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MEASURABLE DOMINATING FUNCTIONS

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Abstract: In this paper the concept of a measurable dominating function on the vertex set of a graph is introduced and condition for such a function to be minimal is obtained.

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1. Introduction

A *graph* [3] G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges, together with an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = \{u, v\}$, then e is said to join u and v, and the vertices u and v are called the ends of e. In this case we also denote the edge by uv. A graph G is *finite* [3] if both its vertex set $V(G)$ and edge set $E(G)$ are finite. A set of two or more edges of a graph G is called a set of *multiple edges*[1] if they have the same ends. An edge with identical ends is called a *loop*[1]. A graph is *simple*[1] if it has no loops and multiple edges.

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In this paper we consider only finite simple graphs. If uv is an edge of the graph G, then u and v are called *adjacent vertices*[4]. Two adjacent vertices are referred to as neighbors of each other. The set of neighbors of a vertex v is called the *open neighborhood* [4] of v and is denoted by $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is called the *closed neighborhood* [4] of v.

Another concept we used in this paper is that of an algebra. A distinguished collection $\mathcal{R}$ of subsets of a set X is called an *algebra* [5] if the following axioms are satisfied

(i) If $E \in \mathcal{R}$ and $F \in \mathcal{R}$, then $E \cup F \in \mathcal{R}$

(ii) If $E \in \mathcal{R}$, then $E^c \in \mathcal{R}$, where $E^c := X \setminus E$ is the complement of E in X.

An algebra $\mathcal{R}$, of subsets of a set X is called a *sigma algebra* [8] if $\bigcup_{i=1}^{\infty} E_i \in \mathcal{R}$, whenever $E_1, E_2, ... \in \mathcal{R}$

Proposition 1. [8] If $\mathcal{F}$ is any family of subsets of a set X, there exists a smallest sigma algebra containing $\mathcal{F}$, called the *sigma-algebra generated by* $\mathcal{F}$.

A set X together with a sigma algebra $\mathcal{R}$ of subsets of X is called a *measurable space*[8], and the members of $\mathcal{R}$ are called the *measurable sets*[8] in X. Let X be a measurable space and Y be a topological space. A mapping f from X into Y is said to be *measurable*[8] if $f^{-1}(V)$ is a measurable set in X for every open set V in Y. A *measure*[8] is a function μ, defined on a sigma algebra $\mathcal{R}$, whose range is in $[0, \infty]$ and which is countably additive. This means that if $\{E_i\}$ is a disjoint countable collection of members of $\mathcal{R}$ then $\mu(\bigcup_{i=1}^{\infty} E_i) = \Sigma_{i=1}^{\infty} \mu(E_i)$. A *measure space*[8] is a measurable space which has a measure defined on the sigma algebra of its measurable sets. A function s on a measure space X whose range consists of only finitely many points is called a *simple function*[8]. If $\alpha_1, \alpha_2, ..., \alpha_n$ are the distinct values of a simple function s, and if we set $A_i = \{x : s(x) = \alpha_i\}$ then $s = \Sigma_{i=1}^{n} \alpha_i \chi_{A_i}$, where χ_{A_i} is the characteristic function of A_i. It can be proved that s is measurable if and only if each of the sets A_i is measurable. Suppose $\mathcal{R}$ is a sigma algebra on the set X and μ is a measure on $\mathcal{R}$. If $s = \Sigma_{i=1}^{n} \alpha_i \chi_{A_i}$ is a measurable simple function from X into $[0, \infty)$, where $\alpha_1, \alpha_2, ..., \alpha_n$ are the distinct values assumed by s and if $E \in \mathcal{R}$, then $\int_E s d\mu$ is defined by $\Sigma_{i=1}^{n} \alpha_i \mu(A_i \cap E)$[8].

2. Measurable Dominating Functions

The mathematical study of dominating sets in graphs began around 1960. The concept of domination was first studied by O. Ore and C. Berge. C. Berge in his book "Theory of Graphs and its Applications"[2], has introduced the concept of

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dominating sets and he called it as the "externally stable sets ". O. Ore in his book "Theory of Graphs "[7], used first time, the names "dominating set "and "domination number ".

