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ON N-NORMED CESARO SEQUENCE SPACE $Ces_{n,\phi}$

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Abstract: In this paper, it is introduced and studied n-normed cesaro sequence spaces with its some algebraical and topological properties, like having a vector space, Banach space, conditions not to be empty, etc.

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1. Introduction

Throughout this work, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and real numbers, respectively. Let $n \in \mathbb{N}$ and X be a $\mathbb{R}$-linear space of the dimension $d \geq n$. A n-norm is a function satisfying following four properties on X^n(see, [2],[4],[7]): For all $z_1, ..., z_n \in X$:

1. $\|(z_1, ..., z_n)\|_n = 0$ if and only if $z_1, ..., z_n$ are linearly depended,
2. $\|(z_1, ..., z_n)\|_n$ is constant under permutation,
3. $\|(z_1, ..., \alpha z_n)\|_n = |\alpha| \|(z_1, ..., z_n)\|_n$ for any $\alpha \in R$,
4. $\|(z_1, ..., z_{n-1}, x + y)\|_n \leq \|(z_1, ..., z_{n-1}, x)\|_n + \|(z_1, ..., z_{n-1}, y)\|_n$.

In this case, a double $(X, \|.\|_n)$ is called a n-normed space. Let $(x(k))$ be a sequence in $(X, \|.\|_n)$ and x be an element in $(X, \|.\|_n)$. If for each $\varepsilon > 0$ and $z_1, ..., z_{n-1} \in X$ there is one $n_\varepsilon \in \mathbb{N}$ such that $\|(z_1, ..., z_{n-1}, x(k) - x)\|_n < \varepsilon$ wherever $k > n_\varepsilon$, then it is said to converge to x of $(x(k))$. If for each $\varepsilon > 0$ and $z_1, ..., z_{n-1} \in X$ there is one $n_\varepsilon \in \mathbb{N}$ such that $\|(z_1, ..., z_{n-1}, x(k) - x(l))\|_n < \varepsilon$ wherever $k, l > n_\varepsilon$, then it is said to be a Cauchy sequence in X of (x_k). If every

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Cauchy sequence in a n-normed space X is convergent, then this space is called a n-Banach space.

An Orlicz function is a function $\phi : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $\phi(0) = 0$, $\phi(x) > 0$ for $x > 0$ and $\phi(x) \to \infty$ as $x \to \infty$.

An Orlicz function ϕ can always be represented in the following integral form: $\phi(x) = \int_0^x \eta(t)\,dt$, where η is known as the kernel of ϕ, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$ for $t > 0$, η is nondecreasing and $\eta(t) \to \infty$ as $t \to \infty$.

An Orlicz function is said to be satisfied Δ_2-condition for all values of u, if there exists a constant $T > 0$ such that $\phi(2u) \leq T\phi(u)$ for all $u \geq 0$.

2. Main Results

Definition 1. Let X be a n-normed real linear space, $w(n,X)$ be the space of X-valued sequences, and ϕ be a Orlicz function. A set $Ces_{n,\phi}$ is defined in the form

$$\begin{aligned} Ces_{n,\phi} &= \{x = (x(k)) \in w(n,X) : \exists\lambda > 0, \quad \forall z_1, z_2, ..., z_{n-1} \in X, \quad . \\ &\quad \sum_{i=1}^{\infty} \phi\left(\frac{\lambda}{i}\sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, x(k)\|_n\right) < \infty\right\}. \end{aligned}$$

Theorem 2. *$Ces_{n,\phi}$ is a real linear space under usual addition and scalar multiplication operations for sequences.*

