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LEFSCHETZ COINCIDENCE AND FIXED POINT THEORY FOR MORPHISMS ON HAUSDORFF TOPOLOGICAL SPACES

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Abstract: Using Lefschetz coincidence spaces we establish a variety of coincidence and α-coincidence type results for compact morphisms.

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1. Introduction

This paper, motivated in part by the papers [1, 3, 5, 6], presents general Lefschetz type results for compact morphisms (in the sense of Gorniewicz and Granas) on Hausdorff topological spaces. Our spaces include NES(compact) and ANES(compact) spaces and we present a number of coincidence and α-coincidence (and α-fixed point) type results. Noncompact morphisms will be discussed briefly at the end of the paper.

Now we present some ideas needed in Section 2. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

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Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;

(ii) p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are continuous maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. Note if (p,q), $(p_1,q_1) \in D(X,Y)$ (where $X \overset{p}{\leftarrow} \Gamma \overset{q}{\to} Y$ and $X \overset{p_1}{\leftarrow} \Gamma' \overset{q_1}{\to} Y$) and $(p,q) \sim (p_1,q_1)$ then it is easy to see (use $q \circ g = q_1$ and $p \circ g = p_1$ where $g : \Gamma' \to \Gamma$) that for $x \in X$ we have $q_1\left(p_1^{-1}(x)\right) = q\left(p^{-1}(x)\right)$. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\,p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ. Let $\phi \in M(X,Y)$ and (p,q) a representative of ϕ. We define $\phi(X) \subseteq Y$ by $\phi(X) = q\left(p^{-1}(X)\right)$. Note $\phi(X)$ does not depend on the representative of ϕ. Now $\phi \in M(X,Y)$ is called compact provided the set $\phi(X)$ is relatively compact in Y. Note we will identify a map $f : X \to Y$ with the morphism $f = \{X \overset{Id_X}{\leftarrow} X \overset{f}{\to} Y\} : X \to Y$. Let $X \subseteq Y$. A point $x \in X$ is called a fixed point of a morphism $\phi \in M(X,Y)$ if $x \in \phi(x)$.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \backslash N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\,\tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q\, Tr\,(f_q).$$

A linear map $f : E \to E$ of a vector space E into itself is called weakly nilpotent provided for every $x \in E$ there exists n_x such that $f^{n_x}(x) = 0$. Assume that

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$E = \{E_q\}$ is a graded vector space and $f = \{f_q\} : E \to E$ is an endomorphism. We say that f is weakly nilpotent iff f_q is weakly nilpotent for every q. It is well known [4, pp 53] that any weakly nilpotent endomorphism $f : E \to E$ is a Leray endomorphism and $\Lambda(f) = 0$.

With Čech homology functor extended to a category of morphisms (see [5 pp. 364]) we have the following well known result (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y\} : X \to Y$$

we define the induced map

$$H(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$).

Recall the following result [3, 4 pp. 227].

Theorem 1.1. *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then*

$$(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star.$$

Definition 1.2. Now $\phi \in M(X, X)$ is said to be a Lefschetz morphism if $\phi_\star : H(X) \to H(X)$ is a Leray endomorphism; for such ϕ we define the generalized Lefschetz number by putting $\Lambda(\phi) = \Lambda(\phi_\star)$.

Let $\phi = \{X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y\} : X \to Y$ be a morphism. We define the coincidence set

$$Coin(p, q) = \{y \in \Gamma : p(y) = q(y)\}.$$

We say ϕ has a coincidence provided the set $C(\phi) = p(Coin(p, q))$ is nonempty (i.e. there exists $x \in p(Coin(p, q))$ i.e. there exists $y \in \Gamma$ with $x = p(y) = q(y)$). Let (p', q') be another representation of ϕ, say $\phi = \{X \overset{p'}{\leftarrow} \Gamma' \overset{q'}{\rightarrow} Y\}$. Note $p(Coin(p, q)) = p'(Coin(p', q'))$; to see this note if $x \in p(Coin(p, q))$ then $x = p(y) = q(y)$ for some $y \in \Gamma$ and now since $(p, q) \sim (p', q')$ and with $f : \Gamma \to \Gamma'$ we have $x = q(y) = q'(f(y))$ and $x = p(y) = p'(f(y))$ so $f(y) \in \Gamma'$ and $x = q'(f(y)) = p'(f(y))$ i.e. $x \in p'(Coin(p', q'))$. Thus the above definition does not depend on the choice of a representation (p, q). Also $C(\phi) \neq \emptyset$ iff $Coin(p, q) \neq \emptyset$ for any representation (p, q) of ϕ.

