SOME MORE RESULTS ON ONE MODULO THREE ROOT SQUARE MEAN GRAPHS

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Abstract: A graph G is said to be one modulo three root square mean graph if there is an injective function $\phi$ from the vertex set of G to the set $\{0, 1, 3, \ldots, 3q - 2, 3q\}$ where $q$ is the number of edges of G and $\phi$ induces a bijection $\phi^*$ from the edge set of G to $\{1, 4, \ldots, 3q - 2\}$ given by $\phi^*(uv) = \left\lceil \sqrt{\phi(u)^2 + \phi(v)^2} \right\rceil$ or $\left\lfloor \sqrt{\phi(u)^2 + \phi(v)^2} \right\rfloor$ and the function $\phi$ is called one modulo three root square mean labeling of G. The concept of one modulo three root square mean labeling was introduced by C. Jayasekaran and M. Jaslin Melbha and they investigated some path related graphs are one modulo three root square mean graphs.

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1. Introduction

We begin with simple, finite, connected and undirected graph. For standard terminology and notations we follow Harary [3]. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain condition(s). If the domain of the mapping is the set of vertices (edges) then the labeling is called a vertex labeling (an edge labeling). Several types of graph labeling and a detailed survey is available in [2].

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C. Jayasekaran and C. David Raj was introduced the concept of one modulo three harmonic mean labeling of graphs in [1]. P. Jeyanthi and A. Maheswari was introduced the concept of one modulo three mean labeling of graphs in [5]. Root square mean labeling was introduced by S.S. Sandhya, S. Somasundaram and S. Anusa in [6]. C. Jayasekaran and M. Jaslin Melbha introduce the concept one modulo three root square mean labeling of graphs in [4]. They investigated the one modulo three root square mean labeling of path related graphs. Not every graph is one modulo three root square mean. For example, star graph $K_{1,n}$, where $n \geq 4$ is not a one modulo three root square mean graph. We are interested to study different classes of graphs, which are one modulo three root square mean graphs.

We will provide a brief summary of definitions and other informations which are necessary for our present investigation.

**Definition 1.1.** A graph $G$ is said to be one modulo three root square mean graph if there is an injective function $\phi$ from the vertex set of $G$ to the set $\{0, 1, 3, \ldots, 3q - 2, 3q\}$ where $q$ is the number of edges of $G$ and $\phi$ induces a bijection $\phi^*$ from the edge set of $G$ to $\{1, 4, \ldots, 3q - 2\}$ given by $\phi^*(uv) = \lceil \sqrt{\phi(u)^2 + \phi(v)^2} \rceil$ or $\lfloor \sqrt{\phi(u)^2 + \phi(v)^2} \rfloor$ and the function $\phi$ is called one modulo three root square mean labeling of $G$.

**Definition 1.2.** A graph $G$ is said to be complete, if every pair of its distinct vertices are adjacent. A complete graph on $p$ vertices is denoted by $K_p$.

**Definition 1.3.** The corona of two graphs $G_1$ and $G_2$ is the graph $G = G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$ where $i^{th}$ vertex of $G_1$ is adjacent to every vertices in the $i^{th}$ copy of $G_2$.

**Definition 1.4.** The graph $P_n \circ K_1$ is called a comb.

**Definition 1.5.** The product $P_2 \times P_n$ is called a ladder, and it is denoted by $L_n$. The ladder graph $L_n$ is a planar undirected graph with $2n$ vertices and $3n - 2$ edges.

**Definition 1.6.** The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a graph $G = G_1 \cup G_2$ with vertex set $V = V_1 \cup V_2$ and the edge set $E = E_1 \cup E_2$.

**Definition 1.7.** The graph $W_n = C_n - 1 + K_1$ is called a Wheel with $n$ spokes. A wheel graph $W_n$ is obtained from a cycle $C_n$ by adding a new vertex and joining it to all the vertices of the cycle by an edge, the new edges are called the spokes of the wheel.