Proof. Given $x,\ y \in Ces_{n,\phi}$ and $\alpha,\ \beta \in \mathbb{R} - \{0\}$. Then there exist $\lambda_1 > 0$ and $\lambda_2 > 0$ such that

$$\sum_{i=1}^{\infty} \phi\left(\frac{\lambda_1}{i}\sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, x(k)\|_n\right) < \infty$$

and

$$\sum_{i=1}^{\infty} \phi\left(\frac{\lambda_2}{i}\sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, y(k)\|_n\right) < \infty$$

for all $z_1, z_2, ..., z_{n-1} \in X$. Let $\lambda = \min\left\{\frac{\lambda_1}{2|\alpha|}, \frac{\lambda_2}{2|\beta|}\right\}$. Since ϕ nondecreasing and convex, we have

$$\sum_{i=1}^{\infty} \phi\left(\frac{\lambda}{i}\sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, \alpha x(k) + \beta y(k)\|_n\right)$$

$$\begin{aligned}
&\leq \sum_{i=1}^{\infty}\phi\left(\frac{\lambda\left|\alpha\right|}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},x\left(k\right)\right\|_n\right.\\
&\quad\left.+\frac{\lambda\left|\beta\right|}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},y\left(k\right)\right\|_n\right)\\
&\leq \sum_{i=1}^{\infty}\phi\left(\frac{\lambda_1}{2i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},x\left(k\right)\right\|_n\right.\\
&\quad\left.+\frac{\lambda_2}{2i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},y\left(k\right)\right\|_n\right)\\
&\leq \frac{1}{2}\sum_{i=1}^{\infty}\phi\left(\frac{\lambda_1}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},x\left(k\right)\right\|_n\right)\\
&+\frac{1}{2}\sum_{i=1}^{\infty}\phi\left(\frac{\lambda_2}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},y\left(k\right)\right\|_n\right)<\infty.
\end{aligned}$$

This shows that $\alpha x+\beta y\in Ces_{n,\phi}$ and whereat $Ces_{n,\phi}$ is a linear space. □

Theorem 3. *$Ces_{n,\phi}$ is a normed space by the norm*

$$\begin{aligned}
\left\|x\right\|_{n,\phi} &= \inf\Bigg\{\lambda>0:\ \forall z_1,z_2,...,z_{n-1}\in X,\\
&\quad\sum_{i=1}^{\infty}\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},x\left(k\right)\right\|_n}{\lambda}\right)\leq 1\Bigg\}.
\end{aligned}$$

Proof. N1) If $x=0$, then clearly $\left\|x\right\|_{n,\phi}=0$. Conversely let $\left\|x\right\|_{n,\phi}=0$. Then for any $z_1,z_2,...,z_{n-1}\in X$ there exists a sequence (λ_N) with $\lambda_N\to 0$ as $N\to\infty$ such that

$$\sum_{i=1}^{\infty}\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},x\left(k\right)\right\|_n}{\lambda_N}\right)\leq 1.$$

We now suppose $x\left(k\right)\neq 0$ for at least one k. Then including linear independent the vectors $z_1,z_2,...,z_{n-1},x\left(k\right)$ we obtain $\left\|z_1,z_2,...,z_{n-1},\frac{x(k)}{\lambda_N}\right\|_n\to\infty$ where $N\to\infty$ as a contradiction. For this reason it must be $x\left(k\right)=0$ for all $k\in\mathbb{N}$, and hence $x=0$.

N2) Let any $x\in Ces_{n,\phi}$ and $t\in\mathbb{R}$ be given. If $t=0$, then clearly $\left\|tx\right\|_{n,\phi}=\left|t\right|\left\|x\right\|_{n,\phi}$. We are now able to accept $t\neq 0$. In this case we have

$$\begin{aligned}
\left\|tx\right\|_{n,\phi} &= \inf\Bigg\{\lambda>0:\forall\ z_1,z_2,...,z_{n-1}\in X,\\
&\quad\sum_{i=1}^{\infty}\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},tx\left(k\right)\right\|_n}{\lambda}\right)\leq 1\Bigg\}
\end{aligned}$$

