

## A GENERAL ELLIPTIC NONLINEAR SYSTEM OF MULTIPLE FUNCTIONS WITH APPLICATION

Timothy Robertson<sup>1</sup>, Joon Hyuk Kang<sup>2 §</sup>

<sup>1,2</sup>Department of Mathematics

Andrews University

Berrien Springs, MI. 49104, USA

**Abstract:** The purpose of this paper is to give a sufficient condition for the existence, nonexistence and uniqueness of positive solutions to a rather general type of elliptic system of the Dirichlet problem on a bounded domain  $\Omega$  in  $R^n$ . We also investigate the effects of perturbation on the positive solutions to the system.

The techniques used in this paper are upper-lower solutions, eigenvalues of operators, the maximum principles and spectrum estimates. The arguments also rely on some detailed properties for the solution of logistic equations. This result yields an algebraically computable criterion for the positive coexistence of competing species of animals in many biological models.

**AMS Subject Classification:** 35A05, 35A07, 35B50, 35G30, 35J25, 35K20

**Key Words:** general elliptic system, positive solution

### 1. Introduction

The most general type of elliptic interacting system with homogeneous boundary condition is

$$\begin{cases} \Delta u_i + g_i(u_1, u_2, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_N) = 0 & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, \end{cases} \quad (1)$$

for  $i = 1, 2, \dots, N$ , where we assume that the  $C^2$  functions  $g_i$ 's are relative growth rates satisfying the following so-called growth rate conditions:

$$(G_1) \quad (g_i)_{u_j} < 0, (g_i)_{u_i u_j} < 0 \text{ for } i, j = 1, 2, \dots, N,$$

---

Received: August 21, 2016

© 2016 Academic Publications, Ltd.

<sup>§</sup>Correspondence author

(G<sub>2</sub>)  $g_i(u_1, u_2, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_N) = 0$  for  $i = 1, 2, \dots, N$ ,

(G<sub>3</sub>) There exist constants  $c_0 > 0$  such that  $g_i(0, \dots, 0, u_i, 0, \dots, 0) < 0$ ,  $(g_i)_{u_i}(0, \dots, 0, u_i, 0, \dots, 0) < 0$  for  $u_i > c_0$  for  $i = 1, \dots, N$ .

The goal of this paper is to answer the following questions about positive solutions of (1).

*Problem 1:* What are sufficient conditions for the existence of a positive solution?

*Problem 2:* What are sufficient conditions for the nonexistence of positive solutions?

*Problem 3:* What is the effect of perturbation on the positive solution?

## 2. Preliminaries

In this section we state some preliminary results which will be useful for our later arguments.

**Definition 2.1.** (Upper and Lower solutions) The vector functions  $(\bar{u}^1, \dots, \bar{u}^N)$ ,  $(\underline{u}^1, \dots, \underline{u}^N)$  form an upper/lower solution pair for the system

$$\begin{cases} \Delta u^i + g^i(u^1, \dots, u^N) = 0 & \text{in } \Omega, \\ u^i = 0 & \text{on } \partial\Omega, \end{cases}$$

if for  $i = 1, \dots, N$

$$\begin{cases} \Delta \bar{u}^i + g^i(u^1, \dots, u^{i-1}, \bar{u}^i, u^{i+1}, \dots, u^N) \leq 0, \\ \Delta \underline{u}^i + g^i(u^1, \dots, u^{i-1}, \underline{u}^i, u^{i+1}, \dots, u^N) \geq 0, \\ \text{in } \Omega \text{ for } \underline{u}^j \leq u^j \leq \bar{u}^j, j \neq i, \end{cases}$$

and

$$\begin{aligned} \underline{u}^i &\leq \bar{u}^i && \text{on } \Omega, \\ \underline{u}^i &\leq 0 \leq \bar{u}^i && \text{on } \partial\Omega. \end{aligned}$$

**Lemma 2.1.** (see [1]) *If  $g^i$  in the Definition 2.1 are in  $C^1$  and the system admits an upper/lower solution pair  $(\underline{u}^1, \dots, \underline{u}^N)$ ,  $(\bar{u}^1, \dots, \bar{u}^N)$ , then there is a solution of the system in 2.1 with  $\underline{u}^i \leq u^i \leq \bar{u}^i$  in  $\bar{\Omega}$ . If*

$$\begin{aligned} \Delta \bar{u}^i + g^i(\bar{u}^1, \dots, \bar{u}^N) &\neq 0, \\ \Delta \underline{u}^i + g^i(\underline{u}^1, \dots, \underline{u}^N) &\neq 0 \end{aligned}$$

*in  $\Omega$  for  $i = 1, \dots, N$ , then  $\underline{u}^i < u^i < \bar{u}^i$  in  $\Omega$ .*

**Lemma 2.2.** (The First Eigenvalue)

$$\begin{cases} -\Delta u + q(x)u = \lambda u & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \quad (2)$$

where  $q(x)$  is a smooth function from  $\Omega$  to  $R$  and  $\Omega$  is a bounded domain in  $R^n$ .

