International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 10, No. 2 (2016)

International Electronic Journal of Pure and Applied Mathematics

Volume 10 No. 2 2016, 161-166
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v10i2.6

AN ALTERNATIVE CONSTRUCTION OF THE KAN LOOP GROUP

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Abstract: Waldhausen described the construction of the Kan loop group using simplicial graphs. He proves that the homotopy group of the simplicial graph he constructed is the Kan loop group. Motivated by his construction, we introduce a new construction of the Kan loop group using bisimlicial sets. We construct a sequence of bisimplicial sets, one for each dimension, then prove the properties similar to Waldhausen's simplicial graphs. Then we prove that in case of dimension 1, our construction defines the Kan loop group as with Waldhausen's.

AMS Subject Classification: 55P35, 55U10
Key Words: loop group, skeleton functor, bisimplicial set

1. Introduction

A simplicial group G is called a *loop group* of a simplicial set X if there is a principal G-bundle over X with contractible total space. It has the property that $\pi_n(G) \cong \pi_{n+1}(X)$ for $n \geq 0$. In his classic paper [3], Kan constructed a loop group of a simplicial set combinatorially and proved related properties. Recently, Stevenson [5] proved Kan's theorem from a different point of view.

Another approach to Kan loop group was made previously by Waldhausen in [7]. He described the construction of the Kan loop group in terms of simplicial graphs. In order to explain his construction, let's introduce some terminology from [7]. An ordered graph is a triple of sets (P, N, E) with structure maps $N \leftarrow E \rightarrow P$, where N and P are sets of "vertices" and E is a set of "edges". A *pointed* ordered graph is an ordered graph equipped with a basepoint $x \in P$. Any set can be regarded as the set of vertices of a discrete graph, as a consequence, a simplicial set can be regarded as a simplicial (discrete) graph. Given a simplicial set X, Waldhausen constructed

Received: May 26, 2016 © 2016 Academic Publications, Ltd.

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a simplicial pointed ordered graph ΓX with the following properties:

- If X is connected, then so is $\Gamma_n X$, for each $n \geq 0$.
- If we regard X as a simplicial discrete graph, there is a natural homotopy equivalence $X \to \Gamma X$.
- The simplicial group $n \mapsto \pi_1(\Gamma_n X)$ is a *loop group* of X.

In this paper, we present yet another approach to Kan loop group in a way of generalizing Waldhausen's construction. Given a simplicial set X and an integer $d \geq 0$, we construct a *simplicial space* $\Sigma^{(d)} X$. (In this paper, a *space* is, by definition, a simplicial set, so a simplicial space is just a bisimplicial set.) We will prove the following properties.

- For each n, dim $\Sigma_n^{(d)} X \leq d$.
- If X is $(d-1)$-connected, then so is $\Sigma_n^{(d)} X$ for each $n \geq 0$.
- If we regard X as a *constant* simplicial space, there is a natural homotopy equivalence $X \to \Sigma^{(d)} X$.
- For $d = 1$, the simplicial group $n \mapsto \pi_1(\Sigma^{(1)} X)$ is a loop group of X.

An ordered graph can be regarded as a space of dimension ≤ 1 in a canonical way. So a simplicial graph is considered as a simplicial space of dimension ≤ 1. In that sense, we have generalized Waldhausen's construction. This new construction might be useful in analyzing iterated loop spaces.

We use [1] as a standard reference to the theory of simplical sets.

2. Construction

Let *SSet* be the category of simplicial sets and Δ the category of finite nonempty ordered sets with order-preserving maps. We write Δ^n for the standard n-simplex $\mathrm{Hom}_\Delta(-, [n])$. Any simplicial set X can be regarded as a (constant) simplicial space $[n] \mapsto X$ with every arrow being the identity on X.

The construction of the space with the properties in the introduction is as follows. We define an auxiliary simplicial set first. Let $\mathrm{Maps}(\Delta^n, X)$ be the simplicial set $[p] \mapsto \mathrm{Hom}_{SSet}(\Delta^n \times \Delta^p, X)$. A map $g : [p] \to [q]$ in Δ induces a map

$$(1 \times g)^* : \mathrm{Hom}_{SSet}(\Delta^n \times \Delta^q, X) \to \mathrm{Hom}_{SSet}(\Delta^n \times \Delta^p, X).$$

