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TUBULAR SURFACES AROUND A TIMELIKE FOCAL CURVE IN MINKOWSKI 3-SPACE

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Abstract: A canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(t)$ (spine curve) of its center and a radius $r(t)$ and it is parametrized through Frenet frame of the spine curve $C(t)$. If the radius function $r(t) = r$ is a constant, then the canal surface is called a tube or tubular surface [1]. The focal curve of an immersed smooth curve $\alpha : I \longrightarrow E^3$ in Euclidean space consists of the centres of its osculating hyperspheres. This curve may be parametrised in terms of the Frenet frame of α (T, N, B), as $C_\alpha = (\alpha + c_1 N + c_2 B)$, where the coefficients c_1, c_2 are smooth functions that we call the focal curvatures of α [6]. In this work, we initially gave characterization of canal and tubular surfaces around a timelike focal curves in Minkowski 3-Space, afterwards we investigated the curvatures of tubular surfaces around a timelike focal curves in Minkowski 3-Space.

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1. Preliminaries

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, and let $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The Lorentz scalar

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product of x and y is defined by

$$< x, y >= x_1 y_1 + x_2 y_2 - x_3 y_3,$$

$E_1^3 = \left(\mathbb{R}^3, <,>\right)$ is called 3-dimensional Lorentzian space, Minkowski 3-Space or 3−dimensional semi-Euclidean space. A vector $x \in \mathbb{R}_1^3$ is called

$$\text{spacelilke if} \quad < x, x >> 0 \quad \text{or } x = 0,$$

$$\text{null if} \quad < x, x >= 0 \quad \text{and } x \neq 0,$$

$$\text{timelike if} \quad < x, x >< 0,$$

the norm $\|x\|$ of a vector $x \in \mathbb{R}_1^3$ is $|< x, x >|^{\frac{1}{2}}$, two vectors x and y in $\mathbb{R}_1^3$ are said to be orthogonal, if $< x, y >= 0$. x is called a unit vector if $\|x\| = 1$. Similarly, an arbitrary curve α in E_1^3 is locally spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(s) = T(s)$ are respectively spacelike, timelike or null, for each $s \in I \subset \mathbb{R}$. For any $x, y \in \mathbb{R}_1^3$,Lorentzian vectoral product of x and y is defined by

$$x \times y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_2 y_1 - x_1 y_2).$$

The Lorentzian sphere of center $m = (m_1, m_2, m_3)$ and radius $r \in \mathbb{R}^+$ in the space E_1^3 is defined by [4]

$$S_1^2 = \left\{a = (a_1, a_2, a_3) \in \mathbb{E}_1^3 \,|< a - m, a - m >= r^2, a_1, a_2, a_3 \in \mathbb{R}\right\}.$$

Consider a curve α in $\mathbb{E}_1^3$, parameterized by its arclength s. Let $T(s)$ be its tangent vector, i.e., $T(s) = \alpha'(s) = \dfrac{d\alpha(s)}{ds}$. The arclength parameterization of the curve makes $T(s)$ a unit vector, i.e.,$\|T(s)\| = 1$, therefore its derivative is orthogonal to T. The principal normal vector N is defined as $N = \dfrac{T'(s)}{\|T'(s)\|}$. The binormal vector B is defined as the cross product $B = N \times T$. The Frenet-Serret equations, express the rate of change of the moving orthonormal triad $\{T, N, B\}$ along a base spacelike curve $\alpha(s)$ with a spacelike normal in Minkowski 3-Space,

$$\left.\begin{array}{l} T' = \kappa_\alpha N, \\ N' = -\kappa_\alpha T + \tau_\alpha B, \\ B' = \tau_\alpha N \end{array}\right\} \tag{1}$$

where the coefficients κ_α and τ_α are curvature and torsion of the α, respectively [6].

