International Electronic Journal of Pure and Applied Mathematics

Volume 11 No. 1 2017, 25-38
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v11i1.3

MONOTONE-ITERATIVE TECHNIQUE FOR THE PERIODIC BOUNDARY VALUE PROBLEM FOR DIFFERENCE EQUATIONS WITH NON-INSTANTANEOUS IMPULSES

S. Hristova[1 §], R. Terzieva[2]
[1,2]University of Plovdiv "Paisii Hilendarski"
24 Tzar Asen, 4000, Plovdiv, BULGARIA
snehri@gmail.com

Abstract: An algorithm for constructing two monotone sequences of upper and lower solutions of the periodic boundary value problem for nonlinear difference equations with non-instantaneous impulses is given. The impulses start abruptly at some points and their action continue on given finite discrete intervals. It is proved both functional sequences are pointwise convergent and their limits are minimal and maximal solutions of the considered problem.

AMS Subject Classification: 39A22, 65Q10
Key Words: difference equations, non-instantaneous impulses, periodic boundary value problem,, lower and upper solutions, monotone-iterative technique

1. Introduction

One of the problems in difference equations is approximate obtaining of the solution ([1], [3]). It is very important specially in the case when the unknown function in the present time is involved in both side parts of the nonlinear equation. One of the approximate method is based on the method of upper and lower solutions combined with a monotone–iterative technique. This method allow us to be constructed two monotonous sequences of upper and lower solutions of the nonlinear non-instantaneous impulsive difference equation. Also any term of the sequence is obtained a formula for explicit solution. Note this method is applied for difference equations in [5], [6], [7], for impulsive difference equations in [2] and for difference equations with maxima in [4].

Received: December 10, 2016

[§]Correspondence author

The idea of the method is to use given upper and lower solutions for initial iteration and to construct successive approximation from the corresponding non-instantaneous impulsive linear equation. These functional sequences converge monotonically to the minimal and maximal solutions of the nonlinear equation. In Section 3, we obtain an explicit formula for solution of a periodic boundary value problem for linear non–instantaneous impulsive difference equation and some comparison results. Finally, by use of the monotone iterative technique and the method of upper and lower solutions we obtain the existence theorem of extremal solutions of the nonlinear equation.

2. Statement of the Problem

Let $\mathbb{Z}_+$ denote the set of all nonegative integers. Let the increasing sequence $\{n_i\}_{i=0}^{p+1}: \ n_i \in \mathbb{Z}_+, \ n_i \geq n_{i-1}+3, i=1,2,\dots,p$ and the sequence $\{d_i\}_{i=1}^{p}: \ d_i \in \mathbb{Z}_+, \ 1 \leq d_i \leq n_{i+1}-n_i-2, \ i=1,2,\dots,p$ be given. We denote $\mathbb{Z}[a,b]=\{z \in \mathbb{Z}_+ : a \leq z \leq b\}, \ a,b \in \mathbb{Z}_+, \ a<b$ and $I_k=\mathbb{Z}[n_k+d_k, n_{k+1}-2], \ k \in \mathbb{Z}[0,p-1], \ I_p = \mathbb{Z}[n_p+d_p, n_{p+1}-1]$ and $J_k=\mathbb{Z}[n_k+1, n_k+d_k], \ k \in \mathbb{Z}[1,p]$ where $d_0=0$.

Consider the *periodic boundary value problem (PBVP)* for the nonlinear *difference equation with non–instantaneous impulses* (NIDE)

$$\begin{aligned} & x(n+1) = f(n, x(n), x(n+1)) \text{ for } n \in \bigcup_{k=0}^{p} I_k, \\ & x(n_k) = F(k, x(n_k-1)), \ k \in \mathbb{Z}[1,p], \\ & x(n) = g(n, x(n), x(n_k)) \text{ for } n \in \bigcup_{k=1}^{p} J_k, \\ & x(n_0) = x(n_{p+1}), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}$, $f: \bigcup_{k=0}^{p} I_k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ F: \mathbb{Z}[1,p] \times \mathbb{R} \to \mathbb{R}$, and $g: \bigcup_{k=1}^{p} J_k \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

3. Preliminaries Results

Definition 1. We will say that the function $\alpha : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ is a minimal(maximal) solution of the PBVP for NIDE (1) in $\mathbb{Z}[n_0, n_{p+1}]$ if it is a solution of (1) and for any solution $u(n), n \in \mathbb{Z}[n_0, n_{p+1}]$ the inequality $\alpha(n) \leq u(n)$ $(\alpha(n) \geq u(n))$ holds on $\mathbb{Z}[n_0, n_{p+1}]$.

Definition 2. The function $\alpha : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ is called lower (upper) solutions of PBVP for NIDE (1), if:

$$\begin{aligned}
&\alpha(n+1) \leq (\geq) f(n, \alpha(n), \alpha(n+1)),\\
&\alpha(n_k) \leq (\geq) F(k, \alpha(n_k - 1)),\\
&\alpha(n) \leq (\geq) g(n, \alpha(n), \alpha(n_k)),\\
&\alpha(n_0) \leq (\geq) \alpha(n_{p+1}).
\end{aligned}$$

Consider the linear NIDE of the type

$$\begin{aligned}
&u(n+1) = Q_n u(n) + \sigma_n, \quad n \in \bigcup_{k=0}^{p} I_k,\\
&u(n_k) = T_k u(n_k - 1) + \mu_k, \ k \in \mathbb{Z}[1,p],\\
&u(n) = M_n u(n) + L_n u(n_k) + \gamma_n, \quad n \in \bigcup_{k=1}^{p} J_k,\\
&u(n_0) = u(n_{p+1}),
\end{aligned} \tag{2}$$

where $u \in \mathbb{R}$.