(A) The first eigenvalue  $\lambda_1(q)$  of (2), denoted by simply  $\lambda_1$  when  $q \equiv 0$ , is simple with a positive eigenfunction.

(B) If  $q_1(x) < q_2(x)$  for all  $x \in \Omega$ , then  $\lambda_1(q_1) < \lambda_1(q_2)$ .

(C) (Variational Characterization of the First Eigenvalue)

$$\lambda_1(q) = \min_{\phi \in W_0^1(\Omega), \phi \neq 0} \frac{\int_{\Omega} (|\nabla\phi|^2 + q\phi^2) dx}{\int_{\Omega} \phi^2 dx}.$$

**Lemma 2.3.** (Maximum Principles)

$$Lu = \sum_{i,j=1}^n a_{ij}(x)D_{ij}u + \sum_{i=1}^n a_i(x)D_iu + a(x)u = f(x) \text{ in } \Omega,$$

where  $\Omega$  is a bounded domain in  $R^n$ .

(M1)  $\partial\Omega \in C^{2,\alpha}$  ( $0 < \alpha < 1$ ).

(M2)  $|a_{ij}(x)|_{\alpha}, |a_i(x)|_{\alpha}, |a(x)|_{\alpha} \leq M$  ( $i, j = 1, \dots, n$ ).

(M3)  $L$  is uniformly elliptic in  $\bar{\Omega}$ , with ellipticity constant  $\gamma$ , i.e., for every  $x \in \bar{\Omega}$  and every real vector  $\xi = (\xi_1, \dots, \xi_n)$

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq \gamma \sum_{i=1}^n |\xi_i|^2.$$

Let  $u \in C^2(\Omega) \cap C(\bar{\Omega})$  be a solution of  $Lu \geq 0$  ( $Lu \leq 0$ ) in  $\Omega$ .

(A) If  $a(x) \equiv 0$ , then  $\max_{\bar{\Omega}} u = \max_{\partial\Omega} u$  ( $\min_{\bar{\Omega}} u = \min_{\partial\Omega} u$ ).

(B) If  $a(x) \leq 0$ , then  $\max_{\bar{\Omega}} u \leq \max_{\partial\Omega} u^+$  ( $\min_{\bar{\Omega}} u \geq -\max_{\partial\Omega} u^-$ ), where  $u^+ = \max(u, 0)$ ,  $u^- = -\min(u, 0)$ .

(C) If  $a(x) \equiv 0$  and  $u$  attains its maximum (minimum) at an interior point of  $\Omega$ , then  $u$  is identically a constant in  $\Omega$ .

(D) If  $a(x) \leq 0$  and  $u$  attains a nonnegative maximum (nonpositive minimum) at an interior point of  $\Omega$ , then  $u$  is identically a constant in  $\Omega$ .

**Lemma 2.4.**

$$\begin{cases} \Delta u + uf(u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, u > 0, \end{cases}$$

where  $f$  is a decreasing  $C^1$  function such that there exists  $c_0 > 0$  such that  $f(u) \leq 0$  for  $u \geq c_0$  and  $\Omega$  is a bounded domain in  $R^n$ .

(1) If  $f(0) > \lambda_1$ , then the above equation has a unique positive solution, where  $\lambda_1$  is the first eigenvalue of  $-\Delta$  with homogeneous boundary condition. We denote this unique positive solution as  $\theta_f$ .

(2) If  $f(0) \leq \lambda_1$ , then the above equation does not have any positive solution.

### 3. Existence and Nonexistence

We consider the system (1) with conditions  $(G_1)$ ,  $(G_2)$  and  $(G_3)$ .

**Theorem 3.1.** (A) If  $(g_i)_{u_i}(c_0, \dots, c_0, 0, c_0, \dots, c_0) > \lambda_1$  for  $i = 1, \dots, N$ , then (1) has a solution  $(u_1, \dots, u_N)$  with  $u_1 > 0, \dots, u_N > 0$ .

Any solution  $(u_1, \dots, u_N)$  of (1) with  $u_1 > 0, \dots, u_N > 0$  in  $\Omega$  satisfies

$$\theta_{(g_i)_{u_i}(c_0, \dots, c_0, \cdot, c_0, \dots, c_0)} < u \leq c_0.$$

(B) If  $(g_i)_{u_i}(0, \dots, 0) \leq \lambda_1$  for some  $i = 1, \dots, N$ , then (1) does not have any positive solutions.

*Proof.* (A) By the result in [7], there is a positive solution  $(u_1, \dots, u_N)$  to

$$\begin{cases} \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_N) = 0 & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0 \end{cases}$$

for  $i = 1, \dots, N$ .