The curve's torsion is a function of the curve's third derivative as evidenced by the explicit formulas [4].

$$\kappa = \frac{\|x' \wedge x''\|}{\|x'\|}, \qquad \tau = \frac{\det(x', x'', x''')}{\|x' \wedge x''\|^2}.$$

2. Characterization of Canal and Tubular Surfaces Around a Timelike Focal Curve

Our purpose in this section, we will obtain the tubular surface from the canal surface around a timelike focal curve in Minkowski 3-space. A canal surface is defined as the envelope of a family of one parameter spheres. Alternatively, a canal surface is the envelope of a moving sphere with varying radius, defined by the trajectory $C(t)$ of its center and a radius function $r(t)$. This moving sphere $S(t)$ touches the canal surface at a characteristic circle $K(t)$. If the radius function $r(t) = r$ is a constant, then the canal surface is called a tube or pipe surface [1].

Definition 2.1. Let (x_1, x_2, x_3) be semi-Euclidean coordinate system in $\mathbb{E}_1^3$. We take a Lorentzian sphere $< X - C, X - C >= r^2$, with origin and radius C_α and r, respectively, where $X = (x_1, x_2, x_3)$. Let $f(s) = \frac{1}{2}\left(\|C_\alpha - \alpha\|^2 - r^2\right)$. If we have the following equations

$$f = f^{'} = f^{''} = f^{'''} = 0, \tag{2}$$

then we say that the sphere passing trough four consecutive points the curve is called osculating Lorentzian sphere.

Definition 2.2. For any unit speed curve $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}_1^3$, the focal curve C_α may be parameterized using the Frenet frame $(T(s), N(s), B(s))$ of α as follows:

$$C_\alpha = \alpha + c_1 N + c_2 B, \tag{3}$$

where the coefficients c_1, c_2 are smooth functions that are called focal curvatures of α.

Lemma 2.3. *[5] Let $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}_1^3$ is any unit speed spacelike curve with a spacelike principal normal , the focal curvatures are*

$$c_1 = \frac{1}{\kappa_\alpha}, c_2 = -c_1^{'}\frac{1}{\tau_\alpha}, \kappa_\alpha \neq 0, \tau_\alpha \neq 0 \tag{4}$$

Definition 2.4. A vertex of a curve in $\mathbb{E}_1^3$ is a point at which the curve has at least 5-point contact with its osculating Lorentzian sphere.

Lemma 2.5. *A non-flattening point of any unit speed spacelike curve with a spacelike principal normal $\alpha = \alpha(s)$ in $\mathbb{E}_1^3$ is a vertex if and only if*

$$c_2^{\prime} + c_1\tau_\alpha = 0,$$

at that point .

Proof. The sphere of radius r with centre at C_α is defined by the equation.

$$f(s) = \frac{1}{2}\left(\|C_\alpha - \alpha\|^2 - r^2\right).$$

So a point C_α is defined as the centers of the osculating spheres of α having 5-point contact with α at $\alpha(s_0)$ if and only if the function $f o \alpha$ has a zero of multiplicity 5 at $s = s_0$, i.e. if and only if the function $f(s)$ has a critical point of multiplicity 5 at s_0, i.e.

$$f(s_0) = f'(s_0) = f''(s_0) = f'''(s_0) = f^{(4)}(s_0) = 0.$$

If the operations is made, we obtain $c_2' + c_1\tau_\alpha = 0$. □

Definition 2.6. A parametrised curve $\alpha = \alpha(s)$ in $\mathbb{E}_1^3$ is said to be good if its derivatives of order $1, 2,$ are linearly independent at any point.

Theorem 2.7. *Let $\alpha : I \longrightarrow \mathbb{E}_1^3$ be a good unit speed spacelike curve with a spacelike principal normal without its flattening. Write κ_α and τ_α for its curvatures and $\{T, N, B\}$ for its Frenet frame. For each non-vertex $\alpha(s)$ of α, write $\varepsilon_{\mathbf{t}} = \dfrac{c_2' + c_1\tau_\alpha}{|c_2' + c_1\tau_\alpha|}$, $\varepsilon_{\mathbf{n}} = \dfrac{\tau_\alpha}{|\tau_\alpha|}$ and $\varepsilon_{\mathbf{b}} = \dfrac{\tau_c}{|\tau_c|}$. For any non-vertex of α the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ of C_α is well-defined and its vectors are given by*

$$\begin{aligned} \mathbf{t} &= \varepsilon_{\mathbf{t}} B, \\ \mathbf{n} &= \varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}} N, \\ \mathbf{b} &= -\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}\varepsilon_{\mathbf{b}} T. \end{aligned} \tag{5}$$

