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MOORE-PENROSE RINGS AS ELEMENTAL ANNIHILATOR RINGS

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Abstract: This paper investigates the elemental annihilator ring (e.a.r) structure of Moore-Penrose One (MP1) rings and Moore-Penrose Two (MP2) rings. For clarity, R is MP1 (or Von Neumann regular) if, whenever a $\in$ R, a $\neq$ 0, then there exists x $\in$ R, x $\neq$ 0 such that axa = a. Likewise, R is MP2 if, whenever a $\in$ R, a $\neq$ 0, then there exists x $\in$ R, x $\neq$ 0 such that xax = x.

Key Words: annihilator, semisimple, completely primary ideal, Wedderburn, Jacobson radical

1. Definitions

1. Let R be a ring. If S $\subset$ R, the annihilator of S, denoted ann_R(S), is given by $\text{ann}_R(\text{S}) = \{\text{r} \in \text{R} | \text{ rs} = 0 \ \forall \text{ s} \in \text{S}\}$.

   Sometimes it is best to distinguish between left and right.

2. For a ring R and S $\subset$ R, the left annihilator of S, denoted ann_ℓ(S), is given by $\text{ann}_\ell(\text{S}) = \{\text{r} \in \text{R} \mid \text{rs} = 0 \ \forall \text{ s} \in \text{S} \}$.

3. For a ring R and S $\subset$ R, the right annihilator of S, denoted ann_r(S), is given by $\text{ann}_r(\text{S}) = \{\text{r} \in \text{R} \mid \text{sr} = 0 \ \forall \text{ s} \in \text{S} \}$.

4. R is a left elemental annihilator ring (l.e.a.r) if, whenever L is a left ideal of R, then there exists a$\in$R such that L = ann_ℓ(a). A right elemental ring (r.e.a.r) is defined analogously.

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2. Deductions and Historical Conceptions

Assuming R is a ring with unity and S is a subset of R, then $\mathrm{ann}_\ell(\mathrm{S}) = \mathrm{ann}_\ell(\mathrm{ann}_r(\mathrm{S}))$ and $\mathrm{ann}_r(\mathrm{S}) = \mathrm{ann}_r(\mathrm{ann}_\ell(\mathrm{S}))$. It is well known that if R is semisimple Artinian, then R is a direct sum of matrix ring over division rings (Wedderburn Theorem) [2]. In such a ring, if L is a left ideal of R, then L = Re where is is an idempotent. Hence, L = ann_ℓ(1-e). The algebraist Cleon R. Yohe has proven the following pertinent historical algebraic assertion:

"Let R be a commutative ring with unit element. Then R is an elemental annihilator ring if and only if R is a direct sum of completely primary principal ideal rings.", see [3].

3. Examples of Annihilator Rings

1. Z_4 is a completely primary principal ideal ring. Z_4 has maximal (prime) ideal P = (2) with P^2 = O. Z_4 is an e.a.r. whose only ideals are O and P = $(\overline{2})$. Both ideals are the annihilators of $\overline{2}$.
2. Consider the matrix ring R = $\begin{bmatrix} Z_4 & 0 \\ 0 & Z_9 \end{bmatrix}$. Note that Z has maximal ideal $(\overline{3})$. The only ideals of R are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} Z_4 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & Z_9 \end{bmatrix}$, $\begin{bmatrix} (\overline{2}) & 0 \\ 0 & Z_9 \end{bmatrix}$, $\begin{bmatrix} Z_4 & 0 \\ 0 & (\overline{3}) \end{bmatrix}$, $\begin{bmatrix} (\overline{2}) & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & (\overline{3}) \end{bmatrix}$, $\begin{bmatrix} (\overline{2}) & 0 \\ 0 & (\overline{3}) \end{bmatrix}$, and $\begin{bmatrix} Z_4 & 0 \\ 0 & Z_9 \end{bmatrix}$ which annihilate the elements $\begin{bmatrix} \overline{2} & 0 \\ 0 & \overline{3} \end{bmatrix}$, $\begin{bmatrix} \overline{2} & 0 \\ 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 0 \\ 0 & \overline{3} \end{bmatrix}$.

R is indeed another elemental annihilator ring.

4. Some Moore-Penrose Annihilator Structure Theorems

Theorem. *Let R be a commutative MP2 ring with identity. Then R is an elemental annihilator ring if and only if R is a direct sum of fields.*

Proof. (Necessity) Let R be an elemental annihilator ring. Then by C. R. Yohe's Theorem, R is a direct sum of completely primary principal ideal rings, that is, R $\cong \sum_{Pprime} R_P$ where R_P is a completely primary principal ideal ring with maximal (prime) ideal P. Note that P = J(R), the Jacobson radical of R. Since R is MP2, there exists a nonzero idempotent e∈J(R). Then 1-e is invertible. This is a contradiction since 1-e is a zero-divisor and hence cannot be a unit. Thus, J(R) = P = O, the zero ideal. Since O is maximal this implies that R_P is theoretically a division ring Δ_P. Thereby, R $\cong \sum_{Pprime} \Delta_P$ a direct sum of division rings. But R is commutative which means that each Δ_P is really a field say F_P. In succinct notation then R $\cong \sum_{Pprime} F_P$. Therefore R is a direct sum of fields. For sufficiency, assuming R is a direct sum of the fields then the ideal L represented as $(F_{P_1}, F_{P_2}, \ldots, F_{P_{t-1}}, 0, \ldots, 0)$

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= ann_P(a) for a = (0,. . .,0,f_t,f_{t+1},. . .,f_k) $\in$ R where the integer k $< \infty$. Hence, R is an elemental annihilator ring. □

Note: Since R $\cong \sum_{Pprime} F_P$, a direct sum of fields, then R is obviously MP2. This viewpoint renders the following:

Corollary. *A commutative elemental annihilator MP2 ring is noetherian.*

Theorem. *Let R be a commutative MP1 ring with identity. Then R is an elemental annihilator ring if and only if R is a direct sum of fields.*

Proof. Since R is MP1, then R is also MP2. Hence the results follows from the theorem above. □

The obvious conceptual observation is:

Corollary. *A commutative elemental annihilator MP1 ring with identity is noetherian.*

Theorem. *A direct sum of commutative elemental annihilator rings with identity is an elemental annihilator ring.*

Proof. Let S and T be commutative elemental annihilator rings with identity. Consider the direct sum R = S$\oplus$T. Let L be an ideal of R, that is L $\lhd$ R. Then L = J$\oplus$K where J $\lhd$ S and K $\lhd$ T. Since S is an elemental annihilator ring, $\exists$ a $\in$ S such that J = ann_S(a); similarly, since T is also an elemental annihilator ring, $\exists$ b $\in$ T such that K = ann_T(b); it follows that L = ann_R(a$\oplus$b); hence R is an elemental annihilator ring and a direct sum of commutative elemental annihilator rings with identity is an elemental annihilator ring. □

Theorem. *The following are equivalent for a commutative MP2 ring with identity:*

(a) R is noetherian.

(b) R is a direct sum of fields, i.e. R $\cong \sum_\alpha F_\alpha$.

(c) R is an elemental annihilator ring.

(d) R is a direct sum of completely primary principal ideal rings.

Proof. (a) $\to$ (b) result due to previous paper by G. Battle [1]; (b) $\to$ (c) A direct sum of fields is a degenerate direct sum of completely primary principal ideal rings (hence result follows from paper by C. R. Yohe [3]); (c) $\to$ (d) also follows from [3]; (d) $\to$ (a) Under the assumption that R is commutative MP2 each completely primary principal ideal ring is a field. Hence R is noetherian. □

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