This is an arrow of $\mathrm{Maps}(\Delta^n, X)$. Then, consider its d-skeleton $\mathrm{sk}_d\mathrm{Maps}(\Delta^n, X)$. Now we assemble those spaces into a simplicial space $\Sigma^{(d)} X$ as follows. Its space in degree n is

$$\Sigma_n^{(d)} X = \mathrm{sk}_d\mathrm{Maps}(\Delta^n, X).$$

By the functoriality of Hom, the diagram

$$\begin{array}{ccc}
\mathrm{Maps}(\Delta^m, X)_q & \xrightarrow{(1\times g)^*} & \mathrm{Maps}(\Delta^m, X)_p \\
\Big\downarrow{\scriptstyle (f\times 1)^*} & & \Big\downarrow{\scriptstyle (f\times 1)^*} \\
\mathrm{Maps}(\Delta^n, X)_q & \xrightarrow{(1\times g)^*} & \mathrm{Maps}(\Delta^n, X)_p
\end{array}$$

commutes for all maps $f : [n] \to [m]$ and $g : [p] \to [q]$ in Δ. Apply d-skeleton functor $\mathrm{sk}_d : \mathit{SSet} \to \mathit{SSet}$ to get the following commutative diagram.

$$\begin{array}{ccc}
(\Sigma^{(d)}_m X)_q & \longrightarrow & (\Sigma^{(d)}_m X)_p \\
\Big\downarrow & & \Big\downarrow \\
(\Sigma^{(d)}_n X)_q & \longrightarrow & (\Sigma^{(d)}_n X)_p
\end{array} \qquad (1)$$

Then we see that $\Sigma^{(d)}X$ is indeed a simplicial space by defining its arrows corresponding to f to be the vertical maps in diagram (1).

3. Properties

In this section we prove the properties stated in the introduction.

Lemma. *Suppose Y is a $(d-1)$-connected CW-complex of dimension d. Then Y is homotopy equivalent to a bouquet of d-spheres $\bigvee_\alpha S^d_\alpha$.*

Proof. Let Y^i denote the i-skeleton of Y. Since Y is $(d-1)$-connected, Y^{d-1} is contractible. Since Y is d-dimensional, $Y = Y^d$. Therefore, the map $Y \to Y^d/Y^{d-1}$ is a homotopy equivalence. Y^d/Y^{d-1} is a bouquet of d-spheres. □

Theorem. *For a simplicial set X, the simplicial spaces $\Sigma^{(d)}X$ for $d \geq 0$ have the following properties.*

(i) *$\Sigma^{(0)}X = X$ regarded as a constant simplicial space.*

(ii) *$\Sigma^{(d)}X$ is contained in $\Sigma^{(d+1)}X$.*

(iii) *For each n, $\dim \Sigma^{(d)}_n X \leq d$.*

(iv) *If X is $(d-1)$-connected, then so is $\Sigma^{(d)}_n X$ for each n, and it has the homotopy type of a bouquet of d-spheres.*

(v) *The inclusion $X = \Sigma^{(0)}X \to \Sigma^{(d)}X$ is a homotopy equivalence.*

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Proof. By the definition of 0-skeleton and the Yoneda lemma[4, III.2],

$$\begin{aligned}(\Sigma_n^{(0)}X)_p &= (\mathrm{sk}_0\mathrm{Maps}(\Delta^n, X))_p \\ &= (\mathrm{Maps}(\Delta^n, X))_0 \\ &= \mathrm{Hom}_{SSet}(\Delta^n \times \Delta^0, X) \\ &= \mathrm{Hom}_{SSet}(\Delta^n, X) \\ &= X_n\end{aligned}$$

proving (i). Properties (ii) and (iii) are obvious by definition. Suppose X is $(d-1)$-connected. The simplicial set $\mathrm{Maps}(\Delta^0, X)$ is canonically identified with X by Yoneda's lemma. Note that there is a simplicial homotopy equivalence $\Delta^n \to \Delta^0$. So there is a simplicial homotopy equivalence $X = \mathrm{Maps}(\Delta^0, X) \to \mathrm{Maps}(\Delta^n, X)$. The d-skeleton of a $(d-1)$-connected simplicial set, $\mathrm{sk}_d\mathrm{Maps}(\Delta^n, X) = \Sigma_n^{(d)}X$, is $(d-1)$-connected since the homotopy groups π_i for $i < d$ of a CW-complex depend only on its d-skeleton [2, 4.12]. By the Lemma, it has the homotopy type of a bouquet of d-spheres.