Proof. Write $\sigma(s)$ for the value of the arc length parameter of C_α at $C_\alpha(s)$. We assume that the orientations of the parametrised focal curve. C_α given by the arc length parameter s of α and by the arc length parameter σ of C_α coincide. At a non-vertex of α, the unit tangent vector of the parametrised focal curve C_α is

$$\frac{dC_\alpha}{ds} = \frac{dC_\alpha}{d\sigma}\frac{d\sigma}{ds} = \left[c_2' + c_1\tau_\alpha\right] B, \tag{6}$$

Taking the norm of both sides of 6, we obtain

$$\frac{ds}{d\sigma} = \frac{1}{|c_2' + c_1\tau_\alpha|}, \tag{7}$$

and

$$\frac{dC_\alpha}{d\sigma} = \mathbf{t} = \frac{(c_2' + c_1\tau_\alpha)}{|c_2' + c_1\tau_\alpha|} B = \varepsilon_{\mathbf{t}} B. \tag{8}$$

In order to obtain that

$$\mathbf{n} = \varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}} N \tag{9}$$

and

$$\kappa_c = \frac{|\tau_\alpha|}{|c_2' + c_1\tau_\alpha|},$$

derive equation 8 with respect to σ. In the same way, use equation9 to obtain

$$\mathbf{b} = -\varepsilon_{\mathbf{t}}\varepsilon_{\mathbf{n}}\varepsilon_{\mathbf{b}} T, \qquad \kappa_\alpha = |\tau_c|\,|c_2' + c_1\tau_\alpha|.$$

□

Corollary 2.8. *If radius of osculating sphere r is constant, $c_2 = c_2' = 0$. Then 1) $\varepsilon_{\mathbf{t}} = \varepsilon_{\mathbf{n}}$ end in this case*

$$\begin{aligned} \mathbf{t} &= \varepsilon_{\mathbf{t}} B, \\ \mathbf{n} &= N, \\ \mathbf{b} &= -\varepsilon_{\mathbf{b}} T. \end{aligned}$$

2) $\kappa_c = \dfrac{|\tau|}{|c_2' + c_1\tau|} = \kappa_\alpha$.
3) If the C_α is a right-handed curve $\varepsilon_{\mathbf{t}} = \varepsilon_{\mathbf{n}} = \varepsilon_{\mathbf{b}} = 1$ end in this case

$$\begin{aligned} \mathbf{t} &= B, \\ \mathbf{n} &= N, \\ \mathbf{b} &= -T. \end{aligned} \tag{10}$$

4) If $\alpha : I \longrightarrow \mathbb{E}_1^3$ be a good unit speed spacelike curves with a spacelike principal normal curve, the focal curve C_α, locus of the curvature center for α, is a timelike curve.

After this we will admit that the $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ orthonormal frame field forms right handed system and is {time, space, space} type.

Proposition 2.9. *A curve $\alpha = \alpha(s) : I \longrightarrow \mathbb{E}_1^3$ is Lorentzian spherical, i.e., it is contained in a Lorentzian sphere of radius r, if and only if*

$$r^2 = c_1^2 - c_2^2. \tag{11}$$

This means that the curve is Lorentzian spherical if and only if

$$c_2' + c_1\tau_\alpha = 0 \tag{12}$$

holds, where

$$\left(r^2\right)' = -2c_2\left(c_2' + c_1\tau_\alpha\right). \tag{13}$$

The focal curvatures c_1, c_2 of α satisfy the following Frenet equations:

$$\begin{bmatrix} 1 \\ c_1' \\ c_2' + \dfrac{\left(r^2\right)'}{2c_2} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & -\tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} 0 \\ c_1 \\ c_2 \end{bmatrix} \tag{14}$$

where r is the radius of the osculating sphere of α. If the curve α is spherical, i.e., lies on a sphere, then the last component of the left hand side vector of equation only consists of c_2'. We can give the following corollaries.