Suppose $\phi \in M(X, X)$ (here $\phi = \{X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} X\}$) has a coincidence point for (p, q) i.e. suppose there exists $y \in \Gamma$ with $p(y) = q(y)$. Now since p is surjective there exists $w \in X$ with $y \in p^{-1}(w)$ (note $p(w) = y$) and so $w \in q(p^{-1}(w)) = \phi(w)$

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(note $p\,p^{-1}(w) = w$ and the set $q\,(p^{-1}(w))$ is the image of w under ϕ) i.e. ϕ has a fixed point. As a result

$$p(y) = q(y),\ y \in \Gamma \ \text{(and let } w = p(y)) \Leftrightarrow w \in q\,(p^{-1}(w));$$

note if $w \in q\,(p^{-1}(w))$ then there exists $y \in p^{-1}(w)$ with $w = q(y)$ so $p(y) \in p\,p^{-1}(w) = w$ (i.e. $p(y) = w$) and so $p(y) = q(y)$. In particular if the morphism $\phi \in M(X, X)$ (here (p, q) is a representation of ϕ) has a fixed point (say w i.e. $w \in q\,(p^{-1}(w))$) then there exists $y \in p^{-1}(w)$ with $q(y) = p(y)$, so ϕ has a coincidence point for (p, q), and now since we can do this argument for any representation (p, q) of ϕ (recall if (p_1, q_1) is another representation of ϕ then $(p, q) \sim (p_1, q_1)$ and as above $q\,(p^{-1}(w)) = q_1\,(p_1^{-1}(w))$ so $w \in q_1\,(p_1^{-1}(w))$ so there exists $y_1 \in p_1^{-1}(w)$ with $q_1(y_1) = p_1(y_1)$) then $Coin\,(p, q) \neq \emptyset$ for any representation (p, q) of ϕ i.e. ϕ has a coincidence.

For a subset K of a topological space X, we denote by $Cov_X\,(K)$ the set of all coverings of K by open sets of X (usually we write $Cov\,(K) = Cov_X\,(K)$). Given a morphism $\phi \in M(X, X)$ and $\alpha \in Cov\,(X)$, a point $x \in X$ is said to be an α–fixed point of ϕ if there exists a member $U \in \alpha$ such that $x \in U$ and $\phi(x) \cap U \neq \emptyset$. Given a morphism $\phi \in M(X, X)$ (here $\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} X\}$) and $\alpha \in Cov\,(X)$ a point $y \in \Gamma$ is said to be a α–coincidence point for (p, q) if there exists a member $U \in \alpha$ with $p(y) \in U$ and $q(y) \in U$. We say ϕ has a α–coincidence if ϕ has a α–coincidence point for each representation (p, q) of ϕ.

Let X and Y be topological spaces. Given two morphisms $\phi \in M(X, Y)$ and $\psi \in M(X, Y)$ and $\alpha \in Cov\,(Y)$, ϕ and ψ are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$ with $\phi(x) \cap U_x \neq \emptyset$ and $\psi(x) \cap U_x \neq \emptyset$. Recall given two single valued maps $f, g : X \to Y$ and $\alpha \in Cov\,(Y)$, f and g are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. Given a morphism $\phi \in M(X, Y)$ and a single valued map $g : X \to Y$ and $\alpha \in Cov\,(Y)$, ϕ and g are said to be strongly α–close if for any $x \in X$ there exists $U_x \in \alpha$ with $\phi(x) \subseteq U_x$ and $g(x) \in U_x$.

Definition 1.3. A space X is said to be a Lefschetz coincidence space if every compact morphism $\phi \in M(X, X)$ is a Lefschetz morphism and $\Lambda(\phi) \neq 0$ implies ϕ has a coincidence.

Remark 1.4. Examples of Lefschetz coincidence spaces can be found in [5].

2. Coincidence Theory

By a space we mean a Hausdorff topological space.

Definition 2.1. We say $X \in NESM$ (w.r.t. the morphism ϕ) if there exists a Lefschetz coincidence space U, a set $V \subseteq X$ with $\overline{\phi(V)} \subseteq V$, $\phi|_W \in M(W, W)$ (here

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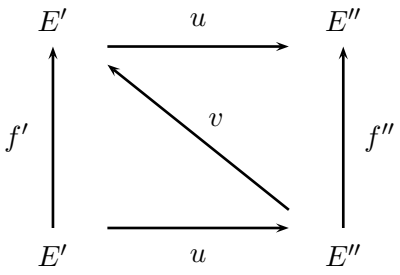
$W = \overline{\phi(V)}$), a compact $\theta \in M(U,W)$, a $\psi \in M(W,U)$, with $\theta_\star \circ \psi_\star = (\phi|_W)_\star$ and with the property that if $\theta\,\psi$ has a coincidence then $\phi|_W$ has a coincidence.