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$$
\begin{aligned}
&= \inf \{\lambda > 0 : \forall\ z_1, z_2, ..., z_{n-1} \in X, \\
&\quad \sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, x(k)\|_n}{\frac{\lambda}{|t|}}\right) \le 1\right\} \\
&= \inf\left\{|t|\,\lambda^{/} > 0 : \forall\ z_1, z_2, ..., z_{n-1} \in X, \right. \\
&\quad \left. \sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, tx(k)\|_n}{\lambda^{/}}\right) \le 1\right\} \\
&= |t| \inf\left\{\lambda^{/} > 0 : \forall\ z_1, z_2, ..., z_{n-1} \in X, \right. \\
&\quad \left. \sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, tx(k)\|_n}{\lambda^{/}}\right) \le 1\right\} \\
&= |t|\,\|x\|_{n,\phi}\,.
\end{aligned}
$$

N3) Let any $x, y \in Ces_{n,\phi}$ be given. Then there exist the numbers $\lambda_1 > 0$, $\lambda_2 > 0$ and $K_1 > 1$, $K_2 > 1$ with

$$
\sum_{i=1}^{\infty} \phi\left(\frac{\lambda_1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, x(k)\|_n\right) \le K_1
$$

and

$$
\sum_{i=1}^{\infty} \phi\left(\frac{\lambda_2}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, y(k)\|_n\right) \le K_2.
$$

We now select $\lambda_1^{/} = \frac{K_1}{\lambda_1}$. Since the Orlicz function ϕ is convex and $0 < \frac{1}{K_1} < 1$, we obtain

$$
\begin{aligned}
&\sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, x(k)\|_n}{\lambda_1^{/}}\right) \\
&= \sum_{i=1}^{\infty} \phi\left(\frac{\frac{\lambda_1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, x(k)\|_n}{K_1}\right) \\
&\le \frac{1}{K_1}\sum_{i=1}^{\infty} \phi\left(\frac{\lambda_1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, x(k)\|_n\right) \\
&< \frac{1}{K_1}K_1 = 1.
\end{aligned}
$$

In exactly the same manner, selecting $\lambda_2^{/} = \frac{K_2}{\lambda_2}$ we have

$$
\sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, x(k)\|_n}{\lambda_2^{/}}\right) \le 1.
$$

We now say $\lambda_3 = \lambda_1^{/} + \lambda_2^{/}$. In this case we find

$$\begin{aligned}
&\sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, x(k) + y(k)\|_n}{\lambda_3}\right) \\
\leq\ & \sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, x(k)\|_n}{\lambda_1^{/} + \lambda_2^{/}} + \frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, y(k)\|_n}{\lambda_1^{/} + \lambda_2^{/}}\right) \\
\leq\ & \frac{\lambda_1^{/}}{\lambda_1^{/} + \lambda_2^{/}} \sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, x(k)\|_n}{\lambda_1^{/}}\right) \\
& + \frac{\lambda_2^{/}}{\lambda_1^{/} + \lambda_2^{/}} \sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\|z_1, z_2, ..., z_{n-1}, y(k)\|_n}{\lambda_2^{/}}\right) \\
\leq\ & \frac{\lambda_1^{/}}{\lambda_1^{/} + \lambda_2^{/}} + \frac{\lambda_2^{/}}{\lambda_1^{/} + \lambda_2^{/}} = 1
\end{aligned}$$

and so

$$\|x + y\|_{n,\phi} \leq \|x\|_{n,\phi} + \|y\|_{n,\phi}.$$

This completes the proof. □

Theorem 4. *If X is a n-Banach space with the n-norm on itself, then*

$$\left(Ces_{n,\phi}, \|.\|_{n,\phi}\right)$$

is a Banach space.