Corollary 2.10. *[7] Let κ_c and τ_c (resp. κ_α and τ_α) be the curvatures of C_α (resp. of α). Then*

$$\frac{\kappa_c}{|\tau_\alpha|} = \frac{|\tau_c|}{\kappa_\alpha} = \frac{1}{|c_2' + c_1\tau_\alpha|} = \frac{2\,|c_2|}{\left|(r^2)'\right|}. \tag{15}$$

Lemma 2.11. *If radius of osculating Lorentzian sphere r is constant, curvature of the $\alpha(s)$, κ_α is constant and $r = c_1$.*

Proof. According to 13, if radius of osculating Lorentzian sphere r is constant, $c_2 = 0$ or $c_2' + c_1\tau_\alpha = 0$. If $c_2' + c_1\tau_\alpha = 0$, the curve is spherical. If $c_2 = 0$, $-c_1' \frac{1}{\tau_\alpha} = 0$. Namly $c_1 = \dfrac{1}{\kappa_\alpha}$ is constant. □

Lemma 2.12. *Let κ_c and τ_c are curvature and torsion of the the focal curve $C_\alpha(s)$, respectively. The derivative variations with respect to s of the Frenet frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ are*

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \begin{bmatrix} 0 & \nu\kappa_c & 0 \\ \nu\kappa_c & 0 & \nu\tau_c \\ 0 & -\nu\tau_c & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}, \tag{16}$$

where for a non-vertex of $\alpha(s)$, $\nu = \dfrac{d\sigma}{ds} = \left|c_2' + c_1\tau_\alpha\right|$. If radius of osculating Lorentzian sphere r is constant,

$$\nu = \frac{d\sigma}{ds} = r\tau_\alpha \tag{17}$$

We can give about characterization of canal and tubular surfaces around a timelike focal curves with the following theorem.

Theorem 2.13. *Suppose $\alpha : I \longrightarrow \mathbb{E}_1^3$ is a unit speed spacelike curve with a spacelike principal normal, nonzero curvature. Then the canal surface around its timelike focal curve can be parametrized as follows,*

$$K(s,t) = C_\alpha(s) + \varepsilon_{\mathbf{t}} r(s) r'(s) B \mp r(s)\sqrt{1 + (r'(s))^2}\,(T(s)\cos t \pm N(s)\sin t)\,. \tag{18}$$

Proof. Let K denote a patch that parametrizes the envelope of Lorentzian spheres defining the canal surface. Since the curvatures of α and C_α are nonzero, the Frenet- Serret frame $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ and $\{T, N, B\}$ are well-defined, and we can write

$$K(s,t) - C_\alpha(s) = -a(s,t)\mathbf{b} + b(s,t)\mathbf{n} + c(s,t)\mathbf{t}$$

or according to 10

$$K(s,t) - C_\alpha(s) = a(s,t)T + b(s,t)N + c(s,t)B \tag{19}$$

because the $K(s,t)-C_\alpha(s)$ vector can always be written in terms of $\{T, N, B\}$ basis vectors, where a,b and c are differentiable on the interval on which α is defined. We must have

$$\|K(s,t) - C_\alpha(s)\|^2 = r^2(s) \tag{20}$$

Equation 20 expresses analytically the geometric fact that $K(s,t)$ lies on a Lorentzian sphere $S_1^2(s)$ of radius $r(s)$ centered at $C_\alpha(s)$. Furthermore, $K(s,t) - C_\alpha(s)$ is a normal vector to the canal surface; this fact implies that

$$< \ K(s,t) - C_\alpha(s), K_s >= 0, \tag{21}$$

$$< \ K(s,t) - C_\alpha(s), K_t >= 0. \tag{22}$$

Equations 21 and 22 say that the vectors K_s and K_t are tangent to $S_1^2(s)$. From 19 and 20 we get

$$\left.\begin{array}{r} a^2+b^2-c^2=r^2, \\ aa_s+bb_s-cc_s=rr', \end{array}\right\} \tag{23}$$

When we differentiate 19 with respect to s and use the Frenet-Serret formulas, we obtain

$$K_s = (a_s - b\kappa_\alpha)\, T + (a\kappa_\alpha + b_s + c\tau_\alpha)\, N + (\nu + b\tau_\alpha + c_s)\, B. \tag{24}$$

Then 23, 24, 19 and 21 imply that

$$c = rr', \tag{25}$$

and from 23 and 24 we get

$$a^2 + b^2 = r^2\left(1 + (r')^2\right). \tag{26}$$

We can write

$$\begin{aligned} a &= \mp r\sqrt{1+(r')^2}\cos t, \\ b &= \mp r\sqrt{1+(r')^2}\sin t. \end{aligned}$$