Theorem 2.2. *Let $X \in NESM$ (w.r.t. ϕ) and let U, V, W, θ, ψ be as described in Definition 2.1. Then $\Lambda(\phi|_W)$ is well defined. Also $\Lambda(\phi|_W) \neq 0$ guarantees that $\phi|_W$ has a coincidence.*

Proof. Let $\phi = \phi|_W = \{W \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} W\}$, $\psi = \{W \overset{p_1}{\leftarrow} \Gamma_1 \overset{q_1}{\rightarrow} U\}$ and $\theta = \{U \overset{p_2}{\leftarrow} \Gamma_2 \overset{q_2}{\rightarrow} W\}$. Let $\eta = \psi\,\theta$. Note [5, see (4.2)] guarantees that $\eta \in M(U,U)$ is compact (note θ is compact and the map $x \mapsto q_1(p_1^{-1}(x))$ is upper semicontinuous with nonempty compact values [4]). Since U is a Lefschetz coincidence space then η is a Lefschetz morphism. Now [5, see (2.2) or see the diagram below] (here $E' = U' = H(U)$, $E'' = W' = H(W)$, $u = \theta_\star$, $v = \psi_\star$, $f' = \eta_\star$ and $f'' = (\phi|_W)_\star$ and note $u\,v = \theta_\star\,\psi_\star = (\phi|_W)_\star$ and $v\,u = \psi_\star\,\theta_\star = \eta_\star$)



guarantees that $\phi|_W$ is a Lefschetz morphism and $\Lambda((\phi|_W)_\star) = \Lambda(\psi_\star\,\theta_\star)$. Thus $\Lambda(\phi|_W)$ is well defined.

Now suppose $\Lambda(\phi|_W) \neq 0$ (i.e. $\Lambda((\phi|_W)_\star) \neq 0$). Then $\Lambda(\psi_\star\,\theta_\star) \neq 0$ and since U is a Lefschetz coincidence space then $\psi\,\theta$ has a coincidence. Then (see [5, Lemma 6.3]) $\theta\,\psi$ has a coincidence. From Definition 2.1, $\phi|_W$ has a coincidence. □

Remark 2.3. Note in Theorem 2.2 (see the proof) we could replace θ compact with $\psi\,\theta$ compact.

Remark 2.4. In Theorem 2.2 to show $\Lambda(\phi|_W)$ is well defined we do <u>not</u> need to assume $\theta\,\psi$ has a coincidence guarantees that $\phi|_W$ has a coincidence.

Remark 2.5. If $\theta\,\psi = \phi|_W$ then trivially we have the property $\theta\,\psi$ has a coincidence guarantees that $\phi|_W$ has a coincidence. Also of course $\theta_\star \circ \psi_\star = (\phi|_W)_\star$.

Remark 2.6. Suppose in Definition 2.1 we have $\phi|_W \in M(W,W)$ replaced by $\phi|_V \in M(V,V)$. Then $\Lambda(\phi|_V)$ is well defined and $\Lambda(\phi|_V) \neq 0$ guarantees that $\phi|_V$ has a coincidence. Let $\phi|_V = \{V \overset{p_0}{\leftarrow} \Gamma \overset{q_0}{\rightarrow} V\}$ and $\phi|_W = \{W \overset{p}{\leftarrow} p_0^{-1}(W) \overset{q}{\rightarrow} W\}$; here p and q denote contractions of the appropriate maps p_0 and q_0 (see [3, pg 214]). Also let $\phi_1 \in M(V,W)$ where $\phi_1(x) = \phi(x)$ for $x \in V$ so $\phi_1 = \{V \overset{p_0}{\leftarrow} \Gamma \overset{q_0'}{\rightarrow} W\}$ where

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q_0' is the contraction of q_0 to (Γ, W). Let $i_0 : W \to V$ be the inclusion and note $\phi|_W = \phi_1\, i_0$. Notice Theorem 2.2 guarantees that $\phi|_W$ is a Lefschetz morphism. Also [5, see (2.2)] (here $E' = W' = H(W)$, $E'' = V' = H(V)$, $u = (i_0)_\star$, $v = (\phi_1)_\star$, $f' = (\phi|_W)_\star$ and $f'' = (\phi|_V)_\star$ and note $u\, v = (i_0)_\star\,(\phi_1)_\star = (i_0)_\star\,(q_0')_\star\,(p_0)_\star^{-1} = (q_0)_\star\,(p_0)_\star^{-1} = (\phi|_V)_\star$ since $i_0\, q_0' = q_0$ and $v\, u = (\phi_1)_\star\,(i_0)_\star = (\phi|_W)_\star$ since $\phi|_W = \phi_1\, i_0$) guarantees that $\phi|_V$ is a Lefschetz morphism (and $\Lambda\,((\phi|_V)_\star) = \Lambda\,((\phi|_W)_\star)$). Now suppose $\Lambda\,(\phi|_V) \neq 0$ (i.e. $\Lambda\,((\phi|_V)_\star) \neq 0$). Then $\Lambda\,((\phi|_W)_\star) \neq 0$ (i.e. $\Lambda\,(\phi|_W) \neq 0$) so from Theorem 2.2 we have that $\phi|_W$ (so $\phi|_V$) has a coincidence.

Suppose in Definition 2.1 we have $\phi \in M(X, X)$ and $\phi \in M(X, V)$ then similar reasoning as above guarantees that $\Lambda\,(\phi)$ is well defined and $\Lambda\,(\phi) \neq 0$ guarantees that ϕ has a coincidence.

