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A NEW FORMULA FOR THE VOLUME OF A SIMPLEX

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Abstract: Cayley-Menger's formula represents the volume of an n-simplex with an $(n+2)\times(n+2)$ matrix based on the distance between its vertices. A simpler formula in terms of an $n \times n$ matrix is given.

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1. Introduction

Cayley-Menger's formula [1] represents the volume of an $(m-1)$-simplex Δ as

$$vol(\Delta) = \frac{\sqrt{(-1)^m \; det \begin{bmatrix} 0 & J^t \\ J & E \end{bmatrix}}}{2^{m-1}(m-1)!} \tag{1}$$

where $E_{ij} = dist(v_i, v_j)^2$ for $1 \le i, j \le m$ and $J = [1, \ldots, 1]^t$, an m-dimensional column vector. In this paper, we give a new formula:

$$vol(\Delta) = \frac{\sqrt{det(-A^t E A/2)}}{(m-1)!} \tag{2}$$

where $A = [-J, I_{m-1}]^t$, the identity matrix I_{m-1} augmented with $-J^t$ for the first row.

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Remark. For the area of a triangle with the edge lengths a, b and c, Cayley-Menger's formula evaluates

$$det \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 \\ 1 & a^2 & 0 & c^2 \\ 1 & b^2 & c^2 & 0 \end{bmatrix}$$

and the formula (2) evaluates

$$det \begin{bmatrix} 2a^2 & a^2+b^2-c^2 \\ a^2+b^2-c^2 & 2b^2 \end{bmatrix}$$

Both formulae generalize Heron's formula $Area = \sqrt{s(s-a)(s-b)(s-c)}$ and its generalizations to tetrahedron (see [2], [3].) The formula (2) is derived by applying the multi-dimensional scaling (MDS).

2. The Proof and Examples

A brief description of the MDS: Given distances $dist(v_i, v_j)$ between vertices of a simplex Δ, MDS finds vectors $\mathbf{x}_i \in \mathbf{R}^p$ such that $\|\mathbf{x}_i - \mathbf{x}_j\| = dist(v_i, v_j)$, i.e. Δ is isometrically mapped onto the simplex generated by $\mathbf{x}_i$. Hence its volume can be computed in terms of $\mathbf{x}_i$. An important observation is that

$$\|\mathbf{x}_i - \mathbf{x}_j\|^2 = \|\mathbf{x}_i\|^2 + \|\mathbf{x}_j\|^2 - 2\mathbf{x}_i \cdot \mathbf{x}_j \tag{3}$$

implies

$$E = \Lambda J^t + J\Lambda^t - 2X^tX \tag{4}$$

where $X = [\mathbf{x}_1, \ldots, \mathbf{x}_m]$ and $\Lambda = [\|\mathbf{x}_1\|^2, \ldots, \mathbf{x}_m\|^2]^t$. Let $H = I_m - JJ^t/m$ (a centering operator.) Then the row means of HE (resp. the the column means of EH) are 0. Since we may assume $HX = X$ (by translation), it follows $HEH = -2X^tX$. The m-dimensional column vectors $\mathbf{y}_k$ of $Y = \sqrt{-HEH/2}$ satisfy the distance condition $\|\mathbf{y}_i - \mathbf{y}_j\|^2 = E_{ij}$. For details, we refer [4].

Proof of the formula (2). We may assume the vertices of Δ are the points $\mathbf{z}_i = \mathbf{y}_i - \mathbf{y}_1$ for $i = 1, \ldots, m$. Multiplication by $A = [-J, I_{m-1}]^t$ on the right of Y subtracts the first column of Y from each column of Y then removes the first column. Hence the matrix $[\mathbf{z}_2, \ldots, \mathbf{z}_m]$ can be written as YA and the $vol(\Delta)$ as

$$\frac{\sqrt{det\,((YA)^t(YA))}}{(m-1)!} = \frac{\sqrt{det\,(A^tY^tYA)}}{(m-1)!}$$

Since $Y^tY = -HEH/2$ and $HA = A$, the euqation (**??**) follows. **q.e.d.**

Example 1. If Δ is a unit regular $(m-1)$-simplex, then $E_{ij} = 1 - \delta_{ij}$ and the formula (2) gives

$$vol(\Delta) = \frac{\sqrt{m}}{\sqrt{2}^{m-1}(m-1)!} \tag{5}$$

The volume of a unit regular tetrahedron, for example, is $1/(6\sqrt{2})$. If the lengths $\|\mathbf{x}_i\|$ and the angles α_{ij} between $\mathbf{x}_i$ and $\mathbf{x}_j$ are given, instead of $dist(v_i, v_j)$, a similar argument shows the volume of Δ generated by $\{\mathbf{x}_i - \mathbf{x}_1\}_{i=1}^m$:

$$vol(\Delta) = \frac{\sqrt{det(A^tBA)}}{(m-1)!} \tag{6}$$

where B is the matrix with $B_{ij} = \|\mathbf{x}_i\|\|\mathbf{x}_j\|\cos(\alpha_{ij})$.

Example 2. If Δ is the $(m-1)$-simplex generated by the standard unit basis vectors $\{\mathbf{u}_i\}_{i=1}^m$, then $B = I_m$ and the equation(6) gives

$$vol(\Delta) = \frac{\sqrt{det(A^tA)}}{(m-1)!} = \frac{\sqrt{m}}{(m-1)!} \tag{7}$$

The area of the equilateral generated by three orthonormal vectors, for example, is $\sqrt{3}/2$.

Matlab code for the example 1:

```
m=4; % for (m-1)-simplex
E=ones(m,m)-eye(m)
J=ones(m-1,1);
A=[-J,eye(m-1)]';
vol=sqrt(det(-A'*E*A)/2^(m-1))/factorial(m-1)
```

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