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TIMELIKE MINIMAL SURFACE FAMILY PRESCRIBED BY A SPACELIKE CURVE IN MINKOWSKI 3-SPACE

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Abstract: In this paper, we investigate parametric representation of timelike minimal surface family prescribed a given spacelike curve in three dimensional Minkowski space. Also, we signify the condition of a given curve is a geodesic curve or asymptotic curve on timelike minimal surface. We derive some timelike minimal surfaces prescribed some spacelike curves as examples.

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Key Words: Minkowski space, timelike minimal surface, Frenet frame

1. Introduction

A surface which has zero constant mean curvature is called minimal surface in Euclidean space. In Minkowski 3-space R^3_1, a timelike surface with vanishing mean curvature is called timelike minimal surface. In [6], Weierstrass type representation formula for timelike minimal surface has been derived.

Minimal surfaces prescribed by a geodesic curve or asymptotic curve is known as Björling problem. Using the Frenet frame of a curve, minimal surface family prescribed a curve was studied in [2] in Euclidean space. In [1], Björling problem for timelike minimal surface has been studied in Minkowski 3-space.

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In this paper, we derive parametric representation of timelike minimal surface family prescribed a given spacelike curve. Utilizing the Frenet frame of a given curve we derive necessary and sufficient conditions for timelike minimal surface. We point out the condition of a given spacelike curve to be a geodesic curve or asymptotic curve on timelike minimal surface. Also, we present some examples for timelike minimal surface family prescribed by a given spacelike curve.

2. Preliminaries

In this section, we give some properties of Minkowski 3-space.

Let R_1^3 be three dimensional Minkowski space with metric $\langle , \rangle$. The metric $\langle , \rangle$ is expressed as

$$\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2$$

in terms of natural coordinates.

The vector product of vectors $U = (u_1, u_2, u_3)$ and $V = (v_1, v_2, v_3)$ in R_1^3 defined by

$$U \times V = \begin{vmatrix} x_1 & x_2 & -x_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Let $\alpha = \alpha(s)$ be unit speed curve in R_1^3. The Frenet frame of the curve α is defined by orthonormal basis $\{T, N, B\}$ where $T = \alpha'$, $N = \frac{\alpha''}{\|\alpha''\|}$, $B = T \times N$, they are called tangent, normal and binormal of the curve α, respectively. The curve α is called spacelike (timelike, null) if its tangent vector is spacelike (timelike, null) in everywhere. Frenet equations for a spacelike curve are

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \eta\kappa & 0 \\ -\kappa & 0 & -\eta\tau \\ 0 & -\eta\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

where $\eta = \langle N, N \rangle$ and $\kappa = \langle T', N \rangle$, $\tau = \langle N', B \rangle$ are curvature and torsion of the curve α [7, 5].

By utilizing the Frenet frame of the curve α, we express parametrization of the surface $P \subset R_1^3$ as

$$P(s,t) = \alpha(s) + u(s,t)T(s) + v(s,t)N(s) + w(s,t)B(s).$$

For $t = t_0$, $\alpha(s) = P(s, t_0)$ is a parameter curve on P and is called isoparametric curve on the surface P. If we calculate partial derivatives of $P(s,t)$ then we have,

$$P_s = (1 + u_s - \kappa v)T + (v_s + \eta(\kappa u - \tau w))N + (w_s - \eta\tau v)B$$

$$P_t = u_s T + v_t N + w_t B$$
$$P_{tt} = v_{tt} T + v_{tt} N + w_{tt} B$$

$$\begin{aligned} P_{ss} &= [(1 + u_s - \kappa v)_s - \kappa(v_s + \eta(\kappa u - \tau w))]T \\ &+ [(v_s + \eta(\kappa u - \tau w))_s + \eta\kappa(1 + u_s - \kappa v) - \eta\tau(v_s - \eta\tau v)]N \\ &+ [(w_s - \eta\tau v)_s - \eta\tau(v_s + \eta(\kappa u - \tau w))]B \end{aligned}$$