Alternate Definition and Results: We say $X \in GNESM$ (w.r.t. ϕ) if there exists a Lefschetz coincidence space U, a set $A \subseteq X$ with $\phi(A) \subseteq A$, $\phi|_W \in M(W, W)$ (here $W = \phi(A)$), a compact $\theta \in M(U, W)$, a $\psi \in M(W, U)$, with $\theta_\star \circ \psi_\star = (\phi|_W)_\star$ and with the property that if $\theta\,\psi$ has a coincidence then $\phi|_W$ has a coincidence.

(1) The reasoning in Theorem 2.2 guarantees that if $X \in GNESM$ (w.r.t. ϕ) (let U, A, W, θ and ψ be as described above) then $\Lambda\,(\phi|_W)$ is well defined and also $\Lambda\,(\phi|_W) \neq 0$ guarantees that $\phi|_W$ has a coincidence.

(2) Suppose in the statement above we have $\phi|_W \in M(W, W)$ replaced by $\phi|_A \in M(A, A)$ then the reasoning in Remark 2.6 guarantees that $\Lambda\,(\phi|_A)$ is well defined and $\Lambda\,(\phi|_A) \neq 0$ guarantees that $\phi|_A$ has a coincidence.

Suppose in the statement above we have $\phi \in M(X, X)$ and $\phi \in M(X, A)$ then the reasoning in Remark 2.6 guarantees that $\Lambda\,(\phi)$ is well defined and $\Lambda\,(\phi) \neq 0$ guarantees that ϕ has a coincidence.

(3) We note here that an analogue of Remark 2.3 and Remark 2.4 hold.

Next we discuss some special cases of Theorem 2.2.

Definition 2.7. We say $X \in NESM_1$ (w.r.t. the morphism ϕ) if there exists a Lefschetz coincidence space U, $\phi|_W \in M(W, W)$ (here $W = \overline{\phi(X)}$), a compact $\theta \in M(U, W)$, a $\psi \in M(W, U)$, with $\theta_\star \circ \psi_\star = (\phi|_W)_\star$ and with the property that if $\theta\,\psi$ has a coincidence then $\phi|_W$ has a coincidence.

Corollary 2.8. *Let $X \in NESM_1$ (w.r.t. ϕ) and $W = \overline{\phi(X)}$. Then $\Lambda\,(\phi|_W)$ is well defined. Also $\Lambda\,(\phi|_W) \neq 0$ guarantees that $\phi|_W$ has a coincidence.*

Proof. Take $V = X$ in Theorem 2.2. □

We note here that an analogue of Remark 2.3, Remark 2.4, Remark 2.5 and Remark 2.6 hold for Corollary 2.8.

Definition 2.9. We say $X \in NESM_2$ (w.r.t. the morphism ϕ) if there exists a Lefschetz coincidence space U, $\phi \in M(X, X)$, a compact $\theta \in M(U, X)$, a $\psi \in$

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$M(X,U)$, with $\theta_\star \circ \psi_\star = \phi_\star$ and with the property that if $\theta\,\psi$ has a coincidence then ϕ has a coincidence.

Corollary 2.10. *Let $X \in NESM_2$ (w.r.t. ϕ). Then $\Lambda(\phi)$ is well defined. Also $\Lambda(\phi) \neq 0$ guarantees that ϕ has a coincidence.*

Proof. The proof is word for word that in Theorem 2.2 with W replaced by X. □

We note here that an analogue of Remark 2.3, Remark 2.4 and Remark 2.5 (if $\theta\,\psi = \phi$ then trivially we have the property $\theta\,\psi$ has a coincidence guarantees that ϕ has a coincidence) hold for Corollary 2.10.

Example 2.11. Let X be a space and $\phi \in M(X,X)$ and $W = \overline{\phi(X)}$ is compact. We know [7] that W can be embedded as a closed subset $K^\star$ of T (Tychonoff cube i.e. cartesian product of copies of the unit interval); let $s : W \to K^\star$ be a homeomorphism. Let $i : W \hookrightarrow X$ be the inclusion. Now assume

$$\begin{cases} \text{there exists an open neighborhood } V \text{ of } K^\star \text{ in } T \text{ and a} \\ \text{continuous extension } h_V : V \to X \text{ of } i\,s^{-1}\,(: K^\star \to X). \end{cases} \tag{2.1}$$

Let $j_V : K^\star \hookrightarrow V$ be the natural embedding so $h_V\,j_V = i\,s^{-1}$. Now consider $span\,(T)$ in a Hausdorff locally convex topological vector space containing T. We know [3, page 221] that there exists a retraction $r : span\,(T) \to T$. Let $i_1 : V \hookrightarrow r^{-1}(V)$ be the inclusion. Let $U = r^{-1}(V)$, $\psi = i_1\,j_V\,s\,\phi$ and $\theta = h_V\,r$. We now show $X \in NESM_2$ (w.r.t. ϕ).