Lemma 1. *Let the following conditions be fulfilled:*

1. $Q_n \neq 0, \ \sigma_n : \ n \in \bigcup_{k=0}^{p} I_k$ *and* $\sigma_n = 0, \ Q_n = 1$ *for* $n \in \mathbb{Z}[n_0, n_{p+1}]/\bigcup_{k=0}^{p} I_k$, $L_n \neq 0, \ M_n \neq 1, \gamma_n : \ n \in \bigcup_{k=1}^{p} J_k$, *and* $T_k \neq 0, \mu_k : \ k \in \mathbb{Z}[1,p]$ *are given real constants;*

2. *The real constants* Q_n, T_k, L_n, *and* M_n *from the condition 1 are such that:*

$$\Big(\prod_{i=1}^{p} T_i \frac{L_{n_i+d_i}}{1 - M_{n_i+d_i}}\Big)\Big(\prod_{i=n_0}^{n_{p+1}-1} Q_i\Big) \neq 1.$$

Then the PBVP for NIDE (2) has an unique solution given by

$$\begin{aligned}
u(n) = N(n) \sum_{j=n_0-1}^{n-1} (\prod_{i=j+1}^{n} R(i))\sigma_j \prod_{i=j+1}^{n-1} Q_i + N(n) \sum_{j=n_0}^{n} (\prod_{i=j+1}^{n} R(i))\zeta(j) \prod_{i=j}^{n-1} Q_i \\
+ \tau(n) \quad \text{for } n \in \mathbb{Z}[n_0, n_{p+1}]
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
\sigma_{n_0-1} = \frac{1}{1 - \prod_{i=n_0}^{n_{p+1}} R(i) \prod_{i=n_0}^{n_{p+1}-1} Q_i} \Big(\sum_{j=n_0}^{n_{p+1}-1} (\prod_{i=j+1}^{n_{p+1}} R(i))\sigma_j \prod_{i=j+1}^{n_{p+1}-1} Q_i \\
+ \sum_{j=n_0}^{n_{p+1}} (\prod_{i=j+1}^{n_{p+1}} R(i))\zeta(j) \prod_{i=j}^{n_{p+1}-1} Q_i \Big),
\end{aligned}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 11, No. 1 (2017)

$$N(n) = \begin{cases} \frac{L_n}{1-M_n} & \textit{for} \quad n \in \bigcup_{k=1}^{p} J_k \\ 1 & \textit{otherwise,} \end{cases} \tag{4}$$

$$\tau(n) = \begin{cases} \frac{\gamma_n}{1-M_n} & \textit{for} \quad n \in \bigcup_{k=1}^{p} J_k \\ 0 & \textit{otherwise,} \end{cases} \tag{5}$$

$$R(n) = \begin{cases} N(n_k + d_k) = \frac{L_{n_k+d_k}}{1-M_{n_k+d_k}} & \textit{for} \quad n = n_k + d_k + 1, \qquad k \in \mathbb{Z}[1,p] \\ T_k \quad \textit{for} \quad n = n_k, \qquad k \in \mathbb{Z}[1,p] \\ 1 \quad \textit{otherwise,} \end{cases} \tag{6}$$

$$\zeta(n) = \begin{cases} \tau(n_k + d_k) = \frac{\gamma_{n_k+d_k}}{1-M_{n_k+d_k}} & \textit{for} \quad n = n_k + d_k + 1, \qquad k \in \mathbb{Z}[1,p] \\ \mu_k \quad \textit{for} \quad n = n_k, \qquad k \in \mathbb{Z}[1,p] \\ 0 \quad \textit{otherwise.} \end{cases} \tag{7}$$

Note that $\prod_{i=1}^{p} T_i \frac{L_{n_i+d_i}}{1-M_{n_i+d_i}} = \prod_{i=n_0}^{n_{p+1}} R(i)$.

Proof. We will use an induction with respect to the interval.

Let $n \in I_0 = \mathbb{Z}[n_0, n_1 - 2]$. Then we obtain

$$u(n) = u(n_0) \prod_{i=n_0}^{n-1} Q_i + \sum_{j=n_0}^{n-1} \sigma_j \prod_{i=j+1}^{n-1} Q_i, \ n \in \mathbb{Z}[n_0 + 1, n_1 - 1].$$

Let $n = n_1$. Then using $\sigma_{n_1-1} = 0, \ \ Q_{n_1-1} = 1$ we get

$$\begin{aligned} u(n_1) &= T_1 u(n_1 - 1) + \mu_1 = u(n_0) T_1 \prod_{i=n_0}^{n_1-1} Q_i + T_1 \sum_{j=n_0}^{n_1-1} \sigma_j \prod_{i=j+1}^{n_1-1} Q_i + \mu_1 \\ &= u(n_0) (\prod_{i=n_0}^{n_1} R(i)) \prod_{i=n_0}^{n_1-1} Q_i + \sum_{j=n_0}^{n_1-1} (\prod_{i=j+1}^{n_1} R(i)) \sigma_j \prod_{i=j+1}^{n_1-1} Q_i + \mu_1. \end{aligned}$$

Let $n \in J_1 = \mathbb{Z}[n_1 + 1, n_1 + d_1]$. Then using $\sigma_j = 0, \ \ Q_j = 1, \ j \in \mathbb{Z}[n_1, n_1 + d_1]$ we get

$$\begin{aligned} u(n) &= u(n_0) \frac{L_n}{1 - M_n} (\prod_{i=n_0}^{n_1} R(i)) \prod_{i=n_0}^{n-1} Q_i + \frac{L_n}{1 - M_n} \sum_{j=n_0}^{n-1} (\prod_{i=j+1}^{n_1} R(i)) \sigma_j \prod_{i=j+1}^{n-1} Q_i \\ &\quad + \frac{L_n}{1 - M_n} \mu_1 + \frac{\gamma_n}{1 - M_n}. \\ &= u(n_0) N(n) (\prod_{i=n_0}^{n} R(i)) \prod_{i=n_0}^{n-1} Q_i + N(n) \sum_{j=n_0}^{n-1} \sigma_j (\prod_{i=j+1}^{n} R(i)) \prod_{i=j+1}^{n-1} Q_i \\ &\quad + N(n) \mu_1 + \tau(n), \ n \in J_1. \end{aligned}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 11, No. 1 (2017)