Proof. Let $x^l = \left(x_k^l\right)$ be an arbitrary Cauchy sequence in $\left(Ces_{n,\phi}, \|.\|_{n,\phi}\right)$. Then for any $\varepsilon > 0$ and $r > 1$ there exists one $N \in \mathbb{N}$ such that

$$\left\|x^l - x^m\right\|_{n,\phi} = \inf\Bigg\{\lambda > 0 : \forall z_1, z_2, ..., z_{n-1} \in X\ ,$$

$$\sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x^m(k)\right\|_n}{\lambda}\right) \leq 1\Bigg\} < \frac{\varepsilon}{r},$$

whenever $l, m \geq N$. We can also write

$$\sum_{i=1}^{\infty} \phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x^m(k)\right\|_n}{\|x^l - x^m\|_{n,\phi}}\right) \leq 1$$

for all $l, m \geq N$. Since ϕ is a non-negative function, we have

$$\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x^m(k)\right\|_n}{\|x^l - x^m\|_{n,\phi}}\right) \leq 1$$

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for all $l, m \geq N$. If we use that the Orlicz function ϕ satisfies three conditions with non-decreasing, continuous and $\phi(\infty) = \infty$, then for a large enough $r > 1$ and any $i \in \mathbb{N}$ we obtain

$$\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x^m(k)\right\|_n}{\left\|x^l - x^m\right\|_{n,\phi}}\right) \leq 1 \leq \phi\left(\frac{r}{2i}\right)$$

whenever $l, m \geq N$. Again since ϕ is a non-decreasing function, we can write

$$\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x^m(k)\right\|_n}{\left\|x^l - x^m\right\|_{n,\phi}} \leq \frac{r}{2i}.$$

Hence we have

$$\sum_{k=1}^{i}\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x^m(k)\right\|_n \leq \frac{r}{2}\left\|x^l - x^m\right\|_{n,\phi} < \frac{r}{2}\frac{\varepsilon}{r} = \frac{\varepsilon}{2}$$

whenever $l, m \geq N$ and for each $i \in \mathbb{N}$ and $z_1, z_2, ..., z_{n-1} \in X$ we obtain

$$\left\|z_1, z_2, ..., z_{n-1}, x^l(i) - x^m(i)\right\|_n < \frac{\varepsilon}{2}$$

for each $i \in \mathbb{N}$ and $z_1, z_2, ..., z_{n-1} \in X$. According to this, the sequence $(x^m(i))$ we have achieved above is a Cauchy sequence in $n-$Banach space X. Then there exists an element $x_i \in X$ for each $i \in \mathbb{N}$ such that $\|x^m(i) - x(i)\|_n \to 0$ as $m \to \infty$. Let a number $\lambda_0 > 0$ be defined by

$$\lambda_0 = \sup_{l,m \geq N}\left\|x^l - x^m\right\|_{n,\phi}.$$

Clearly

$$\sum_{i=1}^{\infty}\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x^m(k)\right\|_n}{\lambda_0}\right) \leq 1$$

for all $l, m \geq N$. Additionaly since

$$\begin{aligned} &\lim_{l\to\infty}\left|\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x^m(k)\right\|_n - \left\|z_1, z_2, ..., z_{n-1}, x^m(k) - x(k)\right\|_n\right| \\ \leq\ &\lim_{l\to\infty}\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x(k)\right\|_n = 0, \end{aligned}$$

we have $\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x^m(k)\right\|_n \to \left\|z_1, z_2, ..., z_{n-1}, x^m(k) - x(k)\right\|_n$ as $l \to \infty$.

Next we denote that the sequence (x^m) converges to x with the norm $\|.\|_{n,\phi}$. For this, given any $z_1, z_2, ..., z_{n-1} \in X$ and where $l, m \geq N$, we have

$$\sum_{i=1}^{p}\phi\left(\frac{1}{i}\sum_{k=1}^{i}\frac{\left\|z_1, z_2, ..., z_{n-1}, x^l(k) - x(k)\right\|_n}{\lambda_0}\right)$$

$$= \sum_{i=1}^{p}\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\lim_{m\to\infty}\left\|z_1,z_2,...,z_{n-1},x^l(k)-x^m(k)\right\|_n}{\lambda_0}\right)$$
$$= \lim_{m\to\infty}\sum_{i=1}^{p}\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},x^l(k)-x^m(k)\right\|_n}{\lambda_0}\right)\le 1$$