Lastly, we prove (v). The inclusion $\Sigma^{(0)}X \to \Sigma^{(d)}X$ is a map of bisimplicial sets. It is well known that a map of bisimplicial sets is a weak homotopy equivalence if it is a weak homotopy equivalence degree-wise [6, 5.1]. We will show that the map is a homotopy equivalence with respect to the variable p. For notational simplicity, let Y^p and Z^p be

$$Y^p = ([n] \mapsto (\mathrm{sk}_0\mathrm{Maps}(\Delta^n, X))_p)$$

$$Z^p = ([n] \mapsto (\mathrm{sk}_d\mathrm{Maps}(\Delta^n, X))_p).$$

We define a simplicial deformation retraction

$$F : \Delta^1 \times Z^p \to Z^p$$

on n-simplices to be induced by the map

$$F : \Delta^1_n \times \mathrm{Hom}(\Delta^n \times \Delta^p, X) \to \mathrm{Hom}(\Delta^n \times \Delta^p, X)$$

that arises from the contraction of Δ^p onto its first vertex. The formula is defined as follows. Suppose that $t : [n] \to [1]$ and $f : \Delta^n \times \Delta^p \to X$ are given. The map $\mu^t : [n] \times [p] \to [n] \times [p]$ defined by $\mu^t(i, j) = (i, t(i) \cdot j)$ is an order preserving map because t is and so is the multiplication. Hence its nerve gives a simplicial map $\mu^t_* : \Delta^n \times \Delta^p \to \Delta^n \times \Delta^p$. Now define

$$F(t, f) = f \circ \mu^t_*.$$

Let's verify that F is a simplicial map (with respect to the variable n). So suppose $\alpha : [m] \to [n]$ is a map in Δ. We need to show $F(\alpha^* t, \alpha^* f) = \alpha^* F(t, f)$.

$$F(\alpha^* t, \alpha^* f) = F(t \circ \alpha, f \circ (\alpha \times 1)_*)$$

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$$
\begin{aligned}
&= f \circ (\alpha \times 1)_* \circ (\mu^{t\alpha})_* \\
&= f \circ ((\alpha \times 1) \circ (\mu^{t\alpha}))_* \\
&= f \circ ((\mu^{t}) \circ (\alpha \times 1))_* \\
&= f \circ \mu^t_* \circ (\alpha \times 1)_* \\
&= F(t, f) \circ (\alpha \times 1)_* \\
&= \alpha^* F(t, f)
\end{aligned}
$$

Now let's verify that F sends d-skeletons to d-skeletons. So suppose f is a degeneracy of a simplex of dimension $b \leq d$. This means that we have a surjective map $\beta : [p] \to [b]$ and a simplex $g : \Delta^n \times \Delta^b \to X$ so that $f = \beta^* g = g \circ (1 \times \beta)$. Then

$$
\begin{aligned}
F(t, f) &= g \circ (1 \times \beta)_* \circ \mu^t_* \\
&= g \circ ((1 \times \beta) \circ \mu^t)_* \\
&= g \circ (\mu^t \circ (1 \times \beta))_* \\
&= g \circ \mu^t_* \circ (1 \times \beta)_* \\
&= \beta^* (g \circ \mu^t_*)
\end{aligned}
$$

is a degeneracy of a simplex in dimension b. The third equality needs an explanation. Since β is order-preserving and surjective, $\beta(0) = 0$. It follows that

$$\beta(t(i) \cdot j) = t(i) \cdot \beta(j)$$

for $i \in [n]$ and $j \in [p]$ because if $t(i) = 0$, then $\beta(0) = 0$, and if $t(i) = 1$, then it is immediate. Therefore,

$$
\begin{aligned}
((1 \times \beta) \circ \mu^t)(i, j) &= (1 \times \beta)(i, t(i) \cdot j) \\
&= (i, \beta(t(i) \cdot j)) \\
&= (i, t(i) \cdot \beta(j)) \\
&= \mu^t(i, \beta(j)) \\
&= (\mu^t \circ (1 \times \beta))(i, j).
\end{aligned}
$$

We have defined a homotopy(deformation retraction) F from the identity on Z^p to the map $Z^p \to Y^p \subset Z^p$ arising from the first vertex face map $\Delta^0 \to \Delta^p$. Therefore, the inclusion $Y^p \to Z^p$ is a homotopy equivalence. □

Suppose that X is $(d-1)$-connected and that it is equipped with a basepoint x_0. Then $\Sigma^{(d)}_n X$ is also equipped with a basepoint x_n, namely the degeneracy of x_0. Let $G^{(d)}$ be the simplicial group define by

$$[n] \mapsto \pi_d(\Sigma^{(d)}_n X, x_n).$$

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When $d = 1$, the simplicial group $G = G^{(1)}$ is a loop group for X. A principal G-bundle with a contractible total space can be constructed following the same procedure as in [7] since the properties of ΓX used there hold for $\Sigma^{(1)}X$ also. The principal G-bundle for $\Sigma^{(1)}X$ is constructed by using the group action of G on the simplicial space of universal coverings degreewise, which are contractible since $\dim \Sigma_n^{(1)} X \leq 1$, then the principal G-bundle for X is obtained by pulling back the principal bundle for $\Sigma^{(1)}X$ along the homotopy equivalence $X \to \Sigma^{(1)}X$.

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