Let $n \in I_1 = \mathbb{Z}[n_1 + d_1, n_2 - 2]$. Then

$$\begin{aligned}
u(n) = {} & N(n_1 + d_1)u(n_0)\Big(\prod_{i=n_0}^{n_1+d_1} R(i)\Big) \prod_{i=n_0}^{n_1+d_1-1} Q_i \prod_{i=n_1+d_1}^{n-1} Q_i \\
& + N(n_1 + d_1) \sum_{j=n_0}^{n_1+d_1-1} \sigma_j\Big(\prod_{i=j+1}^{n_1+d_1} R(i)\Big) \prod_{i=j+1}^{n_1+d_1-1} Q_i \prod_{i=n_1+d_1}^{n-1} Q_i \\
& + \mu_1 N(n_1 + d_1) \prod_{i=n_1+d_1}^{n-1} Q_i + \tau(n_1 + d_1) \prod_{i=n_1+d_1}^{n-1} Q_i + \sum_{j=n_1+d_1}^{n-1} \sigma_j \prod_{i=j+1}^{n-1} Q_i \\
= {} & u(n_0)\Big(\prod_{i=n_0}^{n} R(i)\Big) \prod_{i=n_0}^{n-1} Q_i + \sum_{j=n_0}^{n-1}\Big(\prod_{i=j+1}^{n} R(i)\Big)\sigma_j \prod_{i=j+1}^{n-1} Q_i \\
& + \sum_{j=n_0}^{n}\Big(\prod_{i=j+1}^{n} R(i)\Big)\zeta(j) \prod_{i=j}^{n-1} Q_i.
\end{aligned}$$

Let $n = n_2$. Then we get

$$\begin{aligned}
u(n_2) = {} & T_2 u(n_0) \prod_{i=n_0}^{n_2-1} R(i) \prod_{i=n_0}^{n_2-1} Q_i + T_2 \sum_{j=n_0}^{n_2-1}\Big(\prod_{i=j+1}^{n_2-1} R(i)\Big)\sigma_j \prod_{i=j+1}^{n_2-1} Q_i \\
& + T_2 \sum_{j=n_0}^{n_2-1}\Big(\prod_{i=j+1}^{n_2-1} R(i)\Big)\zeta(j) \prod_{i=j}^{n_2-1} Q_i + \mu_2 \\
= {} & u(n_0) \prod_{i=n_0}^{n_2} R(i) \prod_{i=n_0}^{n_2-1} Q_i + \sum_{j=n_0}^{n_2-1}\Big(\prod_{i=j+1}^{n_2} R(i)\Big)\sigma_j \prod_{i=j+1}^{n_2-1} Q_i \\
& + \sum_{j=n_0}^{n_2}\Big(\prod_{i=j+1}^{n_2} R(i)\Big)\zeta(j) \prod_{i=j}^{n_2-1} Q_i.
\end{aligned}$$

Let $n \in J_2 = \mathbb{Z}[n_2 + 1, n_2 + d_2]$. Then

$$\begin{aligned}
u(n) = {} & N(n)u(n_0) \prod_{i=n_0}^{n_2} R(i) \prod_{i=n_0}^{n_2-1} Q_i + N(n) \sum_{j=n_0}^{n_2-1}\Big(\prod_{i=j+1}^{n_2} R(i)\Big)\sigma_j \prod_{i=j+1}^{n_2-1} Q_i \\
& + N(n) \sum_{j=n_0}^{n_2}\Big(\prod_{i=j+1}^{n_2} R(i)\Big)\zeta(j) \prod_{i=j}^{n_2-1} Q_i + \tau(n) \\
= {} & N(n)u(n_0) \prod_{i=n_0}^{n} R(i) \prod_{i=n_0}^{n-1} Q_i + N(n) \sum_{j=n_0-1}^{n-1}\Big(\prod_{i=j+1}^{n} R(i)\Big)\sigma_j \prod_{i=j+1}^{n-1} Q_i \\
& + N(n) \sum_{j=n_0}^{n}\Big(\prod_{i=j+1}^{n} R(i)\Big)\zeta(j) \prod_{i=j}^{n-1} Q_i + \tau(n).
\end{aligned}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 11, No. 1 (2017)

Continue this process step by step w.r.t. the interval we prove the solution of NIDE (2) is given with

$$\begin{aligned} u(n) = u(n_0)N(n)\prod_{i=n_0}^{n} R(i)\prod_{i=n_0}^{n-1} Q_i + N(n)\sum_{j=n_0}^{n-1}(\prod_{i=j+1}^{n} R(i))\sigma_j \prod_{i=j+1}^{n-1} Q_i \\ + \sum_{j=n_0}^{n}(\prod_{i=j+1}^{n} R(i))\zeta(j)\prod_{i=j}^{n-1} Q_i + \tau(n). \end{aligned} \tag{8}$$

Let $n = n_{p+1}$ in the above equation. Then we obtain

$$\begin{aligned} u(n_{p+1}) = u(n_0)\prod_{i=n_0}^{n_{p+1}} R(i)\prod_{i=n_0}^{n_{p+1}-1} Q_i + \sum_{j=n_0}^{n_{p+1}-1}(\prod_{i=j+1}^{n_{p+1}} R(i))\sigma_j \prod_{i=j+1}^{n_{p+1}-1} Q_i \\ + \sum_{j=n_0}^{n_{p+1}}(\prod_{i=j+1}^{n_{p+1}} R(i))\zeta(j)\prod_{i=j}^{n_{p+1}-1} Q_i. \end{aligned}$$

Then from the boundary condition $u(n_0) = u(n_{p+1})$ we obtain

$$\begin{aligned} u(n_0) = \frac{1}{1 - \prod_{i=n_0}^{n_{p+1}} R(i)\prod_{i=n_0}^{n_{p+1}-1} Q_i}\Big(\sum_{j=n_0}^{n_{p+1}-1}(\prod_{i=j+1}^{n_{p+1}} R(i))\sigma_j \prod_{i=j+1}^{n_{p+1}-1} Q_i \\ + \sum_{j=n_0}^{n_{p+1}}(\prod_{i=j+1}^{n_{p+1}} R(i))\zeta(j)\prod_{i=j}^{n_{p+1}-1} Q_i\Big). \end{aligned} \tag{9}$$

Substituting (9) in (8) we obtain the solution of NIDE (2) is given by (3) for all $n \in \mathbb{Z}[n_0, n_{p+1}]$.