But, by the Mean Value Theorem, for  $i = 1, \dots, N$ , there is  $\tilde{u}_i$  such that  $0 \leq \tilde{u}_i \leq u_i$  and

$$\begin{aligned} g_i(u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_N) - g_i(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_N) \\ = u_i(g_i)_{u_i}(u_1, \dots, u_{i-1}, \tilde{u}_i, u_{i+1}, \dots, u_N), \end{aligned}$$

so by the monotonicity of  $(g_i)_{u_i}$ ,

$$\begin{aligned} \Delta u_i + g_i(u_1, \dots, u_N) &= \Delta u_i + g_i(u_1, \dots, u_N) - g_i(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_N) \\ &= \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_{i-1}, \tilde{u}_i, u_{i+1}, \dots, u_N) \\ &\geq \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_N) \\ &= 0. \end{aligned}$$

Hence,  $(u_1, \dots, u_N)$  is a subsolution to (1).

But, by the conditions, any  $(M, \dots, M)$  with large enough  $M > 0$  is a supersolution to (1).

Therefore, by Lemma 2.1, there is a positive solution to (1).

Suppose  $(u_1, \dots, u_N)$  is a solution to (1) with  $u_i > 0$  for  $i = 1, \dots, N$ .

By the Mean Value Theorem, for  $i = 1, \dots, N$ , there is  $\bar{u}_i$  such that  $0 \leq \bar{u}_i \leq u_i$  and

$$\begin{aligned} g_i(u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_N) - g_i(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_N) \\ = u_i(g_i)_{u_i}(u_1, \dots, u_{i-1}, \bar{u}_i, u_{i+1}, \dots, u_N), \end{aligned}$$

so by the monotonicity of  $(g_i)_{u_i}$ ,

$$\begin{aligned} \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_N) &\leq \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_{i-1}, \bar{u}_i, u_{i+1}, \dots, u_N) \\ &= \Delta u_i + g_i(u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_N) \\ &\quad - g_i(u_1, \dots, u_{i-1}, 0, u_{i+1}, \dots, u_N) \\ &= \Delta u_i + g_i(u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_N) \\ &= 0. \end{aligned}$$

Hence,  $(u_1, \dots, u_N)$  is a supersolution to

$$\begin{cases} \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_N) = 0, \\ u_i|_{\partial\Omega} = 0, \quad i = 1, \dots, N. \end{cases}$$

For sufficiently large  $n \in N$ , and for  $i = 1, \dots, N$  by the monotonicity of  $(g_i)_{u_i}$ ,

$$\begin{aligned} &\Delta\left[\frac{1}{n}\theta_{(g_i)_{u_i}}(c_0, \dots, c_0, \cdot, c_0, \dots, c_0)\right] + \frac{1}{n}\theta_{(g_i)_{u_i}}(c_0, \dots, c_0, \cdot, c_0, \dots, c_0) \\ &(g_i)_{u_i}\left(\frac{1}{n}\theta_{(g_1)_{u_1}}(\cdot, c_0, \dots, c_0), \frac{1}{n}\theta_{(g_2)_{u_2}}(c_0, \cdot, c_0, \dots, c_0), \dots, \frac{1}{n}\theta_{(g_N)_{u_N}}(c_0, \dots, c_0, \cdot)\right) \\ &= \frac{1}{n}\left[\Delta\theta_{(g_i)_{u_i}}(c_0, \dots, c_0, \cdot, c_0, \dots, c_0) + \theta_{(g_i)_{u_i}}(c_0, \dots, c_0, \cdot, c_0, \dots, c_0)\right. \\ &\quad \left.(g_i)_{u_i}\left(\frac{1}{n}\theta_{(g_1)_{u_1}}(\cdot, c_0, \dots, c_0), \frac{1}{n}\theta_{(g_2)_{u_2}}(c_0, \cdot, c_0, \dots, c_0), \dots, \frac{1}{n}\theta_{(g_N)_{u_N}}(c_0, \dots, c_0, \cdot)\right)\right] \\ &\geq \frac{1}{n}\left[\Delta\theta_{(g_i)_{u_i}}(c_0, \dots, c_0, \cdot, c_0, \dots, c_0) + \theta_{(g_i)_{u_i}}(c_0, \dots, c_0, \cdot, c_0, \dots, c_0)\right. \\ &\quad \left.(g_i)_{u_i}(c_0, \dots, c_0, \theta_{(g_i)_{u_i}}(c_0, \dots, c_0, \cdot, c_0, \dots, c_0), c_0, \dots, c_0)\right] \\ &= 0, \end{aligned}$$

for  $i = 1, \dots, N$ , so  $(\frac{1}{n}\theta_{(g_1)_{u_1}}(\cdot, c_0, \dots, c_0), \frac{1}{n}\theta_{(g_2)_{u_2}}(c_0, \cdot, c_0, \dots, c_0), \dots, \frac{1}{n}\theta_{(g_N)_{u_N}}(c_0, \dots, c_0, \cdot))$  is a subsolution to

$$\begin{cases} \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_N) = 0, \\ u_i|_{\partial\Omega} = 0, \quad i = 1, \dots, N. \end{cases}$$

Hence, by Lemma 2.1, there is a positive solution  $(\tilde{u}_1, \dots, \tilde{u}_N)$  to

$$\begin{cases} \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_N) = 0, \\ u_i|_{\partial\Omega} = 0, \quad i = 1, \dots, N. \end{cases}$$

with  $\tilde{u}_i \leq u_i$  for  $i = 1, \dots, N$ .

