 1 - d_1$. (6)

For example, for fixed $a = 2.5$, $\theta = 1.5$, $\alpha = 0.6$, $\beta = 0.005$, $b = 0.2$, $\mu_1 = 0.8$ and $t_1 = 0.016188$ for the one-sided H-distance we find $d_1 = 0.0707505$ (see, Fig. 3).

**Example.** Storm worm one of the most biggest cyber threats of 2008.

We analyze the following data [3]
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Figure 3: $M(t)$–blue; $M_1(t)$–dashed for fixed $a = 2.5$, $\theta = 1.5$, $\alpha = 0.6$, $\beta = 0.005$, $b = 0.2$ and $\mu_1 = 0.8$; H–distance $d = 0.107085$; H–distance $d_1 = 0.0707505$.

The cdf $M_1(t)$ for $\alpha = 0.6$, $\theta = 11.7798$, $\beta = 0.005$, $a = 2$, $b = 26.1$ and $\mu_1 = 0.6$ is visualized on Fig. 4.

The recurrence generated cumulative distribution functions: $M(t)$, $M_1(t)$, $M_2(t)$ and $M_3(t)$ are visualized on Fig. 5.

Obviously, the new model (3) can also be used for approximating of some "specific data".

From the Fig. 3–5, it can be seen that the "supersaturation" by the (cdf) $M_i(t)$ is faster.

Evidently, $\{d_i\}_{1}^{\infty} \to 0$.

For other approximation and modelling results, see [9]–[28].

We hope that the results will be useful for specialists in this scientific area.

$\text{data\_Storm\_IDs} := \{\{1,0.843\}, \{4,0.926\}, \{5,0.954\}, \{6,0.967\},$
\{7,0.976\}, \{8,0.981\}, \{9,0.985\}, \{10,0.991\}, \{22,0.995\},$
\{38,0.997\}, \{51,0.998\}, \{64,0.9985\}, \{74,0.999\}, \{83,1\}, \{100,1\},$
\{367,1\}\}$

Figure 4: The fitted model $M_1(t)$.

Figure 5: Comparison between $M(t)$–thick, $M_1(t)$–red, $M_2(t)$–green and $M_3(t)$–dashed for fixed $\alpha = 0.6$, $\theta = 1.5$, $\beta = 0.005$, $a = 2.5$, $b = 0.2$ and $\mu_1 = 0.6$, $\mu_2 = 0.7$, $\mu_3 = 0.9$. 
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References