□

Lemma 2. *Let the following conditions be fulfilled:*

1. *The function* $m : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ *satisfies*

$$\begin{aligned} & m(n+1) \le Q_n m(n), \quad n \in \bigcup_{k=0}^{p} I_k, \\ & m(n_k) \le T_k m(n_k - 1), \quad k \in \mathbb{Z}[1, p], \\ & m(n) \le M_n m(n) + L_n m(n_k), \quad n \in \bigcup_{k=1}^{p} J_k, \\ & m(n_0) \le m(n_{p+1}). \end{aligned} \tag{10}$$

2. $Q_n > 0 :\ n \in \bigcup_{k=0}^{p} I_k$, $Q_n = 1$ *for* $n \in \mathbb{Z}[n_0, n_{p+1}]/\bigcup_{k=0}^{p} I_k$, $T_k > 0,\ k \in \mathbb{Z}[1, p]$ *and* $L_n > 0,\ M_n < 1 :\ n \in \bigcup_{k=0}^{p} J_k$ *are given real constants;*

3. *The real constants Q_n, T_k, L_n, and M_n from the condition 2 satisfies the inequality:*

$$\Big(\prod_{i=1}^{p} T_i \frac{L_{n_i+d_i}}{1-M_{n_i+d_i}}\Big)\Big(\prod_{i=n_0}^{n_{p+1}-1} Q_i\Big) < 1.$$

Then $m(n) \le 0$ for every $n \in \mathbb{Z}[n_0, n_{p+1}]$.

Proof. Let $N(n),\ R(n)$ are defined by (4) and (6), respectively.

We will use an induction with respect to the interval.

Let $n \in I_0 = \mathbb{Z}[n_0, n_1 - 2]$. Then from the first equation of (10) it follows step by step the inequality $m(n) \le m(n_0)\prod_{i=n_0}^{n-1} Q_i$ holds.

Let $n = n_1$. Then using $Q_{n_1-1} = 1$ we obtain $m(n_1) \le T_1 m(n_0)\prod_{i=n_0}^{n_1-1} Q_i = m(n_0)\Big(\prod_{i=n_0}^{n_1} R(i)\Big)\Big(\prod_{i=n_0}^{n_1-1} Q_i\Big)$.

Let $n \in J_1 = \mathbb{Z}[n_1 + 1, n_1 + d_1]$. Then we have $m(n) \le M_n m(n) + L_n m(n_1)$ or $m(n) \le m(n_0)N(n)\Big(\prod_{i=n_0}^{n_1} R(i)\Big)\Big(\prod_{i=n_0}^{n_1-1} Q_i\Big)$ for $n \in J_1$.

Let $n \in I_1 = \mathbb{Z}[n_1 + d_1, n_2 - 2]$. Then $m(n + 1) \le Q_n m(n),\ n \in I_1$, and hence $m(n) \le m(n_0)N(n_1 + d_1)\Big(\prod_{i=n_0}^{n_1} R(i)\Big)\Big(\prod_{i=n_0}^{n_1-1} Q_i\Big)\Big(\prod_{i=n_1}^{n-1} Q_i\Big)$ for $n \in I_1$ or $m(n) \le m(n_0)\Big(\prod_{i=n_0}^{n} R(i)\Big)\Big(\prod_{i=n_0}^{n-1} Q_i\Big)$.

Continue this process by induction we prove that

$$m(n) \le m(n_0)N(n)\prod_{i=n_0}^{n} R(i)\prod_{i=n_0}^{n-1} Q_i. \tag{11}$$

It is enough to show $m(n_0) \le 0$. Let $n = n_{p+1}$ in (11). Then we obtain

$$m(n_0) \le m(n_{p+1}) \le m(n_0)\prod_{i=n_0}^{n_{p+1}} R(i)\prod_{i=n_0}^{n_{p+1}-1} Q_i. \tag{12}$$

From (12), we get $m(n_0)\Big[1 - \prod_{i=n_0}^{n_{p+1}} R(i)\prod_{i=n_0}^{n_{p+1}-1} Q_i\Big] \le 0$. Hence, $m(n_0) \le 0$. From (11) we have $m(n) \le 0$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$.

□

4. Main Result

For any pair of function $\alpha, \beta : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ such that $\alpha(n) \leq \beta(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$ we define the sets

$$S(\alpha,\beta) = \{u : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R} : \alpha(n) \leq u(n) \leq \beta(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]\},$$

$$\Omega_1(\alpha,\beta) = \{u \in \mathbb{R} : \alpha(n) \leq u \leq \beta(n), \quad n \in \bigcup_{k=0}^{p} I_k\},$$

$$\Omega_2(\alpha,\beta) = \{u \in \mathbb{R} : \alpha(n+1) \leq u \leq \beta(n+1), \quad n \in \bigcup_{k=0}^{p} I_k\},$$

$$\Lambda(\alpha,\beta) = \{u \in \mathbb{R} : \alpha(n) \leq u \leq \beta(n), \quad n \in \bigcup_{k=1}^{p} J_k\},$$

$$\Gamma(\alpha,\beta) = \{y \in \mathbb{R} : \alpha(n_k) \leq y \leq \beta(n_k), \quad k \in \mathbb{Z}[1,p]\},$$
$$\Upsilon(\alpha,\beta) = \{z \in \mathbb{R} : \alpha(n_k - 1) \leq z \leq \beta(n_k - 1), \quad k \in \mathbb{Z}[1,p]\}.$$