But, by the solution estimates in [7], we conclude

$$\theta_{(g_i)_{u_i}(c_0, \dots, c_0, \cdot, c_0, \dots, c_0)} < u_i, i = 1, \dots, N \quad (3)$$

We prove  $u_i(x) \leq c_0$  for  $x \in \bar{\Omega}$  and  $i = 1, \dots, N$ .

If  $u_i(x) > c_0$  at some  $x \in \Omega$ , then by the continuity of  $u_i$ , there is  $r > 0$  such that  $u_i(y) > c_0$  for all  $y \in B_r(x) = \{y \in R^n \mid \|y - x\| < r\}$ , so by the monotonicity of  $g_i$ ,

$$\Delta u_i = -g_i(u_1, \dots, u_N) > -g(0, \dots, 0, c_0, 0, \dots, 0) \geq 0 \text{ on } B_r(x),$$

which contradicts the Maximum Principles.

Hence, we conclude

$$u_i \leq c_0, i = 1, \dots, N. \quad (4)$$

By (3) and (4), we have the desired inequalities.

(B) Assume  $(g_i)_{u_i}(0, \dots, 0) \leq \lambda_1$  for some  $i = 1, \dots, N$ . Suppose  $(\bar{u}_1, \dots, \bar{u}_N)$  is a positive solution to (1). Then by the same way as in (A), we have a positive solution  $(u_1, \dots, u_N)$  to

$$\begin{cases} \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_N) = 0, \\ u_i|_{\partial\Omega} = 0, i = 1, \dots, N. \end{cases}$$

Hence, we again contradict Lemma 2.4 by the same reason in [7]. So, there is no positive solution to (1).

#### 4. Uniqueness with Perturbation

We consider the model

$$\begin{cases} \Delta u_i + g_i(u_1, u_2, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_N) = 0, \text{ in } \Omega, \\ u_i|_{\partial\Omega} = 0, \quad i = 1, 2, \dots, N, \end{cases} \quad (5)$$

where  $\Omega$  is a smooth, bounded domain in  $R^n$  and (P1)  $g_i \in C_B^2$  for  $i = 1, 2, \dots, N$ , where  $C_B^2$  is the set of all two variables functions  $f(u, v)$  such that all the first-order partial derivatives of  $f$  are decreasing, and all the second-order partial derivatives of  $f$  are bounded and continuous,

(P2) for  $i = 1, 2, \dots, N$ , there is  $c_i > 0$  such that  $(g_i)_{u_i}(0, \dots, 0, u_i, 0, \dots, 0) < 0$  for  $u_i > c_i$ .

The following theorem is our main result about the uniqueness.

**Theorem 4.1.** *Suppose:*

(A)  $\lambda_1[-(g_i)_{u_i}(\theta_{(g_1)u_i}(\cdot, 0, \dots, 0), \dots, \theta_{(g_{i-1})u_{i-1}}(0, \dots, 0, \cdot, 0, \dots, 0), 0, \theta_{(g_{i+1})u_{i+1}}(\cdot, 0, \dots, 0), \dots, \theta_{(g_N)u_N}(0, \dots, 0, \cdot)) < 0$  for  $i = 1, \dots, N$ , where in general,  $\lambda_1(q)$  is the smallest eigenvalue of  $-\Delta + q$  with homogeneous boundary conditions, denoted simply by  $\lambda_1$  when  $q \equiv 0$ ,

(B) (5) has a unique coexistence state  $(u_1, \dots, u_N)$ ,

(C) the Frechet derivative of (5) at  $(u_1, \dots, u_N)$  is invertible.

Then there is a neighborhood  $V$  of  $(g_1, \dots, g_N)$  in  $[C_B^2(R^2)]^N$  such that if  $(\bar{g}_1, \dots, \bar{g}_N) \in V$ , then (5) with  $(\bar{g}_1, \dots, \bar{g}_N)$  has a unique positive solution.

*Proof.* Since the Frechet derivative of (5) at  $(u_1, u_2, \dots, u_N)$  is invertible, then, by the Implicit Function Theorem, there is a neighborhood  $V$  of  $(g_1, \dots, g_N)$  in  $C_B^N$  and a neighborhood  $W$  of  $(u_1, u_2, \dots, u_N)$  in  $[C_0^{2,\alpha}(\bar{\Omega})]^N$  such that for all  $(\bar{g}_1, \bar{g}_2, \dots, \bar{g}_N) \in V$ , there is a unique positive solution  $(\bar{u}_1, \bar{u}_2, \dots, \bar{u}_N) \in W$  of (5). Thus, the local uniqueness of the solution is guaranteed.

