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CONSTRUCTING A MOORE-PENROSE TWO RING NOT MOORE-PENROSE ONE

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Abstract: This paper manufactures from algebraic principles a Moore-Penrose Two (MP2) ring that is not Moore-Penrose One (MP1). Interestingly enough, it shall be shown that the class of MP1 rings is a subclass of the MP2 rings to a point of near saturation. However, the task remains arduous to produce MP2 rings that are not MP1. A combination of classical theorems on Noetherian rings –manipulating ascending chain condition (ACC), and Artinian rings using descending chain condition (DCC), as well as the semisimple property associated with the Jacobson radical will be manipulated. The purpose of the construction is to show that MP2 rings occupy a unique genre. Some hints of exploring this class of rings with geometric algebra will be given.

Key Words: semisimple, Noetherian, Artinian, ACC, DCC

1. Introduction

A great deal of tedious effort and rigorous research has gone into the task of finding an example of an MP2 ring that is not MP1. Recall a ring R is MP2 if for each nonzero a in R there exists a nonzero x in R such that xax= x. Likewise a ring R is MP1 if for each nonzero a in R there exist a nonzero x in R such that axa = a. It is immediately obvious that if R is MP1 then it is certainly MP2 as shall be demonstrated. Hence all MP1 rings are contained in the larger class of MP2 rings. A seemingly innocuous question: "Is this containment of MP1 rings inside MP2 rings proper?" Though this question has been an intractable one to answer either

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way, it has led to many good theorems and observations about the characteristics of both MP2 and MP1 rings. Given below is an excursion through Robert F. Shanny's paper on "Regular Endomorphism Rings Of Free Modules" [6] which surprisingly provide some subtle insight on necessary traits of an MP2 matrix ring not to be MP1 if such a matrix ring exists. It is an established fact that if R is an MP1 ring and F is a free R-module with finite basis, then HomR(F,F) is an MP1 ring. Shanny removes the restriction that the basis be finite but shows that HomR(F,F) is MP1 if and only if R is semisimple Artinian.

2. Semisimplicity and the Moore-Penrose Rings

As a short review, a ring R is semisimple if its Jacobson radical, denoted J(R) is the zero ideal, i.e., J(R) = 0. Since MP1 and MP2 rings abound with idempotents, the following Lemma assures that both MP1 and MP2 rings are semisimple:

Lemma 1. $x \in J(R)$ if and only if 1- xy is a unit for all $y \in R$.

Proof. (Sufficiency) Let $y \in R$; then for every maximal ideal M, 1 - xy$\notin$M as $x \in M$; thereby 1 - xy is a unit. (Necessity) If $x \notin M$ for some maximal ideal M of R, then $<x> + M = R$; if follows that 1 - xy is not a unit, which contradicts the hypothesis. Hence, $x \in J(R)$. □

Since an MP2 ring is semisimple, if there is an MP2 matrix ring which is not MP1, then surely it must not be Artinian for else it would underscore the sufficiency of Shanny's observation. Thus the class of infinite matrix rings come into an interesting focus as perhaps fertile grounds for discovering if there is an MP2 ring not MP1. Before investigating this approach, some cogent examples of MP2 are provided. Interestingly enough, the class of MP1 rings are a subclass of MP2 rings as noted. Thereby, the examples will be also MP1 rings. First, it will be shown that MP1 ring are MP2 rings and the subsequent examples will follow.

Theorem 2. *R MP1 then R is MP2.*

Proof: Suppose $a \in R$ and $a \neq 0$. Since R is MP1 there exists a nonzero x such that axa=a. Now set y=xax. Claim: $y \neq 0$. Otherwise aya=axaxa=axa=a=0 since y=0. "$\rightarrow \leftarrow$", Contradiction! Thus, $y \neq 0$. Now yay=xaxaxax=xaxax=xax=y. Since a was arbitrary, R is MP2!