Theorem 1. *Let the following conditions be fulfilled:*

1. *The functions $\alpha, \beta : \mathbb{Z}[n_0, n_{p+1}] \to \mathbb{R}$ are lower and upper solutions of the BVP for NIDE (1), respectively, and $\alpha(n) \leq \beta(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$.*
2. *The functions $f : \bigcup_{k=0}^{p} I_k \times \Omega_1(\alpha,\beta) \times \Omega_2(\alpha,\beta)$ are continuous in its second and third arguments and there exist functions $K : \bigcup_{k=0}^{p} I_k \to (-\infty, 1)$ and $P : \bigcup_{k=0}^{p} I_k \to (0,\infty)$ such that for any $n \in \bigcup_{k=0}^{p} I_k$ and $x_1, x_2 \in \Omega_1(\alpha,\beta)$, with $x_1 \leq x_2$, and $x_3, x_4 \in \Omega_2(\alpha,\beta)$, with $x_3 \leq x_4$ the inequality*
$$f(n, x_1, x_3) - f(n, x_2, x_4) \leq P(n)(x_1 - x_2) + K(n)(x_3 - x_4)$$
holds.
3. *The functions $F : \mathbb{Z}[1,p] \times \mathbb{R} \to \mathbb{R}$ are continuous in its second argument and there exists a function $T : \mathbb{Z}[1,p] \to (0,\infty)$ such that for any $k \in \mathbb{Z}[1,p]$ and $z_1, z_2 \in \Upsilon(\alpha,\beta)$ with $z_1 \leq z_2$*
$$F(k, z_1) - F(k, z_2) \leq T(k)(z_1 - z_2).$$
4. *The functions $g : \bigcup_{k=1}^{p} J_k \times \Lambda(\alpha,\beta) \times \Gamma(\alpha,\beta) \to \mathbb{R}$ are continuous in its second and third arguments and there exist functions $M : \bigcup_{k=1}^{p} J_k \to (-\infty, 1)$ and $L : \bigcup_{k=1}^{p} J_k \to (0,\infty)$ such that for any $n \in \bigcup_{k=1}^{p} J_k$ and $y_1, y_2 \in \Lambda(\alpha,\beta)$, with $y_1 \leq y_2$, and $y_3, y_4 \in \Gamma(\alpha,\beta)$, with $y_3 \leq y_4$ the inequality*
$$g(n, y_1, y_3) - g(n, y_2, y_4) \leq M(n)(y_1 - y_2) + L(n)(y_3 - y_4)$$
holds.

5. The Lipshitz constants from the conditions 2, 3 and 4 satisfies the inequality

$$\prod_{i=1}^{p} T(i)\frac{L(n_i+d_i)}{1-M(n_i+d_i)}\prod_{i=n_0}^{n_{p+1}-1}\frac{P(i)}{1-K(i)}<1.$$

Then there exist two sequences of functions $\{\alpha^{(j)}(n)\}_0^\infty$ *and* $\{\beta^{(j)}(n)\}_0^\infty$, $n \in \mathbb{Z}[n_0, n_{p+1}]$ *with* $\alpha^{(0)} = \alpha$ *and* $\beta^{(0)} = \beta$ *such that:*

a) The sequences are nondecreasing and nonincreasing, respectively and

$$\alpha(n) \le \alpha^{(j)}(n) \le \beta^{(j)}(n) \le \beta(n), \text{ for } n \in \mathbb{Z}[n_0, n_{p+1}];$$

b) The functions $\alpha^{(j)}(n)$ *and* $\beta^{(j)}(n)$ *are lower and upper solutions of the BVP for NIDE (1), respectively;*

c) Both sequences are pointwise convergent on $\mathbb{Z}[n_0, n_{p+1}]$*;*

d) The limits $\lim_{j\to\infty}\alpha^{(j)}(n) = A(n)$, $\lim_{j\to\infty}\beta^{(j)}(n) = B(n)$ *are the minimal and maximal solutions of BVP for NIDE (1) in* $S(\alpha,\beta)$*, respectively;*

e) If BVP for NIDE has an unique solution $u(n) \in S(\alpha,\beta)$*, then* $A(n) \equiv u(n) \equiv B(n)$ *for* $n \in \mathbb{Z}[n_0, n_{p+1}]$.

Proof. For an arbitrary function $\eta \in S(\alpha,\beta)$ consider the PBVP for the linear NIDE

$$\begin{aligned}
&u(n+1) = P(n)u(n) + K(n)u(n+1) + \psi(n, \eta(n), \eta(n+1)), \quad n \in \bigcup_{k=0}^{p} I_k,\\
&u(n_k) = T(k)u(n_k-1) + \upsilon(k, \eta(n_k-1)), \quad k \in \mathbb{Z}[1,p],\\
&u(n) = M(n)u(n) + L(n)u(n_k) + \xi(n, \eta(n), \eta(n_k)), \quad n \in \bigcup_{k=1}^{p} J_k,\\
&u(n_0) = u(n_{p+1}),
\end{aligned} \tag{13}$$

where $u \in \mathbb{R}$, and

$$\begin{aligned}
&\psi(n,x,y) = f(n,x,y) - P(n)x - K(n)y, \quad n \in \bigcup_{k=0}^{p} I_k,\\
&\upsilon(k,x) = F(k,x) - T(k)x, \ k \in \mathbb{Z}[1,p],\\
&\xi(n,x,y) = g(n,x,y) - M(n)x - L(n)y, \quad n \in \bigcup_{k=1}^{p} J_k.
\end{aligned} \tag{14}$$

According to Lemma 1 the PBVP for linear NIDE (13) has an unique solution given by (3) with $\sigma_n = \frac{\psi(n,\eta(n),\eta(n+1))}{1-K(n)}$, $Q_n = \frac{P(n)}{1-K(n)}$, $\gamma_n = \xi(n,\eta(n),\eta(n_j))$, $\mu_k = \upsilon(k,\eta(n_k-1))$, $T_k = T(k)$, $M_n = M(n)$, $L_n = L(n)$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 11, No. 1 (2017)

For any function $\eta \in S(\alpha, \beta)$ we define the operator $Q : S(\alpha, \beta) \to S(\alpha, \beta)$ by $Q\eta = u$, where u is the unique solution of PBVP for the linear NIDE (13) for the function η. The operator Q has the following properties:

(P1) $\alpha \leq Q\alpha$, $\beta \geq Q\beta$;

(P2) Q is a monotone nondecreasing operator in $S(\alpha, \beta)$.