The unit normal $n(s,t)$ of the surface P is defined by

$$n(s,t) = \frac{P_s \times P_t}{\|P_s \times P_t\|}$$

where

$$P_s \times P_t = \varphi_1(s,t)T(s) + \varphi_2(s,t)N(s) + \varphi_3(s,t)B(s)$$

and

$$\begin{aligned} \varphi_1 &= ((v_s + \eta(\kappa u - \tau w))w_t - (w_s - \eta\tau v)v_t) \\ \varphi_2 &= \eta((w_s - \eta\tau v)u_t - (1 + u_s - \kappa v)w_s) \\ \varphi_3 &= \eta((v_s + \eta(\kappa u - \tau w))u_t - (1 + u_s - \kappa v)v_t). \end{aligned}$$

Kasap and Akyildiz [3] derive the condition the curveα is a geodesic curve on the surface P as

$$\begin{aligned} &\varphi_1(s,t_0) = 0, \\ &\varphi_2(s,t_0) \neq 0, \\ &\varphi_3(s,t_0) = 0. \end{aligned} \tag{2.1}$$

Saffak et al. [8] derive the condition the curve α is an asymptotic curve on the surface P as

$$w_t(s,t_0) = 0. \tag{2.2}$$

The mean curvature H of the surface P is

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} \tag{2.3}$$

where $e = \langle P_{ss}, n\rangle$, $f = \langle P_{st}, n\rangle$, $g = \langle P_{tt}, n\rangle$, $E = \langle P_s, P_s\rangle$, $F = \langle P_s, P_t\rangle$, $G = \langle P_t, P_t\rangle$ are the coefficient of the first and second fundemental forms.

3. Timelike Minimal Surface Prescribed Spacelike Curve

In this section, we investigate the necessary and sufficient condition of timelike minimal surfaces prescribed by a spacelike curve.

Let $P \subset R_1^3$ be a timelike surface. We know that there exist a isothermal parameter $P(s,t)$ on P with

$$\langle P_s, P_s\rangle = -\langle P_t, P_t\rangle \text{ and } \langle P_s, P_t\rangle = 0 \tag{3.1}$$

because the induced metric of a timelike surface is Lorentzian metric [4]. Then the mean curvature of P vanishes if and only if

$$P_{ss} - P_{tt} = 0. \tag{3.2}$$

For a spacelike curve we derive the following theorem and corollaries.

Theorem 1. *Let the spacelike curve $r \subset R_1^3$ be an isoparametric curve on the surface $P \subset R_1^3$, then P is a timelike minimal surface if and only if there exist the functions u, v, w which are satisfiying the following conditions.*

$$(1 + u_s - \kappa v)^2 + \eta(v_s + \eta(\kappa u - \tau w))^2 - \eta(w_s - \eta\tau v)^2 = -u_t^2 - \eta v_t^2 + \eta w_t^2 \tag{3.3}$$

$$(1 + u_s - \kappa v)u_t + \eta(v_s + \eta(\kappa u - \tau w))v_t - \eta(w_s - \eta\tau v)w_t = 0 \tag{3.4}$$

$$(1 + u_s - \kappa v)_s - \kappa(v_s + \eta(\kappa u - \tau w)) = u_{tt} \tag{3.5}$$

$$(v_s + \eta(\kappa u - \tau w))_s + \eta\kappa(1 + u_s - \kappa v) - \eta\tau(w_s - \eta\tau v) = v_{tt} \tag{3.6}$$

$$(w_s - \eta\tau v)_s - \eta\tau(v_s + \eta(\kappa u - \tau w)) = w_{tt} \tag{3.7}$$

Proof. Let the surface P be a timelike minimal surface. From (3.1) we have (3.3) and (3.4). Also, from (3.2) we have (3.5)-(3.7).

Conversely, let the equations (3.3)-(3.7) are satisfied. From (2.3), we have $H = 0$. □

Corollary 1. *Let the spacelike curve α be an isoparametric curve on the timelike minimal surface P. Then the curve α is a geodesic curve on the surface P if and only if*

$$w_t(s, t_0) = \pm 1 \tag{3.8}$$

$$\langle N, N\rangle = 1. \tag{3.9}$$

Proof. From isoparametric condition, we have

$$u(s, t_0) = v(s, t_0) = w(s, t_0) = 0$$

and

$$u_s(s, t_0) = v_s(s, t_0) = w_s(s, t_0) = 0.$$

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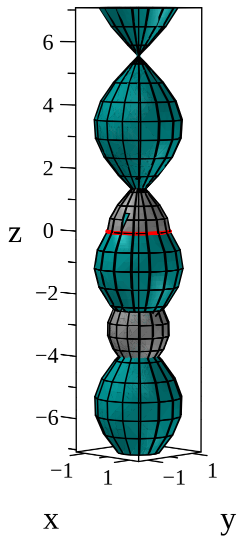

Figure 1: Timelike minimal surfaces $P_1(s,t;1)$, $P_1(s,t;\sqrt{2})$ and the spacelike curve α_1.