Note. The class of MP1 rings were first described structurally by the mathematician John von Neumann as regular rings. [5] Much algebraic literature abounds on the intrinsic properties of regular (MP1) rings. Their characterizations in terms of idempotent elements exist in numerous books especially Goodearl.[4] Even though von Neumann is given credit for his incipient characterizations of regular rings (MP1), the following result is a generalization to matrix rings over regular rings (MP1) :

Theorem 3. *For a ring R with identity, the following statements are equivalent:*

1. *R is MP1.*
2. *For some integer k (k≥2), Mk(R) is MP1.*
3. *For every integer n (n≥2), Mn(R) is MP1.*

Proof. (3)→(2) is immediate. (2)→(1) Let A = diag (a_1, ... , a_k) where a_1 = a_2 = . . . = a_k = a∈R which is nonzero. Then A∈Mk(R). Since M_k(R) is MP1, ∃X∈M_k(R), X nonzero, such that AXA = A. Then for at least one x_{ii}∈R, (1≤i≤k), ax_{ii}a = a where x_{ii} is nonzero. Hence, R is MP1. (1)→ (3) : Let R be MP1. Then M_n(R) = EndR(⊕ΣRi) where each Ri≅R. But the endomorphism ring of a finitely generated projective module over a MP1 ring is a MP1 ring since eM_n(R)e is MP1 for any n and any idempotent e∈M_n(R). □

3. Mosaic of MP2 Rings

Now some illustrations of MP2 rings are given in the form of intriguing MP1 rings:

Example 1. Any domain with unit 1 is MP1 hence MP2, provided that it is a division ring. Thus, the rationals, Q; the reals, R; and the quaternions, **H** are MP1 as well as Z/(p), where p is a prime integer.

Example 2. M_n(F), where F is field, is a noncommutative MP1 ring (n≥2) and thus MP2.

Example 3. Hom_D(V,V) is an MP1 ring (and MP2) where D is a division ring and V is a finite dimensional vector space over D (V is a D module).

Example 4. If R is semisimple artinian then R is MP1 (and MP2). This is the classical Artin-Wedderburn Theorem. [1].

Example 5. A very curious subclass of MP1 or regular rings are called unit regular. The definition of unit-regular is given below:

Definition. A ring R is called unit-regular if for each nonzero a R there exists a unit u such that aua= a. Gertrude Ehrlich offers the following example of a unit-regular ring which is also MP2: "For m>1, the ring Z/(m) of integers modulo m is regular (hence unit-regular) if and only if m is square-free." [2].

Example 6. A McCoy example [5] that is not a direct sum of fields or matrix rings over division rings:

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Let S be the set of all subsets of a given nonempty set A, including the empty set $\emptyset$ and the entire set A. If a,b are in S, define ab to be a$\cap$b, the intersection of a and b. Also, a + b is defined to be the symmetric difference (a - b)$\cup$(b - a). Then S is a ring with unity. Obviously, S is MP1 since every element is idempotent. Also, nonzero idempotents are orthogonal if and only if they correspond to disjoint subsets of A. Note that S is not a direct sum of fields since different sets may have the same intersection with a given set thus violating uniqueness of inverses.

Example 7. Let C_n represent a Clifford algebra with 2^n elements; recall that 2^n 1 of the elements square to 1 and the 2^{nth} element, usually called the pseudoscalar, squares to 1. Now let C(s) denote the ring of s x s matrices with complex entries. C(s) is indeed MP1; hence it is MP2. The notation 2$\oplus$C(s) refer to the diagonal 2s x 2s matrix ring with typical entry: $\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$ where M,N are s x s matrices in C(s) and '0' represents an s x s matrix with zero entries. 2$\oplus$C(s) is also MP1, hence MP2. The algebraist Gerald Hile and Pertti Lounesto show in their research, "Matrix Representation of Clifford Algebras" [8], the following linear isomorphism that preserve the algebraic operations of a Clifford algebra Cn :

$$C_n \cong \begin{cases} C(2^{n/2}) & \text{if } n \text{ is even} \\ 2 \oplus C(2^{(1-n)/2}) & \text{if } n \text{ is odd} \end{cases}$$

After inspecting these examples, the question immediately arises if there are other properties which are sufficient for a ring R to be MP1 (hence MP2); K. R. Goodearl answers this question in the affirmative by noting [3] the following set of equivalent statements:

"**Theorem.** If R is commutative then the following statements are equivalent:

1. R is MP1.
2. RM is a field for all maximal ideals M of R.
3. R has no nonzero nilpotent elements, and all prime ideals of R are maximal.
4. All simple R modules are injective."