Thus 19 becomes

$$K(s,t) = C_\alpha(s) + r(s)r'(s)B \mp r(s)\sqrt{1+(r'(s))^2}\,(T(s)\cos t \pm N(s)\sin t)\,,$$

where $\varepsilon_{\mathbf{t}} = \mp 1$. With the Frenet-Serret system in hand, we can construct a "tubular surface" of radius $r = const.$ about the curve by defining a surface with parameters s and t;

$$L(s,t) = C_\alpha(s) + r(s)T(s)\cos t + r(s)N(s)\sin t. \tag{27}$$

□

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3. The Curvatures Of Tubular Surfaces Around a Timelike Focal Curve

For $L(s,t)$ the tubular surface around a timelike curve of $\alpha(s)$ from 27 the coefficients of the first and second fundamental form are given by

$$\begin{aligned}
L_t &= -r\sin tT + r\cos tN,\ r=\frac{1}{\kappa_\alpha}\\
L_s &= -\sin tT + \cos tN - r\tau_\alpha\left(1+\sin t\right)B,\\
L_{tt} &= -r\cos tT - r\sin tN,\\
L_{ss} &= -\kappa_\alpha\cos tT + \left[-\kappa_\alpha\sin t + r\tau_\alpha\left(1+\sin t\right)\right]N\\
&\quad + \left[\tau_\alpha\cos t + r\tau'_\alpha\left(1+\sin t\right)\right]B\\
L_{st} &= -\cos tT - \sin tN + r\tau_\alpha\cos tB,\\
L_s\times L_t &= r^2\tau_\alpha\left(1+\sin t\right)\left(\cos tT+\sin tN\right)
\end{aligned}$$

the surface normal vector

$$U=\frac{L_s\times L_t}{\|L_s\times L_t\|}=\cos tT+\sin tN$$

and

$$\begin{aligned}
E &= \langle L_s, L_s\rangle = 1 - r^2\tau_\alpha^2\left(1+\sin t\right)^2,\\
F &= \langle L_s, L_t\rangle = r,\\
G &= \langle L_t, L_t\rangle = r^2,\\
e &= \langle U, L_{ss}\rangle = -\kappa_\alpha - r\tau_\alpha\sin t\left(1+\sin t\right),\\
f &= \langle U, L_{st}\rangle = -1,\\
g &= \langle U, L_{tt}\rangle = -r
\end{aligned}$$

and

$$\|L_s\times L_t\|^2 = EG-F^2 = r^4\tau_\alpha^2\left(1+\sin t\right)^2. \tag{28}$$

Corollary 3.1. $L(s,t)=C_\alpha(s)+r(s)T(s)\cos t+r(s)N(s)\sin t$ *tubular surface is a timelike surface.*

Definition 3.2. [1] Let Q is any surface. If $EG-F^2\neq 0$, Q is called a regular surface.

According to definetion and by Eq 28 $L(s,t)$ is a regular tube if and only if $\tau_\alpha\left(1+\sin t\right)\neq 0$. Namely

$$\left.\begin{aligned}\tau_\alpha\neq 0,\\ \sin t\neq -1.\end{aligned}\right\} \tag{29}$$

The Gaussian and mean curvature for a regular tube $L(s,t)$ are computed as

$$\left.\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} = -\frac{\sin t}{r^2 \tau_\alpha (1 + \sin t)}, \\ H &= \frac{eG - 2fF + gE}{2(EG - F^2)} = -\frac{1}{2}\left[\frac{1}{r} - rK\right]. \end{aligned}\right\} \tag{30}$$

Theorem 3.3. *[3] A curve γ lying on a surface is an asymptotic curve if and only if the acceleration vector γ'' is tangent to the surface that is $< U, \gamma'' >= 0$.*

Theorem 3.4. *Let $L(s,t)$ be a regular tube around a timelike focal curve of $\alpha(s)$. In that case, we have the following.*
a) The s-parameter curves of $L(s,t)$ are asymptotic curves if and only if

$$\kappa_\alpha^2 = -\tau_\alpha \sin t (1 + \sin t).$$

b) The $t-$parameter curves of $L(s,t)$ cannot be asymptotic curves. In that case, neither the s-parameter curves of $L(s,t)$ nor the $t-$parameter curves of $L(s,t)$ be asymptotic curves.