Let $G = (V(G), E(G))$ be a graph. A function $f : V(G) \to \{0, 1\}$ is called a *dominating function*[6] of G if $\sum_{u \in N[v]} f(u) \geq 1$ for all $v \in V(G)$. As we know every nonempty set X can be made into a measure space by taking the power set $\mathcal{P}(X)$ of X as the sigma algebra and the counting measure as the measure, the vertex set $V(G)$ of G can also be made into a measure space by taking $\mathcal{P}(V(G))$ as the sigma algebra and the counting measure μ, as the measure. Also as V(G) is finite, every function $f : V(G) \to \{0, 1\}$ is simple. With these notions we can redefine the dominating function as the function $f : V(G) \to \{0, 1\}$ such that $\int_{N[v]} f d\mu \geq 1$ for all $v \in V(G)$.

Now at this stage we can think about the extension of this notion to an arbitrary measure space, $(V(G), \mathcal{R}, \mu)$. There arises two questions. The first question is that 'for all $v \in V(G)$, $N[v] \in \mathcal{R}$ or not 'and the second is that 'though $\mathcal{R}$ contains all $N[v]$, is $\int_{N[v]} f d\mu$ meaningful'. From the theory of measures, the integral of a real valued function is defined only if the function is measurable. Taking all these into consideration, we consider only those sigma algebra $\mathcal{R}$ which contains all $N[v]$, $v \in V(G)$ and only those functions defined on $V(G)$ which are measurable. To make this theory more effective we take the sigma algebra which is generated by the collection $\{N[v] : v \in V\}$ and functions $f : V(G) \to [0, 1]$ which are measurable with respect to this sigma algebra. We build up our theory with this sigma algebra. Such a sigma algebra exists by the proposition[1].

Definition 2. Let $G = (V(G), E(G))$ be a graph. The sigma algebra generated by $\{N[v] : v \in V(G)\}$ on V(G) is called the neighborhood sigma algebra of G and it is denoted by $\mathcal{A}$.

Since we are considering only finite graphs, the vertex set $V(G)$ is finite. Therefore any countable union of subsets of $V(G)$ is just finite unions. So $\mathcal{A}$ is just an algebra.

Throughout this paper by a graph G, we mean a graph with the neighborhood algebra $\mathcal{A}$ on the vertex set $V(G)$ and a measure μ on $\mathcal{A}$.

We define a measurable dominating function of a graph G as follows.

Definition 3. Let G be a graph with vertex set $V(G)$. A function $f : V(G) \to [0, 1]$ is called a measurable dominating function of G if,

(i) f is measurable

(ii) $\int_{N[v]} f d\mu \geq 1$ for all $v \in V(G)$.

Remark 4. Let f be a measurable dominating function of a graph G. Then for all $v \in V(G)$, $f(N[v]) > 0$ and $\mu(N[v]) > 0$, where $f(N[v]) = \sum_{u \in N[v]} f(u)$.

Theorem 5. *Let f and g be two measurable dominating functions of a graph G. Then all convex linear combinations of f and g are measurable dominating functions of G.*

Proof. Let $0 \leq \alpha \leq 1$ and let $h = \alpha f + (1-\alpha)g$. Since f and g are measurable functions, h is also measurable. Let $V(G)$ be the vertex set of G. Then for $v \in V(G)$

$$\begin{aligned}
\int_{N[v]} h d\mu &= \int_{N[v]} [\alpha f + (1-\alpha)g] d\mu \\
&= \int_{N[v]} \alpha f d\mu + \int_{N[v]} (1-\alpha) g d\mu \\
&= \alpha \int_{N[v]} f d\mu + (1-\alpha) \int_{N[v]} g d\mu \\
&\geq \alpha + (1-\alpha) \\
&= 1
\end{aligned}$$

Therefore, h is a measurable dominating function of G. Hence the theorem. □

Notation 6. Let G be a graph with vertex set $V(G)$. For $v \in V(G)$, E_v denotes the smallest measurable set containing v. That is the intersection of all measurable sets containing v.