and

$$\sum_{i=1}^{\infty}\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},x^l(k)-x(k)\right\|_n}{\lambda_0}\right)\le 1,$$

since this inequality satisfies for all $p\in\mathbb{N}$. Then we obtain

$$\left\|x^l-x\right\|_{n,\phi}=\inf\{\lambda>0:\forall z_1,z_2,...,z_{n-1}\in X\ ,$$
$$\left.\sum_{i=1}^{\infty}\phi\left(\frac{\frac{1}{i}\sum_{k=1}^{i}\left\|z_1,z_2,...,z_{n-1},x^l(k)-x(k)\right\|_n}{\lambda}\right)\le 1\right\}<\lambda_0\le\frac{\varepsilon}{r}<\varepsilon.$$

Hence we obtain $\left\|x^l-x\right\|_{n,\phi}\to 0$ as $l\to\infty$. Furthermore one can easily see that $x\in Ces_{n,\phi}$. Thus the proof is completed. □

Theorem 5. *Following conditions hold:*

(i)$Ces_{n,\phi}\neq\{0\}$ ⇒(ii) $\exists i_1\in\mathbb{N}$, $\sum_{k=i_1}^{\infty}\phi\left(\frac{1}{k}\right)<\infty$ ⇒(iii)$\forall\lambda>0$, $\exists i_\lambda\in\mathbb{N}$, $\sum_{k=i_\lambda}^{\infty}\phi\left(\frac{\lambda}{k}\right)<\infty$.

Proof. (i)⇒(ii): Let $t\in Ces_{n,\phi}$ with $t\neq 0$. Then there exists at least one $l\in\mathbb{N}$ with $t(l)\neq 0$ and $\lambda_1>0$ with

$$\sum_{i=1}^{\infty}\phi\left(\frac{\lambda_1}{i}\sum_{k=1}^{i}\|z_1,z_2,...,z_{n-1},t(k)\|_n\right)<\infty$$

for all $z_1,z_2,...,z_{n-1}\in X$. We now define a sequence $y=(y(k))$ with

$$y(k)=\begin{cases} t(k), & k=l\\ 0, & \text{otherwise}\end{cases}.$$

Then we find

$$\sum_{i=1}^{\infty}\phi\left(\frac{\lambda_1}{i}\sum_{k=1}^{i}\|z_1,z_2,...,z_{n-1},y(k)\|_n\right)\le\sum_{i=1}^{\infty}\phi\left(\frac{\lambda_1}{i}\sum_{k=1}^{i}\|z_1,z_2,...,z_{n-1},t(k)\|_n\right)<\infty$$

and thus $y\in Ces_{n,\phi}$. Hence

$$\sum_{i=l}^{\infty}\phi\left(\frac{\lambda_1}{i}\|z_1,z_2,...,z_{n-1},t(l)\|_n\right)=\sum_{i=l}^{\infty}\phi\left(\frac{\lambda_1}{i}\sum_{k=1}^{i}\|z_1,z_2,...,z_{n-1},y(k)\|_n\right)<\infty.$$

If the vectors $z_1, z_2, ..., z_{n-1}, t(l)$ in X is selected linear independent and is said $\lambda = \lambda_1 \|z_1, z_2, ..., z_{n-1}, t(l)\|_n$, then we obtain $\sum_{i=l}^{\infty} \phi\left(\frac{\lambda}{i}\right) < \infty$.