First note [5, Theorem 7.6] that U (note U is open) is a Lefschetz coincidence space space. Also note [5] that $\psi \in M(X,U)$ and $\theta \in M(U,X)$ and (see Remark 2.3) $\psi\,\theta$ is compact (since $\overline{\phi(X)}$ is compact). Also notice that

$$\theta_\star\psi_\star = (h_V)_\star\,r_\star\,(i_1)_\star\,(j_V)_\star\,s_\star = \phi_\star$$

since $h_V\,r\,i_1\,j_V\,s(w) = (h_V\,r\,i_1\,j_V)\,s(w) = i(w)$ for $w \in W$ (note $h_V\,j_V = i\,s^{-1}$). Finally note (see Remark 2.5) that for $z \in X$ (note $\phi(z) \in W$) we have $\theta\,\psi(z) = h_V\,r\,i_1\,j_V\,s\,\phi(z) = i\,\phi(z) = \phi(z)$. Thus $X \in NESM_2$ (w.r.t. ϕ).

Example 2.12. Let X be a space and $\phi \in M(X,X)$ and $W = \overline{\phi(X)}$ is compact. Now W can be embedded as a closed subset $K^\star$ of T; let $s : W \to K^\star$ be a homeomorphism. Now assume

$$\begin{cases} \text{there exists an open neighborhood } U \text{ of } K^\star \text{ in } T \\ \text{and a continuous extension } h : U \to W \text{ of } s^{-1}. \end{cases} \tag{2.2}$$

Let $j_U : K^\star \hookrightarrow U$ be the natural embedding. Now let $\psi = j_U\,s$ and $\theta = \phi|_W\,h$. We now show $X \in NESM_1$ (w.r.t. ϕ).

Note $\psi \in M(W,U)$ and $\theta \in M(U,W)$ is compact (since $\overline{\phi(X)}$ is compact). Also notice that

$$\theta_\star\psi_\star = (\phi|_W)_\star\,h_\star\,(j_U)_\star\,s_\star = (\phi|_W)_\star$$

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since $h\, j_U\, s = id$. Finally notice for $z \in W$ that $\theta\, \psi(z) = \phi|_W\, h\, j_U\, s(z) = \phi|_W(z)$. Thus $X \in NESM_1$ (w.r.t. ϕ).

Our next result considers a more general situation.

Definition 2.13. We say $X \in ANESM$ (w.r.t the morphism ϕ) if there exists a set $V \subseteq X$ with $\overline{\phi(V)} \subseteq V$, $\phi|_W \in M(W,W)$ (here $W = \overline{\phi(V)}$), and for each $\alpha \in Cov_W\left(\overline{\phi(W)}\right)$ there exists a Lefschetz coincidence space U_α, a compact $\theta_\alpha \in M(U_\alpha, W)$, a $\psi_\alpha \in M(W, U_\alpha)$, with $(\theta_\alpha)_\star\,(\psi_\alpha)_\star = (\phi|_W)_\star$ and with one of the following properties holding: (a) $\theta_\alpha\, \psi_\alpha$ has a coincidence guarantees $\phi|_W$ has a α–coincidence, or (b) $\theta_\alpha\, \psi_\alpha$ has a fixed point guarantees $\phi|_W$ has a α–fixed point.

Theorem 2.14. *Let $X \in ANESM$ (w.r.t ϕ) and let V, W, α, U_α, θ_α and ψ_α be as described in Definition 2.13. Then $\Lambda\left(\phi|_W\right)$ is well defined. Also if Definition 2.13 part (a) holds and $\Lambda\left(\phi|_W\right) \neq 0$ then for any $\alpha \in Cov_W\left(\overline{\phi(W)}\right)$ we have that $\phi|_W$ has a α–coincidence (if Definition 2.13 part (b) holds and $\Lambda\left(\phi|_W\right) \neq 0$ then for any $\alpha \in Cov_W\left(\overline{\phi(W)}\right)$ we have that that $\phi|_W$ has a α–fixed point).*

Remark 2.15. We say (W, ϕ) has the α–coincidence property if ϕ having an α–coincidence for each $\alpha \in Cov_W\left(\overline{\phi(W)}\right)$ guarantees that ϕ has a coincidence (similarly (W, ϕ) has the α–fixed point property if ϕ having an α–fixed point for each $\alpha \in Cov_W\left(\overline{\phi(W)}\right)$ guarantees that ϕ has a fixed point). One can put conditions on the space W and the morphism ϕ so that (W, ϕ) has the α–coincidence property (or (W, ϕ) has the α–fixed point property). For examples we refer the reader to [2 (Lemma 1.2), 5 (Lemma 6.1), 8 (Theorem 1.4 and Remark 1.6)].