To prove (P1) set $Q\alpha = \alpha^{(1)}$, where $\alpha^{(1)}$ is the unique solution of (13) with $\eta = \alpha$ and let $m(n) = \alpha(n) - \alpha^{(1)}(n),\ n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in \bigcup_{k=0}^{p} I_k$ we obtain the inequality

$$\begin{aligned} m(n+1) &\leq P(n)(\alpha(n) - \alpha^{(1)}(n)) + K(n)(\alpha(n+1) - \alpha^{(1)}(n+1)) \\ &= P(n)m(n) + K(n)m(n+1). \end{aligned}$$

Hence the inequality $m(n+1) \leq \frac{P(n)}{1-K(n)}\, m(n)$ holds for $n \in \bigcup_{k=0}^{p} I_k$.

For any $n = n_k$ we obtain

$$\begin{aligned} m(n_k) \leq\ & F(k, \alpha(n_k - 1)) - T(k)\alpha^{(1)}(n_k - 1) - F(k, \alpha(n_k - 1)) \\ & + T(k)\alpha(n_k - 1) = T(k)m(n_k - 1). \end{aligned}$$

For any $n \in \bigcup_{k=1}^{p} J_k$ we get

$$\begin{aligned} m(n) &\leq g(n, \alpha(n), \alpha(n_k)) - M(n)\alpha^{(1)}(n) - L(n)\alpha^{(1)}(n_k) \\ &\quad - g(n, \alpha(n), \alpha(n_k)) + M(n)\alpha(n) + L(n)\alpha(n_k) \\ &= M(n)m(n) + L(n)m(n_k) \end{aligned}$$

The inequality $m(n_0) = \alpha(n_0) - \alpha^{(1)}(n_0) \leq \alpha(n_{p+1}) - \alpha^{(1)}(n_{p+1}) = m(n_{p+1})$ holds.

Therefore, the function $m(n)$ satisfies the inequalities (10) with $Q_n = \frac{P(n)}{1-K(n)}$, $T_k = T(k), M_n = M(n), L_n = L(n)$. According to Lemma 2 the function $m(n)$ is non-positive in $\mathbb{Z}[n_0, n_{p+1}]$, i.e. $\alpha \leq Q\alpha$. Analogously it can be proved that the inequality $\beta \geq Q\beta$ holds for $\eta = \beta$.

To prove (P2) we consider two arbitrary function $\eta_1,\ \eta_2 \in S(\alpha, \beta)$ such that $\eta_1(n) \leq \eta_2(n)$ for $n \in \mathbb{Z}[n_0, n_{p+1}]$. Let $u^{(1)} = Q\eta_1$ and $u^{(2)} = Q\eta_2$. Denote $m(n) = u^{(1)}(n) - u^{(2)}(n)$,
$n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in \bigcup_{k=0}^{p} I_k$ we obtain the inequality

$$m(n+1) \leq P(n)m(n) + K(n)m(n+1).$$

Hence the inequality $m(n+1) \leq \frac{P(n)}{1-K(n)}\, m(n)$ holds for $n \in I_k,\ k \in \mathbb{Z}[0, p]$.

For any $n = n_k, k \in \mathbb{Z}[1, p]$ we get

$$\begin{aligned} m(n_k) =\ & T(k, u^{(1)}(n_k - 1) - u^{(2)}(n_k - 1)) - T(k)(\eta_1(n_k - 1) - \eta_2(n_k - 1)) \\ & + F(k, \eta_1(n_k - 1)) - F(k, \eta_2(n_k - 1)) \leq T(k)m(n_k - 1). \end{aligned}$$

For any $n \in \bigcup_{k=1}^{p} J_k$ we obtain

$$\begin{aligned} m(n) &= M(n)u^{(1)}(n) + L(n)u^{(1)}(n_k) + g(n, \eta_1(n), \eta_1(n_k)) \\ &\quad - M(n)\eta_1(n) - L(n)\eta_1(n_k) - M(n)u^{(2)}(n) - L(n)u^{(2)}(n_k) \\ &\quad - g(n, \eta_2(n), \eta_2(n_k)) + M(n)\eta_2(n) + L(n)\eta_2(n_k) \\ &\le M(n)m(n) + L(n)m(n_k). \end{aligned}$$

The inequality $m(n_0) = u^{(1)}(n_0) - u^{(2)}(n_0) = u^{(1)}(n_{p+1}) - u^{(2)}(n_{p+1}) = m(n_{p+1})$ holds.

According to Lemma 2 with $Q_n = \frac{P(n)}{1-K(n)}$, $T_k = T(k), M_n = M(n), L_n = L(n)$ the function $m(n) \le 0$, i.e. $Q\eta_1 \le Q\eta_2$, for $\eta_1(n) \le \eta_2(n), n \in \mathbb{Z}[n_0, n_{p+1}]$.