**Definition 1.8.** A triangular snake $T_n$ is obtained from a path $v_1v_2 \ldots v_n$ by joining $v_i$ and $v_{i+1}$ to a new vertex $w_i$ for $1 \leq i \leq n - 1$. That is, every edge of the path is replaced by a triangle $C_3$. 
Definition 1.9. A double triangular snake consists of two triangular snakes that have a common path. That is, a double triangular snake $DT_n$ is a graph obtained from a path $u_1u_2 \ldots u_n$ by joining $u_i$ and $u_{i+1}$ to two new vertices $v_i$ and $w_i$ for $1 \leq i \leq n - 1$.

Definition 1.10. A triangular ladder $TL_n, n \geq 2$ is a graph obtained from $L_n$ by adding the edges $u_iv_{i+1}, 1 \leq i \leq n - 1$ where $u_i$ and $v_i$ are the vertices of $L_n$ such that $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ are two paths of length $n$ in the graph $L_n$.

Definition 1.11. A friendship graph $F_n$ is a one point union of $n$ copies of cycle $C_3$.

Theorem 1.12. Any path $P_n$ is a one modulo three root square mean graph \cite{4}.

2. Main Result

Theorem 2.1. The complete graph $K_n$ is a one modulo three root square mean graph if and only if $n \leq 2$.

Proof. Suppose $K_n$ is a one modulo three root square mean graph for $n \geq 4$. To get the edge label $3q - 2$, we must have $3q - 2$ and $3q$ as the vertex labels. Let $u$ and $v$ be the vertices whose labels are $3q - 2$ and $3q$ respectively. Again, to get the edge label 1, we must have 0 and 1 as the vertex labels. Let $w$ and $z$ be the vertices whose labels are 0 and 1 respectively then the edges $uw, vz$ receive the same induced label which should not happen. Hence $K_n$ is not a one modulo three root square mean graph for $n \geq 4$. If $n = 2$, the complete graph $K_2$ is a path $P_2$, which is a one modulo three root square mean graph. If $n = 3$, the complete graph $K_3$ is a triangle. By theorem 2.17, triangle is not a one modulo three root square mean graph. Hence The complete graph $K_n$ is a one modulo three root square mean graph if and only if $n \leq 2$.

Example 2.2. One modulo three root square mean labeling of $K_2$ is given in figure 1.

![Figure 1: K_2](image)

Theorem 2.3. $C_4 = L_2$ is not a one modulo three root square mean graph.

Proof. Let $V(L_2) = \{u, v, x, y\}$ and $E(L_2) = \{uv, vx, xy, yu\}$. Then $L_2$ has 4 vertices and 4 edges. Define a function $\phi : V(L_2) \rightarrow \{0, 1, 3, 4, 6, 7, 9, 10, 12\}$ by $\phi(u) = 0, \phi(v) = 1, \phi(x) = 6, \phi(y) = 7$. Then $\phi^* : E(L_2) \rightarrow \{1, 4, 7, 10\}$, where
\[ \phi^*(uv) = 1, \phi^*(vx) = 4, \phi^*(xy) = 7, \phi^*(yu) = 4. \] Thus two edges get same labeling. Hence \( L_2 \) is not a one modulo three root square mean graph.