Theorem 4. *Assume R has a unit and no nonzero nilpotent elements. R is MP1 if and only if for all proper ideals I, R/I is MP2.*

Proof. Since R is MP1, any homomorphic image is MP1. Hence, for all proper ideals I, R/I is MP2. For any prime ideal P, R/P is MP2 also. Hence, R/P is an MP2 domain. Thus, R/P is a field and P is maximal. By Edward T. Wong's results, R is

MP1. [7] Indeed, it is curious to note that if there is an MP2 ring that is not MP1, then a candidate may be an MP2 ring with no nonzero nilpotent elements and a proper homomorphic image that is not MP2. Also, to violate the Artin-Wedderburn Theorem such a ring will necessarily be semisimple and most definitely not Artinian for else it will be a direct sum of matrix rings over division rings that will make it MP1 and thereby MP2. After considerable deliberations, the following examples are offered as MP2 rings not MP1: □

4. The Construction of an MP2 Ring Not MP1

Let M_2(F) be the ring of 2 x 2 matrices over the field F with charF≠2. Let R be (A_1,A_2, . . . ,A_n, B,B,. . .), the set of all countable sequences with A_1, . . . , A_n, B in M2(F), where n arbitrary, and B = $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$. For r in R, let T(r) = B. Obviously, R is a ring under component-wise addition and multiplication. However, R is not MP1. To see this, if T(x) = $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, then T(xrx) = 0 and x is not an element of xRx. Hence, R is not MP1. Note that any nonzero element of R has a multiple which is idempotent. To show R is MP2, first observe that for each A_i, there is an X_i such that $X_iA_iX_i$= X_i. If T(x) = $\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}$, set y = (X_1, X_2 , . . . , X_{n+1}, 0, . . .) where X_{n+1} = $\begin{bmatrix} b^{-1} & b^{-1} \\ b^{-1} & b^{-1} \end{bmatrix}$; (n+1 index uses premise n is arbitrary) then T(yxy-y) = 0 and R is MP2. Note that R is not Artinian since {(A_1, . . . , A_n, B, . . .)} ⊇ {(0, A_2, . . . , A_n, B, . . .)} ⊇ {(0, 0, A3, . . .,An,B,...)} ⊇ .....
Here's another scenario for an MP2 ring not MP1:

Theorem 5. *Let R be an MP1 ring and proper ideal of a ring S which is not MP1. The direct sum of R and S, R⊕S is MP2 but not MP1.*

Proof. Set R' = R⊕S and let a' = a⊕s (a≠0) be an arbitrary element of R' where a lies in R and s lies in S. Since R is MP1 (hence MP2), there exists an x in R such that xax = x. Designating x = x ⊕ 0 in R', then naturally x'a'x' = (x⊕0) (a⊕s) (x⊕0) = xax⊕0s0 = xax⊕0 = x⊕0 = x' . Now assume a = 0 and a' = 0⊕s in R' where s lies in S. For some r in R, rs lies in R since R is an ideal of S. Since R is MP1, there is an r' in R such that r'rsr' = r'. Let x' = 0⊕r'r. Then x'a'x' =

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(0⊕r'r) (0⊕s) (0⊕r'r) = 0⊕r'rsr'r = 0⊕r'r = x'. Hence, R' is MP2 in both cases. Furthermore, if R' is MP1 then S is MP1 since a direct summand of an MP1 ring is MP1. $\rightarrow \leftarrow$ Contradiction! Thus, R' is MP2 but not MP1. □

Since there are MP2 rings not MP1, then the investigation of MP2 rings is a credible pursuit especially to develop structural theorems of MP2 rings and there greater characterizations, for example, as matrix rings over that property. The challenge to offer MP2 rings not MP1 remain an enticing one with many theoretical implications.

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