To prove global uniqueness, suppose that the conclusion of Theorem 4.1 is false. Then there are sequences  $((g_1)_n, \dots, (g_N)_n, (u_1)_n, \dots, (u_N)_n), ((\bar{g}_1)_n, \dots, (\bar{g}_N)_n, (\bar{u}_1)_n, \dots, (\bar{u}_N)_n)$  in  $V \times [C_0^{2,\alpha}(\bar{\Omega})]^N$  such that  $((u_1)_n, \dots, (u_N)_n)$  and  $((\bar{u}_1)_n, \dots, (\bar{u}_N)_n)$  are positive solutions of (5) with  $((g_1)_n, \dots, (g_N)_n)$  and  $((u_1)_n, \dots, (u_N)_n) \neq ((\bar{u}_1)_n, \dots, (\bar{u}_N)_n)$  and  $((g_1)_n, \dots, (g_N)_n) \rightarrow (g_1, \dots, g_N)$ . By Schauder's estimate in elliptic theory, the convergence of  $((g_1)_n, \dots, (g_N)_n)$ , and the solution estimate in the proof of Theorem 3.1, for  $i = 1, \dots, N$ , there are constants  $k_i > 0, l_i > 0$  such that

$$|(u_i)_n|_{2,\alpha} \leq k_i[|(g_i)_n((u_1)_n, \dots, (u_N)_n)|_\alpha + \sup_{x \in \bar{\Omega}}((u_i)_n(x))] \leq k_i(l_i + c_0),$$

for all  $n = 1, 2, \dots$ , and so we conclude that  $|(u_i)_n|_{2,\alpha}$  is uniformly bounded for each  $i = 1, \dots, N$ . Therefore, there are uniformly convergent subsequence of  $(u_i)_n$  for each  $i = 1, \dots, N$ , which again will be denoted by  $(u_i)_n$ .

Thus, let

$$\begin{aligned} ((u_1)_n, \dots, (u_N)_n) &\rightarrow (\hat{u}_1, \dots, \hat{u}_N), \\ ((\bar{u}_1)_n, \dots, (\bar{u}_N)_n) &\rightarrow (\bar{u}_1, \dots, \bar{u}_N) \end{aligned}$$

Then  $(\hat{u}_1, \dots, \hat{u}_N), (\bar{u}_1, \dots, \bar{u}_N) \in (C^{2,\alpha})^N$  are also solutions of (5) with  $(g_1, \dots, g_N)$ . We claim that  $\hat{u}_1 > 0, \dots, \hat{u}_N > 0, \bar{u}_1 > 0, \dots, \bar{u}_N > 0$ .

By the same proof as in Section 3, for each  $n = 1, 2, \dots$ , there is a positive solution  $((\tilde{u}_1)_n, \dots, (\tilde{u}_N)_n)$  to

$$\begin{cases} \Delta u_i + u_i((g_i)_n)_{u_i}(u_1, \dots, u_N) = 0 & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, \dots, N \end{cases} \tag{6}$$

such that  $(\tilde{u}_1)_n \leq (u_1)_n, \dots, (\tilde{u}_N)_n \leq (u_N)_n$ .

By the exactly same proof as in [6], there are uniformly convergent subsequences of  $(\tilde{u}_1)_n, \dots, (\tilde{u}_N)_n$ , which again will be denoted by  $(\tilde{u}_1)_n, \dots, (\tilde{u}_N)_n, (\tilde{u}_1)_n \rightarrow$

$\tilde{u}_1, \dots, (u_N)_n \rightarrow \tilde{u}_N, (\tilde{u}_1, \dots, \tilde{u}_N)$  is a solution to (6) with

$$((g_1)_{u_1}, \dots, (g_N)_{u_N}), \text{ and } \tilde{u}_1 > 0, \dots, \tilde{u}_n > 0.$$

But, since  $(\tilde{u}_1)_n \leq (u_1)_n, \dots, (\tilde{u}_N)_n \leq (u_N)_n$ , it follows that

$$0 < \tilde{u}_1 \leq \hat{u}_1, \dots, 0 < \tilde{u}_N \leq \hat{u}_N.$$

By the same procedure to the sequence  $((u_1^-)_n, \dots, (u_N^-)_n)$ , we also have

$$\bar{u}_1 > 0, \dots, \bar{u}_N > 0.$$

In conclusion, both  $(\hat{u}_1, \dots, \hat{u}_N)$  and  $(\bar{u}_1, \dots, \bar{u}_N)$  are positive solutions to (6) with  $(g_1, \dots, g_N)$ . But, by the condition (B),  $\hat{u}_1 = \bar{u}_1 = u_1, \dots, \hat{u}_N = \bar{u}_N = u_N$ , which is a contradiction to the Implicit Function Theorem since

$$((u_1)_n, \dots, (u_N)_n) \neq ((u_1^-)_n, \dots, (u_N^-)_n).$$

### 5. Uniqueness with Perturbation of Region

Consider the model

$$\Delta u_i + u_i g_i(u_1, u_2, \dots, u_i, u_{i+1}, \dots, u_N) = 0 \text{ in } \Omega, u_i|_{\partial\Omega} = 0, \tag{7}$$

for  $i = 1, \dots, N$ . Here  $\Omega$  is a bounded smooth domain in  $R^n$  and

$$g_i \in C_B^2 = \left\{ f : R^N \rightarrow R \in C^1 \mid \begin{array}{l} \text{All the first order partial derivatives of } f \\ \text{are strictly decreasing with respect to each } u_i, f \\ \text{and all of its second order partial derivatives are continuous and bounded} \end{array} \right\}.$$

The following theorem is the main result.