**Theorem 2.4.** The ladder \( L_n = P_n \times K_2 \) is a one modulo three root square mean graph for \( n \neq 2 \).

**Proof.** Let \( V(L_n) = \{u_i, v_i / 1 \leq i \leq n\} \) and \( E(L_n) = \{u_iu_{i+1}, v_iv_{i+1}, u_iv_i, u_nv_n/1 \leq i \leq n-1\} \). Then \( L_n \) has \( 2n \) vertices and \( 3n - 2 \) edges. Define a function \( \phi : V(L_n) \rightarrow \{0, 1, 3, \ldots, 3q - 2, 3q\} \) by \( \phi(u_1) = 0, \phi(u_2) = 1, \phi(u_3) = 6, \phi(u_i) = 9i - 6, 4 \leq i \leq n; \phi(v_i) = 6i + 3, 1 \leq i \leq 3; \phi(v_i) = 9i - 9, 4 \leq i \leq n \). Then \( \phi \) induces a bijection \( \phi^* : E(L_n) \rightarrow \{1, 4, \ldots, 3q - 2\} \), where \( \phi^*(u_1u_2) = 1, \phi^*(u_2u_3) = 4, \phi^*(u_3u_4) = 22, \phi^*(u_iu_{i+1}) = 9i - 2, 4 \leq i \leq n - 1; \phi^*(v_iu_{i+1}) = 6i + 7, 1 \leq i \leq 3; \phi^*(v_iv_{i+1}) = 9i - 5, 4 \leq i \leq n - 1; \phi^*(u_1v_1) = 7, \phi^*(u_2v_2) = 10, \phi^*(u_3v_3) = 16, \phi^*(u_iv_i) = 9i - 8, 4 \leq i \leq n \). For \( n = 2 \), by Theorem 2.3, \( L_2 \) is not a one modulo three root square mean graph. Hence the ladder \( L_n \) is a one modulo three root square mean graph for \( n \neq 2 \).

**Example 2.5.** A one modulo three root square mean labeling of \( L_9 \) is given in figure 2.

\[ \begin{array}{cccccccccc}
\phi^*(u_1) = 0, & \phi^*(u_2) = 1, & \phi^*(v_1) = 3, & \phi^*(v_2) = 6, & \phi^*(v_3) = 9, & \phi^*(v_4) = 12, & \phi^*(u_5) = 3, & \phi^*(v_5) = 6, & \phi^*(v_6) = 9, & \phi^*(v_7) = 12, \\
\phi^*(u_8) = 3, & \phi^*(v_8) = 6, & \phi^*(v_9) = 9, & \phi^*(u_10) = 3, & \phi^*(v_{10}) = 6, & \phi^*(v_{11}) = 9, & \phi^*(v_{12}) = 12.
\end{array} \]

![Figure 2: L_9](image)

**Theorem 2.6.** \( P_m \cup (P_n \circ K_1) \) admits one modulo three root square mean labeling.

**Proof.** Let \( P_n \) be the path \( v_1v_2 \ldots v_n \). Let \( w_i \) be the vertex adjacent to \( v_i, 1 \leq i \leq n \). The resultant graph is \( P_n \circ K_1 \). Let \( u_1u_2 \ldots u_m = \) the path \( P_m \). Let \( G = P_m \cup (P_n \circ K_1) \) with \( V(G) = \{u_i, v_j, w_j / 1 \leq i \leq m, 1 \leq j \leq n\} \) and \( E(G) = \{u_iu_{i+1}, v_jw_j, v_jv_{j+1}, v_iw_i, w_iw_{i+1} / 1 \leq i \leq m - 1, 1 \leq j \leq n - 1\} \). Then \( G \) has \( m + 2n \) vertices and \( m + 2n \) edges. Define a function \( \phi : V(G) \rightarrow \{0, 1, 3, \ldots, 3q - 2, 3q\} \) by \( \phi(u_1) = 0, \phi(u_2) = 1, \phi(u_i) = 3i - 3, 3 \leq i \leq m - 1, \phi(u_m) = 3m - 5; \phi(v_i) = 3m + 6i - 6 \) for odd \( i \) and \( \phi(v_i) = 3m + 6i - 9 \) for even \( i \), \( 1 \leq i \leq n; \phi(w_i) = 3m + 6i - 9 \) for even \( i \), \( 1 \leq i \leq n \). Then \( \phi \) induces a bijection \( \phi^* : E(L_n) \rightarrow \{1, 4, \ldots, 3q - 2\} \), where \( \phi^*(u_iu_{i+1}) = 3i - 2, 1 \leq i \leq n - 1; \phi^*(u_iv_i) = 3m + 6i - 5, 1 \leq i \leq n - 1; \phi^*(v_{i+1}) = 3m + 6i - 8, 1 \leq i \leq n \). Therefore, \( \phi \) is a one modulo three root square mean labeling. Hence \( P_m \cup (P_n \circ K_1) \) admits one modulo three root square mean labeling.
Example 2.7. A one modulo three root square mean labeling of $P_7 \cup (P_5 \odot K_1)$ is given in figure 3.