Proof. a) For the $s-$parameter curves we have

$$e =< U, L_{ss} >= -\kappa_\alpha - r\tau_\alpha \sin t (1 + \sin t).$$

From this, we get

$$\kappa_\alpha^2 = -\tau_\alpha \sin t (1 + \sin t).$$

for $s-$parameter curves.
b) For the $t-$parameter curves we have

$$g =< U, L_{tt} >= -r \neq 0,$$

$t-$parameter curves cannot be asymptotic
Here, the equation $\kappa_\alpha^2 = -\tau_\alpha \sin t (1 + \sin t)$ is satisfied for a circular helix $C_\alpha(s)$. In the case of general helix, we get the curvature of spine curve $C_\alpha(s)$ as

$$\kappa_\alpha = -\tan\beta \sin t (1 + \sin t), \tag{31}$$

where β is the angle between tangent line T and the fixed direction of the general helix. $\dfrac{\tau_\alpha}{\kappa_\alpha} = \tan\beta$ is a constant for a general helix. Hence, if we substitute this in the equation $\kappa_\alpha^2 = -\tau_\alpha \sin t (1 + \sin t)$, it gathers that

$$\kappa_\alpha \cot\beta = -\sin t (1 + \sin t). \tag{32}$$

In this situation, we obtain the curvature as $\kappa_\alpha = -\tan\beta \sin t\,(1+\sin t)$. Because t and β are constants, it follows that $\kappa_\alpha(s)$ is a constant. Therefore,

$$\tau_\alpha = \kappa_\alpha \tan\beta = -\tan^2\beta \sin t\,(1+\sin t) \tag{33}$$

is also a constant. We see that the general helix becomes a circular helix and finally the equation is satisfied for a circular helix. □

Theorem 3.5. *[2] A curve γ lying on a surface is a geodesic curve if and only if the acceleration vector γ'' is normal to the surface. This means that γ'' and the surface normal U are linearly dependent namely $U \times \gamma'' = 0$.*

Theorem 3.6. *Let $L(s,t)$ be a regular tube around timelike focal curves of $\alpha(s)$. In that case, we have the following.*
a) The $t-$parameter curves of $L(s,t)$ are geodesic curves.
b) The $s-$parameter curves of $L(s,t)$ are geodesic curves if and only if the curvatures of $\alpha(s)$

$$\tau_\alpha = e^{-\frac{\cos t}{r(1+\sin t)}s+c}, c \in \mathbb{R}.$$

Proof. For the $s-$and $t-$ parameter curves we conclude

$$\begin{aligned} U \times L_{tt} &= (\cos tT + \sin tN) \times (-r\cos tT - r\sin tN) = 0, \\ U \times L_{ss} &= \sin t\left(\tau_\alpha \cos t + r\tau'_\alpha (1+\sin t)\right) T \\ &\quad - \cos t\left(\tau_\alpha \cos t + r\tau'_\alpha (1+\sin t)\right) N \\ &\quad - \cos t\left(-\kappa_\alpha \sin t + r\tau_\alpha (1+\sin t)\right) B, \end{aligned}$$

a) As immediately seen above, $t-$parameter curves of $L(s,t)$ are geodesics.
b) Since $\{T,N,B\}$ is an orthonormal basis and $U \times L_{ss} = 0$, there are

$$\begin{aligned} \sin t\left(\tau_\alpha \cos t + r\tau'_\alpha (1+\sin t)\right) &= 0, \\ \cos t\left(\tau_\alpha \cos t + r\tau'_\alpha (1+\sin t)\right) &= 0, \\ \cos t\left(-\kappa_\alpha \sin t + r\tau_\alpha (1+\sin t)\right) &= 0. \end{aligned} \tag{34}$$

By the first two equations we have

$$\tau_\alpha \cos t + r\tau'_\alpha (1+\sin t) = 0. \tag{35}$$

and by using a simple calculation

$$\tau_\alpha = e^{-\frac{\cos t}{r(1+\sin t)}s+c}, c = \text{constant}. \tag{36}$$

□

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