**Theorem 5.1.** *Suppose:*

(A)  $\Gamma$  is a closed, bounded, convex region in  $[C_B^2]^N$  such that for all  $(g_1, \dots, g_N) \in \Gamma$ ,

$$\lambda_1((-g_i)_{u_i}(\theta_{(g_1)_{u_1}}(\cdot, 0, \dots, 0), \dots, \theta_{(g_{i-1})_{u_{i-1}}}(0, \dots, 0, \cdot, 0, \dots, 0), 0, \theta_{(g_{i+1})_{u_{i+1}}}(0, \dots, 0, \cdot, 0, \dots, 0), \dots, \theta_{(g_N)_{u_N}}(0, \dots, 0, \cdot))) < 0$$

for  $i = 1, \dots, N$ ,



(B) there exist  $c_i > 0$  such that for all  $(g_1, \dots, g_N) \in \Gamma$ ,  $(g_i)_{u_i}(0, \dots, 0, u_i, 0, \dots, 0) < 0$  for  $u_i > c_i, i = 1, \dots, N$ .

(C) (7) has a unique positive solution for every  $(g_1, \dots, g_N) \in \partial_L \Gamma$ , where

$$\begin{aligned} \partial_L \Gamma &= \{(\lambda_{g_2, \dots, g_N}, g_2, \dots, g_N) \in \Gamma \mid \text{for any fixed } g_2, \dots, g_N, \|\lambda_{g_2, \dots, g_N}\| \\ &= \inf\{\|g_1\| \mid (g_1, \dots, g_N) \in \Gamma\}\}, \end{aligned}$$

(D) for all  $(g_1, \dots, g_N) \in \Gamma$ , the Frèchet derivative of (7) at every positive solution to (7) is invertible.

Then for all  $(g_1, \dots, g_N) \in \Gamma$ , (7) has a unique positive solution. Furthermore, there is an open set  $W$  in  $[C_B^2]^N$  such that  $\Gamma \subseteq W$  and for every  $(g_1, \dots, g_N) \in W$ , (7) has a unique positive solution.

Theorem 5.1 goes even further than Theorem 4.1. Theorem 5.1 states uniqueness in the whole region of  $(g_1, \dots, g_N)$  whenever we have uniqueness on the left boundary and invertibility of the linearized operator at any particular solution inside the domain.

*Proof.* For each fixed  $g_2, \dots, g_N$ , consider  $(g_1, g_2, \dots, g_N) \in \partial_L \Gamma$  and  $(\bar{g}_1, g_2, \dots, g_N) \in \Gamma$ . We need to show that for all  $0 \leq t \leq 1$ , (7) with  $(1-t)(g_1, \dots, g_N) + t(\bar{g}_1, g_2, \dots, g_N)$  has a unique positive solution. Since (7) with  $(g_1, \dots, g_N)$  has a unique positive solution  $(u_1, \dots, u_N)$  and the Frechet derivative of (7) at  $(u_1, \dots, u_N)$  is invertible, Theorem 4.1 implies that there is an open neighborhood  $V$  of  $(g_1, \dots, g_N)$  in  $(C_B^1)^N$  such that if  $(g_{10}, g_{20}, \dots, g_{N0}) \in V$ , then (7) with  $(g_{10}, g_{20}, \dots, g_{N0})$  has a unique positive solution.

Let

$$\begin{aligned} \lambda_s &= \sup\{0 \leq \lambda \leq 1 \mid (7) \text{ with } (1-t)(g_1, \dots, g_N) + t(\bar{g}_1, g_2, \dots, g_N) \\ &\text{has a unique positive solution for } 0 \leq t \leq \lambda\}. \end{aligned}$$

We need to show that  $\lambda_s = 1$ . Suppose  $\lambda_s < 1$ . From the definition of  $\lambda_s$ , there is a sequence  $\{\lambda_n\}$  such that  $\lambda_n \rightarrow \lambda_s^-$  and there is a sequence  $(u_{1n}, u_{2n}, \dots, u_{Nn})$  of the unique positive solution of (7) with  $(1 - \lambda_n)(g_1, \dots, g_N) + \lambda_n(\bar{g}_1, g_2, \dots, g_N)$ . Then by the elliptic theory, there is  $(u_{10}, u_{20}, \dots, u_{N0})$  such that  $(u_{1n}, u_{2n}, \dots, u_{Nn})$  converges to  $(u_{10}, u_{20}, \dots, u_{N0})$  uniformly and  $(u_{10}, u_{20}, \dots, u_{N0})$  is a solution of (7) with  $(1 - \lambda_s)(g_1, \dots, g_N) + \lambda_s(\bar{g}_1, g_2, \dots, g_N)$ . We claim that  $u_{i0}$  is positive for  $i = 1, \dots, N$ .