![Diagram of $P_7 \cup (P_5 \odot K_1)$]

**Theorem 2.8.** $(P_m \odot K_2) \cup P_n$ admits one modulo three root square mean labeling.

*Proof.* Let $P_m$ be the path $u_1u_2 \ldots u_m$. Let $v_i, w_i$ be the vertices of $i^{th}$ copy of $K_2$. Join $v_i$ and $w_i$ with the vertex $u_i, 1 \leq i \leq m$. The resultant graph is $P_m \odot K_2$. Let $x_1x_2 \ldots x_n$ be the path $P_n$. Let $G = (P_m \odot K_2) \cup P_n$ with $V(G) = \{u_i, v_i, w_i, x_j | 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{u_iu_{i+1}, u_iw_i, w_iv_i, x_jx_{j+1}, u_mw_m, u_mv_m/1 \leq i \leq m-1, 1 \leq j \leq n-1\}$. Define a function $\phi: V(G) \rightarrow \{0, 1, 3, \ldots, 3q - 2, 3q\}$ by $\phi(u_1) = 1, \phi(w_2) = 9, \phi(u_i) = 9i - 6, 3 \leq i \leq m; \phi(v_1) = 0; \phi(v_2) = 10; \phi(v_i) = 9i - 11, 3 \leq i \leq n; \phi(w_1) = 6; \phi(w_2) = 15; \phi(w_i) = 9i - 5, 3 \leq i \leq n; \phi(x_i) = 9m + 3i - 6, 1 \leq i \leq n$. Then $\phi$ induces a bijection $\phi^*: E(G) \rightarrow \{1, 4, \ldots, 3q - 2\}$, where $\phi^*(u_iu_{i+1}) = 9i - 2, 1 \leq i \leq m - 1; \phi^*(u_iw_i) = 9i - 8, 1 \leq i \leq m; \phi^*(u_iw_i) = 9i - 5, 1 \leq i \leq m; \phi^*(x_iw_{i+1}) = 9m + 3i - 5, 1 \leq i \leq n - 1$. Therefore, $\phi$ is a one modulo three root square mean labeling. Hence $(P_m \odot K_2) \cup P_n$ admits one modulo three root square mean labeling. \(\square\)

Example 2.9. A one modulo three root square mean labeling of $(P_5 \odot K_2) \cup P_7$ is given in figure 4.

Theorem 2.10. Let $G$ be a graph obtained by joining a pendent vertex with a vertex of degree two of a comb $P_n \odot K_1$. Then $G$ is one modulo three root square mean graph.
Define a function $\phi$. Let $P_n$ be the path $u_1u_2\ldots u_n$. Let $v_i$ be the vertex adjacent to $u_i$, $1 \leq i \leq n$. The resultant graph is $P_n \odot K_1$. Let $w$ be the vertex joined with $u_n$. The resultant graph is $G$. Here $V(G) = \{u_i, v_i, w/1 \leq i \leq n\}$ and $E(G) = \{u_iu_i, u_iu_{i+1}, u_nv_n, u_nw/1 \leq i \leq n-1\}$. Then $G$ has $2n+1$ vertices and $2n$ edges.