Proof. Let $\phi = \phi|_W = \{W \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} W\}$ and let $\alpha \in Cov_W(K)$ where $K = \overline{\phi(W)}$. Also let $\psi_\alpha = \{W \overset{p_{1,\alpha}}{\Leftarrow} \Gamma_1 \overset{q_{1,\alpha}}{\to} U_\alpha\}$ and $\theta_\alpha = \{U_\alpha \overset{p_{2,\alpha}}{\Leftarrow} \Gamma_2 \overset{q_{2,\alpha}}{\to} W\}$. Let $\eta_\alpha = \psi_\alpha\, \theta_\alpha$. Note [5, see (4.2)] guarantees that $\eta_\alpha \in M(U_\alpha, U_\alpha)$ is compact (note θ_α is compact and the map $x \mapsto q_{1,\alpha}(p_{1,\alpha}^{-1}(x))$ is upper semicontinuous with nonempty compact values [4]). Since U_α is a Lefschetz coincidence space then η_α is a Lefschetz morphism. Now [5, see (2.2)] (here $E' = U'_\alpha = H(U_\alpha)$, $E'' = W' = H(W)$, $u = (\theta_\alpha)_\star$, $v = (\psi_\alpha)_\star$, $f' = (\eta_\alpha)_\star$ and $f'' = (\phi|_W)_\star$, note $u\, v = (\theta_\alpha)_\star(\psi_\alpha)_\star = (\phi|_W)_\star$ and $v\, u = (\psi_\alpha)_\star(\theta_\alpha)_\star = (\eta_\alpha)_\star$) guarantees that $\phi|_W$ is a Lefschetz morphism and $\Lambda\left((\phi|_W)_\star\right) = \Lambda\left((\psi_\alpha)_\star(\theta_\alpha)_\star\right)$. Thus $\Lambda\left(\phi|_W\right)$ is well defined.

Now suppose $\Lambda\left(\phi|_W\right) \neq 0$ (i.e. $\Lambda((\phi|_W)_\star) \neq 0$), then $\Lambda\left((\psi_\alpha)_\star(\theta_\alpha)_\star\right) \neq 0$ and since U_α is a Lefschetz coincidence space then $\psi_\alpha\theta_\alpha$ has a coincidence. Then (see [5, Lemma 6.3]) $\theta_\alpha\, \psi_\alpha$ has a coincidence (also note from Section 1 that $\theta_\alpha\psi_\alpha$ has a fixed point). If Definition 2.13 part (a) holds, then $\phi|_W$ has a α–coincidence (if Definition 2.13 part (b) holds, then $\phi|_W$ has a α–fixed point). □

Remark 2.16. Note in Theorem 2.14 (see the proof) we could replace θ_α compact with $\psi_\alpha\theta_\alpha$ compact.

Remark 2.17. A morphism $\eta \in M(X,Y)$ is said to be strongly α–close (here $\alpha \in Cov\,(Y)$) to $\xi \in M(X,Y)$ if for any $x \in X$ there exists $U_x \in \alpha$ with $\eta(x) \subseteq U_x$ and $\xi(x) \cap U_x \neq \emptyset$. Assume for each $\alpha \in Cov_W\left(\overline{\phi(W)}\right)$ that the morphism $\theta_\alpha\psi_\alpha \in$

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$M(W,W)$ is strongly α–close to $\phi|_W \in M(W,W)$. Then Definition 2.13 part (b) holds. To see this fix $\alpha \in Cov_W(\overline{\phi(W)})$ and suppose $\theta_\alpha \psi_\alpha$ has a fixed point i.e. there exists $w \in W$ with $w \in \theta_\alpha \psi_\alpha(w)$. Then since $\theta_\alpha\psi_\alpha$ is strongly α–close to $\phi|_W$ there exists $V_0 \in \alpha$ with

$$\theta_\alpha\psi_\alpha(w) \subseteq V_0 \quad \text{and} \quad \phi|_W(w) \cap V_0 \neq \emptyset.$$

Thus $w \in V_0$ and $\phi|_W(w) \cap V_0 \neq \emptyset$ i.e. $\phi|_W$ has a α–fixed point.

Remark 2.18. Two morphisms $\eta \in M(X,Y)$ and $\xi \in M(X,Y)$ are said to be very strongly α–close (here $\alpha \in Cov(Y)$) if for any $x \in X$ there exists $U_x \in \alpha$ with $\eta(x) \subseteq U_x$ and $\xi(x) \subseteq U_x$. Assume for each $\alpha \in Cov_W(\overline{\phi(W)})$ that the morphisms $\theta_\alpha\psi_\alpha \in M(W,W)$ and $\phi|_W \in M(W,W)$ are very strongly α–close. Then Definition 2.13 part (a) holds. To see this fix $\alpha \in Cov_W(\overline{\phi(W)})$ and suppose $\theta_\alpha \psi_\alpha$ has a coincidence. From Section 1 we have that $\theta_\alpha \psi_\alpha$ has a fixed point i.e. there exists $w \in W$ with $w \in \theta_\alpha \psi_\alpha(w)$. Then since $\theta_\alpha \psi_\alpha$ and $\phi|_W$ are very strongly α–close there exists $V_0 \in \alpha$ with

$$\theta_\alpha\psi_\alpha(w) \subseteq V_0 \quad \text{and} \quad \phi|_W(w) = q(p^{-1}(w)) \subseteq V_0.$$

In particular $w \in V_0$. Let $a \in p^{-1}(w)$. Then $p(a) = w \in V_0$. Also $q(a) \in V_0$ since $q(p^{-1}(w)) \subseteq V_0$. Thus $p(a) \in V_0$ and $q(a) \in V_0$, so $\phi|_W$ has a α–coincidence (for (p,q)) [We can do the above argument for any representation (p,q) of $\phi|_W$].