Example 1. *For the graph G_1, given in Figure 1: the neighborhood algebra $\mathcal{A}$ is given by*
$\{\phi, \{u, v, w, x\}, \{u, v, x\}, \{v, w, x\}, \{u\}, \{w\}, \{v, x\}, \{u, w\}\}$*and*
$E_u = \{u\}, E_v = \{v, x\}, E_w = \{w\}$ *and* $E_x = \{v, x\}$.

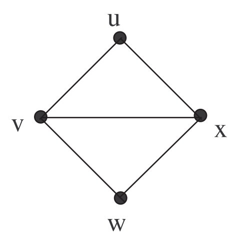

Figure 1: G_1

Definition 7. Let G be a graph with vertex set $V(G)$. Suppose μ is a measure on $\mathcal{A}$. A measurable dominating function f of G is said to be minimal if there does not exist a measurable dominating function $g \neq f$ of G such that $g(v) \leq f(v)$ for all $v \in V(G)$.

Proposition 8. Let G be a graph with vertex set $V(G)$ and let f be a measurable real valued function defined on $V(G)$. Then for each $v \in V(G)$, f is constant on E_v.

Proof. Let $v \in V(G)$ and $f(v) = c$. Suppose $f(u) = d$ for some $u \in E_v$. Let, if possible, $d \neq c$, suppose that $c < d$. Then $f^{-1}(-\infty, d)$ is measurable and $v \in f^{-1}(-\infty, d)$. Therefore $v \in f^{-1}(-\infty, d) \cap E_v$ and $f^{-1}(-\infty, d) \cap E_v$ is a measurable set properly contained in E_v, which contradicts the fact that E_v is the smallest measurable set containing v. A similar kind of contradiction arises when $d < c$. Hence the proposition. □

Proposition 9. Let G be a graph with vertex set $V(G)$ and $u, v \in V(G)$. Then $u \in E_v$ if and only if $v \in E_u$.

Proof. Let $u \in E_v$. If $v \notin E_u$, $E_v \setminus E_u$ is a measurable set containing v and properly contained in E_v, which contradicts the fact that E_v is the smallest measurable set containing v. Hence $v \in E_u$. Also by interchanging the roles of u and v we get $u \in E_v$ whenever $v \in E_u$. □

Lemma 10. *Let G be a graph with vertex set $V(G)$ and $u, v \in V(G)$. Suppose $u \in E_v$. Then $E_u = E_v$.*

Proof. By Proposition 9, $v \in E_u$ because $u \in E_v$. The sets E_u and E_v, being the smallest measurable sets containing u and v respectively, $u \in E_v$ and $v \in E_u$ imply that $E_u \subset E_v$ and $E_v \subset E_u$. Hence $E_u = E_v$. □

Lemma 11. *Let G be a graph with vertex set $V(G)$ and $u, v \in V(G)$. If $E_u \cap E_v \neq \emptyset$, then $E_u = E_v$.*

Proof. Suppose that $E_u \cap E_v \neq \emptyset$. Let $w \in E_u \cap E_v$. Then by Lemma 10, $E_w = E_u = E_v$. □

From Lemma 11, it follows that:

Theorem 12. *Let G be a graph with vertex set $V(G)$. Then $\{E_u : u \in V(G)\}$ forms a partition of $V(G)$.*

Theorem 13. *Let G be a graph with vertex set $V(G)$ and μ be a measure on $\mathcal{A}$. A measurable dominating function f of G is minimal if and only if for every vertex $v \in V(G)$ with $f > 0$ on E_v, there exists a vertex $u \in V(G)$ with $\mu(N[u] \cap E_v) > 0$ such that $\int_{N[u]} f d\mu = 1$.*

Proof. Let f be a minimal measurable dominating function of G. Suppose there exists a vertex v with $f > 0$ on E_v and
$\int_{N[u]} f d\mu > 1$ for all $u \in V(G)$ with $\mu(N[u] \cap E_v) > 0$.