By the same proof as in the previous section, for each  $n = 1, 2, \dots$ , there is a positive solution  $((u_1)_n, \dots, (u_N)_n)$  to

$$\begin{cases} \Delta u_i + u_i(g_i)_{u_i}(u_1, \dots, u_N) = 0, & \text{in } \Omega, \\ u_i|_{\partial\Omega} = 0, i = 1, \dots, N \end{cases}$$

with  $(1 - \lambda_n)(g_1, \dots, g_N) + \lambda_n(\bar{g}_1, g_2, \dots, g_N)$  such that  $(\tilde{u}_1)_n \leq (u_1)_n, \dots, (u_N)_n \leq (u_N)_n$ .

By the exactly same proof as in [5], there are uniformly convergent subsequences of  $(\tilde{u}_1)_n, \dots, (u_N)_n$ , which again will be denoted by  $(\tilde{u}_1)_n, \dots, (u_N)_n, (\tilde{u}_1)_n \rightarrow \tilde{u}_1, \dots, (u_N)_n \rightarrow u_N, (\tilde{u}_1, \dots, u_N)$  is a solution to (6) with

$$((g_1)_{u_1}, \dots, (g_N)_{u_N}), \text{ and } \tilde{u}_1 > 0, \dots, \tilde{u}_n > 0.$$

But, since  $(\tilde{u}_1)_n \leq (u_1)_n, \dots, (u_N)_n \leq (u_N)_n, 0 < \tilde{u}_1 \leq u_{10}, \dots, 0 < \tilde{u}_N \leq u_{N0}$ . We claim that (7) has a unique coexistence state with  $(1 - \lambda_s)(g_1, \dots, g_N) + \lambda_s(\bar{g}_1, g_2, \dots, g_N)$ . In fact, if not, assume that  $(\tilde{u}_{10}, \dots, u_{N0}) \neq (u_{10}, \dots, u_{N0})$  is another coexistence state. By the Implicit Function Theorem, there exists  $0 < \tilde{a} < \lambda_s$  and very close to  $\lambda_s$ , (7) with  $(1 - \tilde{a})(g_1, \dots, g_N) + \tilde{a}(\bar{g}_1, g_2, \dots, g_N)$  has a coexistence state very close to  $(\tilde{u}_{10}, \dots, u_{N0})$  which means that (7) with  $(1 - \tilde{a})(g_1, \dots, g_N) + \tilde{a}(\bar{g}_1, g_2, \dots, g_N)$  has more than one coexistence state. This is a contradiction to the definition of  $\lambda_s$ . But, since (7) with  $(1 - \lambda_s)(g_1, \dots, g_N) + \lambda_s(\bar{g}_1, g_2, \dots, g_N)$  has a unique coexistence state and the Frechet derivative is invertible, Theorem 4.1 concluded that  $\lambda_s$  can not be as defined. Therefore, for each  $(g_1, \dots, g_N) \in \Gamma$ , (7) with  $(g_1, \dots, g_N)$  has a unique coexistence state  $(u_1, \dots, u_N)$ . Furthermore, by the assumption, for each  $(g_1, \dots, g_N) \in \Gamma$ , the Frechet derivative of (7) with  $(g_1, \dots, g_N)$  at the unique solution  $(u_1, \dots, u_N)$  is invertible. Hence, Theorem 4.1 concluded that there is an open neighborhood  $V_{(g_1, \dots, g_N)}$  of  $(g_1, \dots, g_N)$  in  $(C_B^2)^N$  such that if  $(\bar{g}_1, \dots, \bar{g}_N) \in V_{(g_1, \dots, g_N)}$ , then (7) with  $(\bar{g}_1, \dots, \bar{g}_N)$  has a unique coexistence state. Let  $W = \bigcup_{(g_1, \dots, g_N) \in \Gamma} V_{(g_1, \dots, g_N)}$ . Then  $W$  is an open set in  $(C_B^2)^N$  such that  $\Gamma \subseteq W$  and for each  $(\bar{g}_1, \dots, \bar{g}_N) \in W$ , (7) with  $(\bar{g}_1, \dots, \bar{g}_N)$  has a unique coexistence state.