Let $\eta \in S(\alpha, \beta)$ be a lower solution of (1). We consider the functions $Q\eta = m$. According to above $\eta(n) \le m(n),\ n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in \bigcup_{k=0}^{p} I_k$ we get the inequality

$$\begin{aligned} m(n+1) &= P(n)m(n) + K(n)m(n+1) + f(n, \eta(n), \eta(n+1)) \\ &\quad - P(n)\eta(n) - K(n)\eta(n+1) \\ &\le f(n, m(n), m(n+1)). \end{aligned} \tag{15}$$

For any $n = n_k, k \in \mathbb{Z}[1, p]$ we obtain

$$\begin{aligned} m(n_k) &= F(k, m(n_k - 1)) - F(k, m(n_k - 1)) + T(k)m(n_k - 1) \\ &\quad + F(k, \eta(n_k - 1)) - T(k)\eta(n_k - 1) \\ &\le F(k, m(n_k - 1)). \end{aligned} \tag{16}$$

For any $n \in \bigcup_{k=1}^{p} J_k$ we obtain

$$\begin{aligned} m(n) &= g(n, m(n), m(n_k)) - g(n, m(n), m(n_k)) + M(n)m(n) \\ &\quad + L(n)m(n_k) + f(n, \eta(n), \eta(n_k)) - M(n)\eta(n) - L(n)\eta(n_k) \\ &\le g(n, m(n), m(n_k)). \end{aligned} \tag{17}$$

Inequalities (15),(16),(17), and $m(n_0) = m(n_{p+1})$ prove the functions $m(n)$ is lower solution of NIDE (1). Similarly, if $\eta \in S(\alpha, \beta)$ is an upper solution of (1) then the function $m = Q\eta$ is an upper solution of (1).

We define the sequences of functions $\{\alpha^{(j)}(n)\}_0^\infty$ and $\{\beta^{(j)}(n)\}_0^\infty$ by the equalities $\alpha^{(0)} = \alpha,\ \beta^{(0)} = \beta,\ \alpha^{(j+1)} = Q\alpha^{(j)},\ \beta^{(j+1)} = Q\beta^{(j)}$. The functions $\alpha^{(s+1)}(n)$ and $\beta^{(s+1)}(n)$ satisfy the periodic boundary value problem (13) with $\eta(n) = \alpha^{(s)}(n)$ and $\eta(n) = \beta^{(s)}(n),\ n \in \mathbb{Z}[n_0, n_{p+1}]$, respectively.

According to Lemma 1 the following representations are valid:

$$\begin{aligned}\alpha^{(s+1)}(n) = N(n) \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n} R(i)\Big)\frac{\psi(j,\alpha^{(s)}(j),\alpha^{(s)}(j+1))}{1-K(j)} \prod_{i=j+1}^{n-1} \frac{P(i)}{1-K(i)} \\ + \sum_{j=n_0}^{n} \Big(\prod_{i=j+1}^{n} R(i)\Big)\zeta(j) \prod_{i=j}^{n-1} \frac{P(i)}{1-K(i)} + \tau(n), \text{ for } n \in \mathbb{Z}[n_0, n_{p+1}],\end{aligned} \tag{18}$$

where $\tau(n)$ is given by (5) for $\gamma_n = \xi(n, \alpha^{(s)}(n), \alpha^{(s)}(n_j)),\ n \in \bigcup_{k=1}^{p} J_k,\ j \in \mathbb{Z}[1,p]$ and $\zeta(n)$ is given by (7) for $\gamma_{n_k+d_k} = \xi(n_k + d_k, \alpha^{(s)}(n_k + d_k), \alpha^{(s)}(n_k)),\ \mu_k = \upsilon(k, \alpha^{(s)}(n_k - 1)),\ k \in \mathbb{Z}[1,p]$

and

$$\begin{aligned}\beta^{(s+1)}(n) = N(n) \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n} R(i)\Big)\frac{\psi(j,\beta^{(s)}(j),\beta^{(s)}(j+1))}{1-K(j)} \prod_{i=j+1}^{n-1} \frac{P(i)}{1-K(i)} \\ + \sum_{j=n_0}^{n} \Big(\prod_{i=j+1}^{n} R(i)\Big)\zeta(j) \prod_{i=j}^{n-1} \frac{P(i)}{1-K(i)} + \tau(n), \text{ for } n \in \mathbb{Z}[n_0, n_{p+1}],\end{aligned} \tag{19}$$

where $\tau(n)$ is given by (5) for $\gamma_n = \xi(n, \beta^{(s)}(n), \beta^{(s)}(n_j)),\ j \in \mathbb{Z}[1,p]$ and $\zeta(n)$ is given by (7) for $\gamma_{n_k+d_k} = \xi(n_k + d_k, \beta^{(s)}(n_k + d_k), \beta^{(s)}(n_k)),\ \mu_k = \upsilon(k, \beta^{(s)}(n_k - 1)),\ k \in \mathbb{Z}[1,p]$.

According to the above proved, functions $\alpha^{(j)}(n)$ and $\beta^{(j)}(n)$ are lower and upper solutions of NIDE (1), respectively and they satisfy for $n \in \mathbb{Z}[n_0, n_{p+1}]$ the following inequalities

$$\alpha^{(0)}(n) \le \alpha^{(1)}(n) \le \ldots \le \alpha^{(s)}(n) \le \beta^{(s)}(n) \le \ldots \le \beta^{(1)}(n) \le \beta^{(0)}(n) \tag{20}$$

Both sequences of functions being monotonic and bounded are convergent on $\mathbb{Z}[n_0, n_{p+1}]$.

Let $A(n) = \lim_{s\to\infty} \alpha^{(s)}(n), B(n) = \lim_{s\to\infty} \beta^{(s)}(n)$.

Take a limit in (18) for $s \to \infty$ we obtain

$$\begin{aligned}A(n) = N(n) \sum_{j=n_0-1}^{n-1} \Big(\prod_{i=j+1}^{n} R(i)\Big)\frac{\psi(j,A(j),A(j+1))}{1-K(j)} \prod_{i=j+1}^{n-1} \frac{P(i)}{1-K(i)} \\ + \sum_{j=n_0}^{n} \Big(\prod_{i=j+1}^{n} R(i)\Big)\zeta(j) \prod_{i=j}^{n-1} \frac{P(i)}{1-K(i)} + \tau(n), \text{ for } n \in \mathbb{Z}[n_0, n_{p+1}],\end{aligned} \tag{21}$$

where $\tau(n)$ is given by (5) for $\gamma_n = \xi(n, A(n), A(n_j)),\ j \in \mathbb{Z}[1,p]$ and $\zeta(n)$ is given by (7) for $\gamma_{n_k+d_k} = \xi(n_k + d_k, A(n_k + d_k), A(n_k)),\ \mu_k = \upsilon(k, A(n_k - 1)),\ k \in \mathbb{Z}[1,p]$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 11, No. 1 (2017)

From (21) it follows the function $A(n)$ is a solution of NIDE (1), $n \in \mathbb{Z}[n_0, n_{p+1}]$.