**Case 1.** $n$ is odd.
Define a function $\phi : V(G) \rightarrow \{0, 1, 3, \ldots , 3q-2, 3q\}$ by $\phi(u_1) = 1, \phi(u_i) = 6i - 6, 2 \leq i \leq n; \phi(w) = 6n, \phi(v_1) = 0; \phi(v_i) = 6i - 5, 2 \leq i \leq n$. Then $\phi$ induces a bijection $\phi^* : E(G) \rightarrow \{1, 4, \ldots , 3q - 2\}$, where $\phi^*(u_iu_{i+1}) = 6i - 2, 1 \leq i \leq n - 1; \phi^*(u_iw) = 6i - 5, 1 \leq i \leq n$. Thus the edges get distinct labels $1, 4, \ldots , 3q - 2$. In this case $\phi$ is a one modulo three root square mean labeling for $G$.

**Case 2.** $n$ is even.
Define a function $\phi : V(G) \rightarrow \{0, 1, 3, \ldots , 3q-2, 3q\}$ by $\phi(u_1) = 0; \phi(u_i) = 6i - 5, 2 \leq i \leq n; \phi(w) = 6n, \phi(v_1) = 1, \phi(v_i) = 6i - 6, 2 \leq i \leq n$. Then $\phi$ induces a bijection $\phi^* : E(G) \rightarrow \{1, 4, \ldots , 3q - 2\}$, where $\phi^*(u_iu_{i+1}) = 6i - 2, 1 \leq i \leq n - 1; \phi^*(u_iw) = 6i - 5, 1 \leq i \leq n$. Thus the edges get distinct labels $1, 4, \ldots , 3q - 2$. In this case $\phi$ is a one modulo three root square mean labeling for $G$. Therefore, $\phi$ is a one modulo three root square mean labeling for $G$. From case 1 and case 2, we conclude that $G$ is a one modulo three root square mean graph.

**Example 2.11.** One modulo three root square mean labeling of $G$ when $n = 6$ and $n = 7$ is given in figure 5 and figure 6 respectively.

![Figure 4: $(P_3 \odot K_2) \cup P_7$](image-url)
Theorem 2.12. $L_n \odot K_1$ is one modulo three root square mean graph.

Proof. Let $u_1u_2 \ldots u_n$ and $v_1v_2 \ldots v_n$ be two paths of length $n$. Join $u_i$ and $v_i, 1 \leq i \leq n$. The resultant graph is $L_n$. For $1 \leq i \leq n$, let $x_i$ be the pendant vertex adjacent to $u_i$ and $y_i$ be the pendant vertex adjacent to $v_i$. We get the required graph $L_n \odot K_1$ with $V(L_n \odot K_1) = \{u_i, v_i, x_i, y_i/1 \leq i \leq n\}$ and $E(L_n \odot K_1) = \{u_i v_i, u_i x_i, v_i y_i, v_j v_{j+1}, u_j u_{j+1}/1 \leq i, j \leq n-1\}$. Then $G$ has $4n$ vertices and $5n-2$ edges. Define a function $\phi : V(L_n \odot K_1) \rightarrow \{0, 1, 3, \ldots, 3q-2, 3q\}$ by $\phi(u_1) = 1, \phi(u_2) = 15, \phi(u_3) = 33, \phi(u_4) = 46, \phi(u_5) = 15i - 12, 5 \leq i \leq n; \phi(x_1) = 0, \phi(x_2) = 22, \phi(x_3) = 28, \phi(x_4) = 45, \phi(x_5) = 15i - 17, 5 \leq i \leq n; \phi(v_1) = 6, \phi(v_2) = 18, \phi(v_3) = 34, \phi(v_4) = 15i - 9, 4 \leq i \leq n; \phi(y_1) = 9, \phi(y_2) = 24, \phi(y_3) = 15i - 6, 3 \leq i \leq n$. Then $\phi$ induces a bijection $\phi^* : E(L_n \odot K_1) \rightarrow \{1, 4, 7, \ldots, 3q - 2\}$, where $\phi^*(u_1u_2) = 15i - 5, 1 \leq i \leq n - 1; \phi^*(u_1x_1) = 1, \phi^*(u_2x_2) = 19, \phi^*(u_3x_3) = 15i - 14, 3 \leq i \leq n; \phi^*(u_1v_1) = 4, \phi^*(u_2v_2) = 16, \phi^*(u_3v_3) = 15i - 11, 3 \leq i \leq n; \phi^*(v_1y_1) = 7, \phi^*(v_2y_2) = 22, \phi^*(v_3y_3) = 15i - 8, 3 \leq i \leq n; \phi^*(v_1v_2) = 15i - 2, 1 \leq i \leq n - 1$. Hence $L_n \odot K_1$ is one modulo three root square mean graph.