Remark 2.19. Suppose in the statement Definition 2.13 we have $\phi|_W \in M(W,W)$ replaced by $\phi|_V \in M(V,V)$. Then essentially the same reasoning as in Remark 2.6 guarantees that $\Lambda(\phi|_V)$ is well defined. If Definition 2.13 part (a) holds and $\Lambda(\phi|_V) \neq 0$ then for any $\alpha \in Cov_W(\overline{\phi(W)})$ we have that $\phi|_V$ has a α–coincidence (note if $\Lambda(\phi|_V) \neq 0$ then essentially the same argument as in Remark 2.6 guarantees that $\Lambda(\phi|_W) \neq 0$ and apply Theorem 2.14). Also if Definition 2.13 part (b) holds and $\Lambda(\phi|_V) \neq 0$ then for any $\alpha \in Cov_W(\overline{\phi(W)})$ we have that that $\phi|_V$ has a α–fixed point.

We now discuss two special cases of Theorem 2.14.

Definition 2.20. We say $X \in ANESM_1$ (w.r.t the morphism ϕ) if $\phi|_W \in M(W,W)$ (here $W = \overline{\phi(X)}$), and for each $\alpha \in Cov_W(\overline{\phi(W)})$ there exists a Lefschetz coincidence space U_α, a compact $\theta_\alpha \in M(U_\alpha, W)$, a $\psi_\alpha \in M(W,U_\alpha)$, with $(\theta_\alpha)_\star(\psi_\alpha)_\star = (\phi|_W)_\star$ and with one of the following properties holding: (a) $\theta_\alpha\psi_\alpha$ has a coincidence guarantees $\phi|_W$ has a α–coincidence, or (b) $\theta_\alpha\psi_\alpha$ has a fixed point guarantees $\phi|_W$ has a α–fixed point.

Theorem 2.21. *Let $X \in ANESM_1$ (w.r.t ϕ) and $W = \overline{\phi(X)}$. Then $\Lambda(\phi|_W)$ is well defined. Also if Definition 2.20 part (a) holds and $\Lambda(\phi|_W) \neq 0$ then for any $\alpha \in Cov_W(\overline{\phi(W)})$ we have that $\phi|_W$ has a α–coincidence (if Definition 2.20 part (b) holds and $\Lambda(\phi|_W) \neq 0$ then for any $\alpha \in Cov_W(\overline{\phi(W)})$ we have that that $\phi|_W$ has a α–fixed point).*

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Proof. Take $V = X$ in Theorem 2.14. □

Definition 2.22. We say $X \in ANESM_2$ (w.r.t the morphism ϕ) if $\phi \in M(X,X)$ and for each $\alpha \in Cov_X(\overline{\phi(X)})$ there exists a Lefschetz coincidence space U_α, a compact $\theta_\alpha \in M(U_\alpha, X)$, a $\psi_\alpha \in M(X, U_\alpha)$, with $(\theta_\alpha)_\star(\psi_\alpha)_\star = \phi_\star$ and with one of the following properties holding: (a) $\theta_\alpha \psi_\alpha$ has a coincidence guarantees ϕ has a α–coincidence, or (b) $\theta_\alpha \psi_\alpha$ has a fixed point guarantees ϕ has a α–fixed point.

Theorem 2.23. *Let $X \in ANESM_2$ (w.r.t ϕ). Then $\Lambda(\phi)$ is well defined. Also if Definition 2.22 part (a) holds and $\Lambda(\phi) \neq 0$ then for any $\alpha \in Cov_X(\overline{\phi(X)})$ we have that ϕ has a α–coincidence (if Definition 2.22 part (b) holds and $\Lambda(\phi) \neq 0$ then for any $\alpha \in Cov_X(\overline{\phi(X)})$ we have that that ϕ has a α–fixed point).*

Proof. The proof is word for word that in Theorem 2.14 with W replaced by X. □

Alternate Definition and Results: We say $X \in GANESM$ (w.r.t. ϕ) if there exists a set $A \subseteq X$ with $\phi(A) \subseteq A$, $\phi|_W \in M(W,W)$, here $W = \phi(A)$, and for each $\alpha \in Cov_W(\phi(W))$ there exists a Lefschetz coincidence space U_α, a compact $\theta_\alpha \in M(U_\alpha, W)$, a $\psi_\alpha \in M(W, U_\alpha)$, with $(\theta_\alpha)_\star(\psi_\alpha)_\star = (\phi|_W)_\star$ and with one of the following properties holding: (a) $\theta_\alpha \psi_\alpha$ has a coincidence guarantees $\phi|_W$ has a α–coincidence, or (b) $\theta_\alpha \psi_\alpha$ has a fixed point guarantees $\phi|_W$ has a α–fixed point.