Similarly, we prove the function $B(n)$ is a solution of NIDE (1), $n \in \mathbb{Z}[n_0, n_{p+1}]$.

We will prove that the functions $A(n)$ and $B(n)$ are minimal and maximal solutions of PBVP for NIDE (1) in $S(\alpha, \beta)$.

Let $u \in S(\alpha, \beta)$ be a solution of PBVP for NIDE (1). From inequalities (20) it follows there exists a natural number p such that $p \in \mathbb{N}$:

$$\alpha^{(p)}(n) \leq u(n) \leq \beta^{(p)}(n) \quad \text{for} \quad n \in \mathbb{Z}[n_0, n_{p+1}].$$

We introduce the notation $m(n) = \alpha^{(p+1)}(n) - u(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$.

For any $n \in \bigcup_{k=0}^{p} I_k$ we get the inequality

$$\begin{aligned} m(n+1) &= P(n)\alpha^{(p+1)}(n) + K(n)\alpha^{(p+1)}(n+1) + f(n, \alpha^{(p)}(n), \alpha^{(p)}(n+1)) \\ &\quad - P(n)\alpha^{(p)}(n) - K(n)\alpha^{(p)}(n+1) - f(n, u(n), u(n+1)) \\ &\leq P(n)m(n) + K(n)m(n+1). \end{aligned}$$

Hence the inequality $m(n+1) \leq \frac{P(n)}{1-K(n)}\, m(n)$ holds for $n \in \bigcup_{k=0}^{p} I_k$.

For any $n = n_k, \quad k \in \mathbb{Z}[1, p]$ we obtain

$$\begin{aligned} m(n_k) &= T(k)\alpha^{(p+1)}(n_k - 1) + T(k)u(n_k - 1) - T(k)u(n_k - 1) \\ &\quad + F(k, (\alpha^{(p)}(n_k - 1)) - T(k)\alpha^{(p)}(n_k - 1) - F(k, u(n_k - 1)) \\ &\leq T(k)m(n_k - 1). \end{aligned}$$

For any $n \in \bigcup_{k=1}^{p} J_k$ we obtain

$$\begin{aligned} m(n) &= M(n)\alpha^{(p+1)}(n) + L(n)\alpha^{(p+1)}(n_k) + g(n, \alpha^{(p)}(n), \alpha^{(p)}(n_k)) \\ &\quad - M(n)\alpha^{(p)}(n) - K(n)\alpha^{(p)}(n_k) - g(n, u(n), u(n_k)) \\ &\leq M(n)m(n) + L(n)m(n_k). \end{aligned}$$

The inequality

$$m(n_0) = \alpha^{(p+1)}(n_0) - u(n_0) = \alpha^{(p+1)}(n_{p+1}) - u(n_{p+1}) = m(n_{p+1})$$

holds.

According to Lemma 2 with $Q_n = \frac{P(n)}{1-K(n)}$, $T_k = T(k), M_n = M(n), L_n = L(n)$ the function $m(n)$ is nonpositive, i.e. $\alpha^{(p+1)}(n) \leq u(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$. Similarly $\beta^{(p+1)}(n) \geq u(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$, and hence $\alpha^{(j+1)} \leq u(n) \leq \beta^{(j+1)}, \quad n \in \mathbb{Z}[n_0, n_{p+1}]$. Since $\alpha^{(0)}(n) \leq u(n) \leq \beta^{(0)}(n)$ this proves by induction that $\alpha^{(j)}(n) \leq u(n) \leq \beta^{(j)}(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$,for every j.

Taking the limit as $j \to \infty$ we conclude $A(n) \leq u(n) \leq B(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$. Hence $A(n)$ and $B(n)$ are minimal and maximal solutions of PBVP for NIDE (1), respectively.

Let the PBVP for NIDE (1) has an unique solution $u(n) \in S(\alpha, \beta)$.
Then from above it follows $A(n) \equiv u(n) \equiv B(n), \quad n \in \mathbb{Z}[n_0, n_{p+1}]$.

□

Remark 1. The above suggested algorithm for approximate solving of nonlinear difference equations is appropriate for computerizing and easy application to study discrete models.

References

[1] R.P. Agarwal,*Difference Equations and Inequalities*, Singapore, National University of Singapore, 2000.

[2] R.P. Agarwal, S. Hristova, A. Golev, K. Stefanova, Monotone-iterative method for mixed boundary value problems for generalized difference equations with maxima, *J. Appl. Math. Comput.*, **43**, No. 1 (2013), 213-233.

[3] S. Elaydi, *An Introduction to Difference Equations*, San Antonio, Dept. Math., Trinity University, 2005.

[4] A. Golev, S. Hristova, Sv. Nenov, Monotone-iterative method for solving antiperiodic nonlinear boundary value problems for generalized delay difference equations with maxima, *Abstr. Appl. Anal.*, **2013** (2013), Article ID 571954, 9 pages.

[5] C.V. Pao, Monotone iterative methods for finite difference system of reaction-diffusion equations, *Numerische Math.*, **46**, No. 4 (1985), 571-586.

[6] P.Y.H. Pang, R.P. Agarwal, Monotone iterative methods for a general class of discrete boundary value problems, *Comput. Math. Appl.*, **28**, No. 13 (1994), 243254.

[7] P. Wang, Sh. Tian, Yonghong Wu, Monotone iterative method for first-order functional difference equations with nonlinear boundary value conditions, *Appl. Math. Comput.*, **203**, No. 1 (2008), 266-272.