Example 2.13. A one modulo three root square mean labeling of $L_8 \odot K_1$ is given in figure 7.

Construction: Let $L_n = P_n \times K_2$ be the ladder graph. $V(L_n) = \{u_i, v_i/1 \leq i \leq n\}$ and $E(L_n) = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i, u_{i+1} v_{i+1}/1 \leq i \leq n-1\}$. The graph obtained from $L_n$ by deleting the edge $u_1v_1$ is denoted by $G_d$.

Theorem 2.14. The graph $G_d$ is one modulo three root square mean graph.

Proof. Let $V(G_d) = \{u_i, v_i/1 \leq i \leq n\}$. $E(G_d) = \{u_i u_{i+1}, v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{u_i v_i/2 \leq i \leq n\}$. That is, $V(G_d) = V(L_n)$ and $E(G_d) = E(L_n) - \{u_1 v_1\}$. Then $G_d$ has $2n$ vertices and $3(n-1)$ edges. Define a function $\phi : V(G_d) \rightarrow \{1, 3, 4, \ldots, 3q - 2, 3q\}$ by $\phi(u_1) = 0, \phi(u_2) = 1, \phi(u_3) = 15, \phi(u_4) = 22, \phi(u_5) = 9i - 12, 5 \leq i \leq n; \phi(v_1) = 9, \phi(v_2) = 6, \phi(v_3) = 18, \phi(v_4) = 9i - 9, 4 \leq i \leq n$. Then $\phi$ induces a bijective function $\phi^* : E(G_d) \rightarrow \{1, 4, 7, \ldots, 3q - 2\}$, where $\phi^*(u_1 u_2) = 1, \phi^*(u_i u_{i+1}) = 9i - 8, 2 \leq i \leq n - 1; \phi^*(v_1 v_2) = 7, \phi^*(v_i v_{i+1}) = 9i - 5, 2 \leq i \leq n - 1; \phi^*(v_2 v_3) = 4, \phi^*(u_1 v_1) = 9i - 11, 3 \leq i \leq n$. Thus $\phi$ provides
one modulo three root square mean labeling for $G$. Hence $G_d$ is an one modulo three root square mean graph. 

**Example 2.15.** One modulo three root square mean labeling of $G_d$ when $n = 8$ is given in figure 8.

**Theorem 2.16.** If $G$ is a graph in which every edge is an edge of a triangle then $G$ is not a one modulo three root square mean graph.

**Proof.** Let $G$ be a graph in which every edge is an edge of a triangle. Suppose $G$ is a one modulo three root square mean graph. To get 1 as an edge label, there must be two adjacent vertices $x$ and $y$ such that $\phi(x) = 0$ and $\phi(y) = 1$. Let $xyzx$ be a triangle in which the edge $xy$ lies. To get 4 as an edge label, there must be $\phi(z) = 6$, then $xz$ and $yz$ get the same edge label. This is a contradiction to the fact of one modulo three root square mean labeling. Hence $G$ is not a one modulo three root square mean labeling graph.

**Corollary 2.17.** The wheel graph $W_n$, flower graph $FL_n$, triangular snakes, double triangular snakes, triangular ladders, fans $P_n + K_1$, $n \geq 2$, Double fans $P_n +$
$K_2, n \geq 2$, friendship graph $C_n^3$, windmills $K_{m}^n, m > 3$, square graph $B_{n,n}^2$, total graph $T(P_n)$ and composition graph $P_n[P_2]$ are not one modulo three root square mean graph.

References