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Let $m = \min \{ \int_{N[u] \setminus E_v} f d\mu : \mu(N[u] \cap E_v) > 0, u \in V(G)\}$

We consider the cases $m \geq 1$ and $m < 1$ separately.

Case 1. $m \geq 1$

Let $g = f - f\chi_{E_v}$, where χ_{E_v} denotes the characteristic function of E_v. That is for $w \in V(G)$,

$$g(w) = \begin{cases} 0 & \text{if } w \in E_v \\ f(w) & \text{if } w \notin E_v \end{cases}$$

Since the product and difference of measurable functions are measurable [8], the function g is measurable. Also $g \neq f$ and $g(w) \leq f(w)$ for every $w \in V(G)$.
For $u \in V(G)$ with $\mu(N[u] \cap E_v) > 0$,

$$\begin{aligned} \int_{N[u]} g d\mu &= \int_{N[u] \cap E_v} g d\mu + \int_{N[u] \setminus E_v} g d\mu \\ &= \int_{N[u] \setminus E_v} g d\mu \\ &= \int_{N[u] \setminus E_v} f d\mu \\ &\geq m \\ &\geq 1. \end{aligned}$$

Also, for $u \in V(G)$ with $\mu(N[u] \cap E_v) = 0$,

$$\begin{aligned} \int_{N[u]} g d\mu &= \int_{N[u]} f d\mu \\ &\geq 1 \end{aligned}$$

Therefore, g is also a measurable dominating function, a contradiction.

Case 2. $m < 1$

For $u \in V(G)$ with $\mu(N[u] \cap E_v) > 0$, $\int_{N[u]} f d\mu > 1$ by the assumption.

Suppose $f = c$ on E_v. Then,

$$\begin{aligned} \int_{N[u]} f d\mu &= \int_{N[u] \cap E_v} f d\mu + \int_{N[u] \setminus E_v} f d\mu \\ &= c\mu(N[u] \cap E_v) + \int_{N[u] \setminus E_v} f d\mu \\ &> 1 \end{aligned}$$

Since $m < 1$, for atleast one vertex u with $\mu(N[u] \cap E_v) > 0$, $\int_{N[u] \setminus E_v} f d\mu < 1$.

For such a u,

$$\begin{aligned} c\mu(N[u] \cap E_v) &> 1 - \int_{N[u] \setminus E_v} f d\mu \\ &> 0 \end{aligned}$$

This implies

$$c > \frac{1 - \int\limits_{N[u]\setminus E_v} f d\mu}{\mu(N[u] \cap E_v)} = R_u, \text{ say}$$

Let $U = \{u \in V(G) : \mu(N(u) \cap E_v) > 0 \; and \int\limits_{N[u]\setminus E_v} f d\mu \; < 1\}$. Since $V(G)$ is finite, U is also finite. Now choose d so that $c > d > R_u$ for all $u \in U$.
Let $h = f - (f - d)\chi_{E_v}$.
That is for $w \in V(G)$,

$$h(w) = \begin{cases} d & \text{if } w \in E_v \\ f(w) & \text{if } w \notin E_v \end{cases}$$

The function h is measurable, since it is the difference of the measurable functions f and $(f - d)\chi_{E_v}$. Also $g \neq f$ and $g(w) \leq f(w)$ for every $w \in V(G)$.
Let $u \in U$,

$$\begin{aligned} \int\limits_{N[u]} h d\mu &= \int\limits_{N[u]\setminus E_v} h d\mu + \int\limits_{N[u]\cap E_v} h d\mu \\ &= \int\limits_{N[u]\setminus E_v} f d\mu + d\mu(N[u] \cap E_v) \\ &> \int\limits_{N[u]\setminus E_v} f d\mu + R_u\mu(N[u] \cap E_v) \\ &= \int\limits_{N[u]\setminus E_v} f d\mu \; + (1 - \int\limits_{N[u]\setminus E_v} f d\mu) \\ &= 1 \end{aligned}$$