In this instance, let $\lambda \geq 1$. Since Orlicz function ϕ is nondecreasing, we can write

$$\sum_{i=l}^{\infty} \phi\left(\frac{1}{i}\right) \leq \sum_{i=l}^{\infty} \phi\left(\frac{\lambda}{i}\right).$$

This proves the theorem. Let us assume $0 < \lambda < 1$. Then we can select one $m \in \mathbb{N}$ with $\frac{1}{m} < \lambda$. Accordingly the inequality $\frac{1}{m.i} < \frac{\lambda}{i}$ holds for all $i \in \mathbb{N}$. Hence we have

$$\sum_{i=l}^{\infty} \phi\left(\frac{1}{m.i}\right) \leq \sum_{i=l}^{\infty} \phi\left(\frac{\lambda}{i}\right) < \infty$$

and so

$$\sum_{i=m.l}^{\infty} \phi\left(\frac{1}{i}\right) = \left[\phi\left(\frac{1}{m.l}\right) + \phi\left(\frac{1}{m.l+1}\right) + \cdots + \phi\left(\frac{1}{m.l+(m-1)}\right)\right]$$

$$+\left[\phi\left(\frac{1}{m(l+1)}\right) + \phi\left(\frac{1}{m(l+1)+1}\right) + \cdots + \phi\left(\frac{1}{m(l+1)+(m-1)}\right)\right] + \cdots$$

$$\leq \left[\phi\left(\frac{1}{ml}\right) + \phi\left(\frac{1}{ml}\right) + \cdots + \phi\left(\frac{1}{ml}\right)\right] + \left[\phi\left(\frac{1}{m(l+1)}\right) + \phi\left(\frac{1}{m(l+1)}\right)\right.$$

$$\left. + \cdots + \phi\left(\frac{1}{m(l+1)}\right)\right] + \cdots = m\sum_{i=.l}^{\infty} \phi\left(\frac{1}{mi}\right) \leq m\sum_{i=.l}^{\infty} \phi\left(\frac{\lambda}{i}\right) < \infty.$$

In that case we reach the conclusion $\sum_{i=.i_1}^{\infty} \phi\left(\frac{1}{i}\right) < \infty$ with $i_1 = ml$. This is what we search.

(ii)⇒(iii): This proof is exactly given in the reference [6]. □

Theorem 6. *Let ϕ_1 and ϕ_2 be two Orlicz functions. If there are positive numbers b and t_0 with $\phi_2\ (t) \leq \phi_1\ (bt)$ for all $t \in [0, t_0]$, then the inclusion $Ces_{n,\phi_1} \subset Ces_{n,\phi_2}$ hold.*

Proof. Since the Orlicz function ϕ_1 is non-decrasing, we select $b \geq 1$. By the hypothesis, $\phi_2\ \left(\frac{u}{b}\right) \leq \phi_1\ (u)$ for all $u \in [0, bt_0]$. Given $x \in Ces_{n,\phi_1}$, then there is one $\lambda > 0$. Let A_x be defined as follows:

$$A_x = \left\{ i \in \mathbb{N} : \frac{\lambda}{i} \sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, x(k)\|_n > bt_0 \right\}.$$

It is easy to see that A_x is finite. Let $\lambda^{/}$ be defined by $\lambda^{/} = \frac{c}{b}$ for one $0 < c < \lambda$. If it is used $\phi_2\ \left(\frac{u}{b}\right) \leq \phi_1\ (u)$

$$\begin{aligned}
&\sum_{i=l}^{\infty} \phi_2 \left(\frac{\lambda^{/}}{i} \sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, x\,(k)\|_n \right) \\
= &\sum_{i \in A_x} \phi_2 \left(\frac{c}{bi} \sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, x\,(k)\|_n \right) \\
&+ \sum_{i \notin A_x} \phi_2 \left(\frac{c}{bi} \sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, x\,(k)\|_n \right) \\
\leq &\sum_{i \in A_x} \phi_2 \left(\frac{c}{bi} \sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, x\,(k)\|_n \right) \\
&+ \sum_{i=l}^{\infty} \phi_1 \left(\frac{\lambda}{i} \sum_{k=1}^{i} \|z_1, z_2, ..., z_{n-1}, x\,(k)\|_n \right) < \infty.
\end{aligned}$$

This shows that $x \in Ces_{n,\phi_2}$, and the proof is completed. □

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