(1) The reasoning in Theorem 2.14 guarantees that if $X \in GANESM$ (w.r.t. ϕ) (and let A, W, α, U_α, θ_α and ψ_α be as described above) then $\Lambda(\phi|_W)$ is well defined. Also if in the Definition above part (a) holds and $\Lambda(\phi|_W) \neq 0$ then for any $\alpha \in Cov_W(\phi(W))$ we have that $\phi|_W$ has a α–coincidence (if in the Definition above part (b) holds and $\Lambda(\phi|_W) \neq 0$ then for any $\alpha \in Cov_W(\phi(W))$ we have that that $\phi|_W$ has a α–fixed point).

(2) Suppose in the statement above we have $\phi|_W \in M(W,W)$ replaced by $\phi|_A \in M(A,A)$. Then the reasoning in Remark 2.19 guarantees that $\Lambda(\phi|_A)$ is well defined. If in the Definition above part (a) holds and $\Lambda(\phi|_A) \neq 0$ then for any $\alpha \in Cov_W(\phi(W))$ we have that $\phi|_A$ has a α–coincidence (if in the Definition above part (b) holds and $\Lambda(\phi|_A) \neq 0$ then for any $\alpha \in Cov_W(\phi(W))$ we have that that $\phi|_A$ has a α–fixed point).

Motivated in part by [6] we will generalize slightly Theorem 2.2 (see Remark 2.6) and Theorem 2.14 (see Remark 2.19) for noncompact morphisms $\phi \in M(X,X)$.

Definition 2.24. Let X be a space. We say $\phi \in M(X,X)$ is compact absorbing contractive if:

(i) $X \in NESM$ (w.r.t. ϕ) and let U, V, W, θ and ψ be as described in Definition 2.1;

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(ii) $\phi \in M((X,W),(X,W))$ and $\phi_\star : H(X,W) \to H(X,W)$ is a weakly nilpotent endomorphism.

Theorem 2.25. *Let $\phi \in M(X,X)$ be compact absorbing contractive. Then $\Lambda(\phi|_X)$ is well defined and if $\Lambda(\phi|_X) \neq 0$ then $\phi|_X$ has a coincidence.*

Proof. Now Theorem 2.2 and Definition 2.24 (i) guarantees that $\phi|_W$ is a Lefschetz morphism. Note Definition 2.24 (ii) (see [6, Theorem 5.9]) guarantees that ϕ is a Lefschetz morphism and $\Lambda(\phi_\star) = 0$. Now [6, Theorem 5.11] guarantees that $\phi|_X$ is a Lefschetz morphism and $\Lambda((\phi|_X)_\star) = \Lambda((\phi|_W)_\star)$, so $\Lambda(\phi|_X)$ is well defined.

Now suppose $\Lambda(\phi|_X) \neq 0$ (i.e. $\Lambda((\phi|_X)_\star) \neq 0$). Then $\Lambda((\phi|_W)_\star) \neq 0$ (i.e. $\Lambda(\phi|_W) \neq 0$) so Theorem 2.2 enables us to deduce that $\phi|_W$ (so $\phi|_X$) has a coincidence. □

Definition 2.26. Let X be a space. We say $\phi \in M(X,X)$ is approximate compact absorbing contractive if:

(i) $X \in ANESM$ (w.r.t. ϕ), and let V, W, α, U_α, θ_α and ψ_α be as described in Definition 2.13;

(ii) Definition 2.24 (ii) holds.

Theorem 2.27. *Let $\phi \in M(X,X)$ be approximate compact absorbing contractive. Then $\Lambda(\phi|_X)$ is well defined. Also if Definition 2.13 part (a) holds and $\Lambda(\phi|_X) \neq 0$ then for $\alpha \in Cov_W(\overline{\phi(W)})$ we have that $\phi|_X$ has a α–coincidence (if Definition 2.13 part (b) holds and $\Lambda(\phi|_X) \neq 0$ then for $\alpha \in Cov_W(\overline{\phi(W)})$ we have that $\phi|_X$ has a α–fixed point).*

Proof. The same argument as in Theorem 2.25 (with Theorem 2.2 replaced by Theorem 2.14) guarantees that $\phi|_X$ is a Lefschetz morphism and $\Lambda((\phi|_X)_\star) = \Lambda((\phi|_W)_\star)$, so $\Lambda(\phi|_X)$ is well defined.

Now suppose $\Lambda(\phi|_X) \neq 0$ (i.e. $\Lambda((\phi|_X)_\star) \neq 0$). Then $\Lambda((\phi|_W)_\star) \neq 0$ (i.e. $\Lambda(\phi|_W) \neq 0$). If Definition 2.13 part (a) holds then for $\alpha \in Cov_W(\overline{\phi(W)})$ we have from Theorem 2.14 that $\phi|_W$ (so $\phi|_X$) has a α–coincidence (if Definition 2.13 part (b) holds then for $\alpha \in Cov_W(\overline{\phi(W)})$ we have from Theorem 2.14 that $\phi|_W$ (so $\phi|_X$) has a α–fixed point). □

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