Let $u \notin U$.
If $\mu(N[u] \cap E_v) = 0$,

$$\begin{aligned} \int\limits_{N[u]} h d\mu &= \int\limits_{N[u]} f d\mu \\ &\geq 1 \end{aligned}$$

If $\mu(N[u] \cap E_v) > 0$,

$$\int_{N[u]\setminus E_v} f d\mu \geq 1$$

Therefore,

$$\begin{aligned} \int\limits_{N[u]} h d\mu &= \int\limits_{N[u]\setminus E_v} h d\mu + \int\limits_{N[u]\cap E_v} h d\mu \\ &= \int\limits_{N[u]\setminus E_v} f d\mu + d\mu(N[u] \cap E_v) \\ &> 1. \end{aligned}$$

Therefore, h is a measurable dominating function with $h(w) \leq f(w)$ for every $w \in V(G)$ and $h \neq f$, a contradiction.

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Conversely, let f be a measurable dominating function of G such that for every vertex v with $f > 0$ on E_v, there exists a vertex u with $\mu(N[u] \cap E_v) > 0$ such that $\int_{N[u]} f d\mu = 1$. Suppose f is not minimal. Then there exists a measurable dominating function l such that $l(w) \leq f(w)$ for every $w \in V(G)$ and $l \neq f$. So there exists a $v \in V(G)$ such that $l(v) < f(v)$. This implies $f(v) > 0$. Since f is constant on E_v, f is also greater than zero on E_v. Now by assumption, there exists a $u \in V(G)$ with $\mu(N[u] \cap E_v) > 0$ and $\int_{N[u]} f d\mu = 1$.
Therefore,

$$\begin{aligned} 1 &\leq \int\limits_{N[u]} l d\mu \\ &= \int\limits_{N[u]\setminus E_v} l d\mu + \int\limits_{N[u]\cap E_v} l d\mu \\ &< \int\limits_{N[u]\setminus E_v} f d\mu + \int\limits_{N[u]\cap E_v} f d\mu \\ &= 1, \text{a contradiction.} \end{aligned}$$

Therefore, f is a minimal measurable dominating function. Hence the theorem. □

Corollary 14. *Let G be a graph with vertex set $V(G)$ and f be a minimal measurable dominating function of G. Then $f = 0$ on E_v, for every vertex v with $\mu(E_v) = 0$.*

Proposition 15. Let G be a connected graph with vertex set $V(G)$. Then for any end vertex and for any support vertex v, $E_v = \{v\}$, provided $|V(G)| \neq 2$.

Proof. If $|V(G)| = 1$, then the result is trivially true. So assume that $|V(G)| > 2$. Let v be an end vertex of G with support vertex u. Since G is a connected graph of order greater than two, there exists a vertex $w \in N[u] \setminus \{v\}$. Therefore, $N[v] \cap N[w] = \{u\}$ is measurable. Hence $E_u = \{u\}$. Since $\{u\}$ is measurable, $N[v] \setminus \{u\}$ is measurable. That is $\{v\}$ is measurable. Hence $E_v = \{v\}$. □

Remark 16. The converse of Proposition 15 need not be true. That is $E_v = \{v\}$ doesnot imply, v is an end vertex or a support vertex.

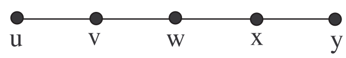

Figure 2: The path P_5

The set E_w of the vertex w of the graph P_5 in Figure 2 is $\{w\}$. But w is neither an end vertex nor a support vertex.

Proposition 17. A graph G is complete if and only if $E_v = V(G)$, for all $v \in V(G)$.

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Proof. For a graph G,

$$\begin{aligned} E_v = V(G) \text{ for all } v \in V(G) &\iff N[v] = V(G) \text{ for all } v \in V(G) \\ &\iff G \text{ is complete.} \end{aligned}$$

□

For the graph G_1, given in Figure 1, $E_v = E_x$. Note also that these vertices v and x have the same closed neighborhoods, that is $N[v] = N[x]$. This result in fact has a general feature.