### 6. Application

Within the academia of mathematical biology, extensive academic work has been devoted to investigation of the simple competition model, commonly known as the Lotka-Volterra competition model. This system describes the competitive interaction of two species residing in the same environment in the following manner:

$$\begin{cases} u_t(x, t) = \Delta u(x, t) + u(x, t)(a - bu(x, t) - cv(x, t)) \\ v_t(x, t) = \Delta v(x, t) + v(x, t)(d - fv(x, t) - eu(x, t)) \\ u(x, t)|_{\partial\Omega} = v(x, t)|_{\partial\Omega} = 0, \end{cases} \text{ in } \Omega \times R^+, \quad (8)$$

where  $\Omega$  is a bounded domain in  $R^n$ . Here,  $u(x, t)$  and  $v(x, t)$  designate the population densities for the two competing species. The positive constant coefficients in this system represent growth rates ( $a$  and  $d$ ), self-limitation rates ( $b$  and  $f$ ) and competition rates ( $c$  and  $e$ ). Furthermore, we assume that both species are not residing on the boundary of  $\Omega$ .

The mathematical community has already established several results for the existence, uniqueness and stability of the positive steady state solution to (8) (see [1], [2], [3], [4]). The positive steady state solution is simply the positive solution to the time-independent system

$$\begin{cases} \Delta u(x) + u(x)(a - bu(x) - cv(x)) = 0 \\ \Delta v(x) + v(x)(d - fv(x) - eu(x)) = 0 \\ u(x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = 0. \end{cases} \quad \text{in } \Omega, \quad (9)$$

One of the important initial results for the time-independent Lotka-Volterra model was obtained by Cosner and Lazer. In 1984, they published the following sufficient conditions for the existence and uniqueness of a positive steady state solution to (9):

**Theorem 6.1.** (see [4]) *Suppose:*

(A)  $a > \lambda_1 + \frac{cd}{f}$ ,  $d > \lambda_1 + \frac{ae}{b}$ , where  $\lambda_1$  is the smallest eigenvalue of  $-\Delta$  with homogeneous boundary conditions,

(B)  $4bf > \frac{fc^2}{b} \sup_{x \in \Omega} [\frac{\omega_a(x)}{\omega_{d - \frac{ae}{b}}(x)}] + 2ce + \frac{be^2}{f} \sup_{x \in \Omega} [\frac{\omega_d(x)}{\omega_{a - \frac{cd}{f}}(x)}]$ , where  $\omega_M(x)$  for  $M > 0$  is the unique positive solution to the logistic equation as mentioned in the next section.

Then (2) has a unique positive solution.

Cosner and Lazer's theorem implies that if the self-reproduction and self-limitation rates are relatively large, and the competition rates are relatively small, then there is a unique positive steady state solution to (9). In other words, the two species will coexist indefinitely at unique population densities.

In 1989, Cantrell and Cosner extended these results by proving that the reproduction and self-limitation rates may vary within bounds without losing the uniqueness result, given certain conditions. Biologically, Cantrell and Cosner's theorem suggests that two species can relax ecologically and maintain a coexistence state. Their primary result is given below:

**Theorem 6.2.** (see [3]) *If  $a = d > \lambda_1$ ,  $b = f = 1$ , and  $0 < c, e < 1$ , then there is a neighborhood  $V$  of  $(a, a)$  such that if  $(a_0, d_0) \in V$ , then (9) with  $(a, d) = (a_0, d_0)$  has a unique positive solution.*

In Theorem 6.2, the condition  $0 < c, e < 1$  biologically implies that the competition rates of both species must be relatively small. This condition plays an important role in the proof of Cantrell and Cosner's theorem by implying the invertibility of the Frechet derivative (linearization) of (9) at a fixed reproduction rate  $(a, a)$ .

### References

- [1] S.W. Ali and C. Cosner, On the uniqueness of the positive steady state for Lotka-Volterra models with diffusion, *Journal of Mathematical Analysis and Application*, **168** (1992), 329-341.
- [2] R.S. Cantrell and C. Cosner, On the steady-state problem for the Volterra-Lotka competition model with diffusion, *Houston Journal of Mathematics*, **13** (1987), 337-352.
- [3] R.S. Cantrell and C. Cosner, On the uniqueness and stability of positive solutions in the Volterra-Lotka competition model with diffusion, *Houston Journal of Mathematics*, **15** (1989), 341-361.
- [4] C. Cosner and A.C. Lazer, Stable coexistence states in the Volterra-Lotka competition model with diffusion, *SIAM J. Appl. Math.*, **44** (1984), 1112-1132.
- [5] J. Kang, Perturbation of a nonlinear elliptic biological interacting model with multiple species, *Communications in Mathematics and Applications*, **2** (2011), no. 2-3, 61-76.
- [6] K. Lizarraga, J. Kang, J. Lee, Perturbation of a nonlinear elliptic biological interacting model, *Dynamics of PDE*, **3**, No. 4 (2006), 281-293.
- [7] J. Lee, J. Kang, Y. Oh, The existence, nonexistence and uniqueness of global positive coexistence of a nonlinear elliptic biological interacting model, *Kangweon-Kyungki Math. Jour.*, **12**, No. 1 (2004), 77-90.