That is for any two vertices v_1 and v_2 of a graph G, $E_{v_1} = E_{v_2}$ if and only if $N[v_1] = N[v_2]$. The proof of this result mainly depends on the neighborhood sigma algebra $\mathcal{A}$. Theorem 18 characterizes the neighborhood sigma algebra.

Theorem 18. *Let G be a graph with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and $\mathcal{A}$, the neighborhood sigma algebra of G. Then a subset A of $V(G)$ belongs to $\mathcal{A}$ if and only if A is a union of subsets of $V(G)$ of the form $\bigcap_{i=1}^{n} A_i$, where $A_i = N[v_i]$ or $N[v_i]^c$.*

Proof. Let $\mathcal{F} = \left\{ \bigcap_{i=1}^{n} A_i : A_i = N[v_i] \text{ or } N[v_i]^c, 1 \leq i \leq n \right\}$. Note that the collection $\mathcal{F}$ forms a partition of $V(G)$. Let $\mathcal{G}$ be the collection of all possible unions of elements of $\mathcal{F}$. Then $\mathcal{G}$ is contained in $\mathcal{A}$ and $\mathcal{G}$ itself is a sigma algebra. Also for any $v_k \in V(G)$, $N[v_k] = \bigcup \left\{ \bigcap_{i=1}^{n} A_i \in \mathcal{F} : A_k = N[v_k] \right\} \in \mathcal{G}$. Therefore, $\mathcal{G} = \mathcal{A}$. Hence the theorem. □

Theorem 19. *Let G be a graph with vertex set $V(G)$. Then for $v_1, v_2 \in V(G)$, $E_{v_1} = E_{v_2}$ if and only if $N[v_1] = N[v_2]$.*

Proof. Assume that $E_{v_1} = E_{v_2}$ for some $v_1, v_2 \in V(G)$. Suppose $N[v_1] \neq N[v_2]$. Without loss of generality, assume that there exists $u \in V(G)$ such that $u \in N[v_1]$ but $u \notin N[v_2]$. Therefore, $N[u] \cap N[v_1]$ is a measurable set containing v_1 but not v_2. This implies that $v_2 \notin E_{v_1}$, since E_{v_1} is the intersection of all measurable sets containing v_1. Hence $E_{v_1} \neq E_{v_2}$, a contradiction.

Conversely, assume that $N[v_1] = N[v_2]$. This implies for any $v \in V(G)$ either $v_1, v_2 \in N[v]$ or $v_1, v_2 \in N[v]^c$. Therefore, by Theorem 18, if B is any measurable set, then either $v_1, v_2 \in B$ or $v_1, v_2 \in B^c$. This implies that $E_{v_1} = E_{v_2}$. □

Corollary 20. Let G be a graph with vertex set $V(G)$ and $v \in V(G)$. Then $E_v = \{u \in V(G) : N[u] = N[v]\}$.

Proof. Suppose $u \in E_v$. Then by Lemma 10, $E_u = E_v$. This implies $N[u] = N[v]$. Thus $E_v \subseteq \{u \in V(G) : N[u] = N[v]\}$. To prove the opposite inclusion. Let $u \in V(G)$ be such that $N[u] = N[v]$. Theorem 19 implies that $E_u = E_v$. Therefore, $u \in E_v$. Thus $\{u \in V(G) : N[u] = N[v]\} \subseteq E_v$. □

Remark 21. Let G be a graph with vertex set $V(G)$ and $v \in V(G)$. In general $E_v \neq \cap\{N[u] : u \in N[v]\}$
Consider the graph P_3 given in Figure 3.

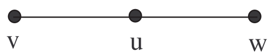

Figure 3: The path P_3

For the vertex v of P_3, $E_v = \{v\}$. But $\cap\{N[u] : u \in N[v]\} = \{v, u\}$

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