Let the conditions (3.8), (3.9) are satisfied. From (3.3) and (3.4) we have $v_t(s,t_0) = 0$. Then the conditions (2.1) are satisifed. Thus the spacelike curve α be a geodesic curve on the timelike minimal surface P.

Conversely, let the spacelike curve α be a geodesic curve on the timelike minimal surface P. From (2.1) we have $v_t(s,t_0) = 0$. Thus, from (3.3),(3.4) we have (3.8),(3.9). □

Corollary 2. *Let the spacelike curve α be an isoparametric curve on the timelike minimal surface P. Then the curve α is an asymptotic curve on the surface P if and only if*

$$v_t(s,t_0) = \pm 1 \tag{3.10}$$

$$\langle N, N\rangle = -1. \tag{3.11}$$

Proof. From (3.3),(3.4) and (2.2) we derive (3.10) and (3.11). □

Utilizing the above theorem and corollaries we present following examples. We take $t_0 = 0$ in the all examples

Example 1. Let $\alpha_1(s) = (a\sin\frac{s}{a}, a\cos\frac{s}{a}, 0)$ be a spacelike curve.

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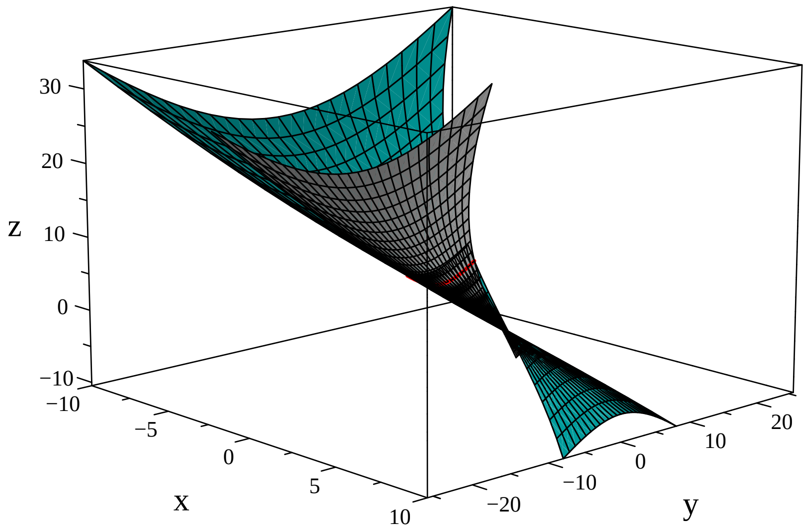

Figure 2: Timelike minimal surfaces $P_2(s,t;1)$, $P_2(s,t;2)$ and the spacelike curve α_2.

If we choose $u(s,t) = u(t)$, $v(s,t) = v(t)$, $w(s,t) = w(t)$ then from (3.6) and (3.7)

$$v(t) = c_1 \sin at - \frac{1}{a}\cos at + \frac{1}{a}$$

and

$$w(t) = ct$$

where $c \neq 0$ and c_1 are constant. From (3.5) and (3.4) $u(t) = 0$.

From (3.3) we have

$$v(t) = \pm\frac{\sqrt{c^2-1}}{a}\sin at - \frac{1}{a}\cos at + \frac{1}{a}.$$

Thus we have a family of catenoid 3nd kind prescribed the spacelike curve α as

$$P_1(s,t;|c|) = (\frac{1}{a}(\sqrt{c^2-1}\sin(at) - \cos(at))(-\sin\frac{s}{a}, -\cos\frac{s}{a}), ct).$$

For $c = \pm1$, the curve r is a geodesic curve on the timelike minimal surface $P_1(s,t;1)$ For a=1, we show two members of the timelike minimal surface family $P_1(s,t;|c|)$ in Figure (1).

Example 2. Let $\alpha_2(s) = (0, a\sinh\frac{s}{a}, a\cosh\frac{s}{a})$ be a spacelike curve.

If we choose $u(s,t) = 0$. Then, from (3.5) and (3.6) we have

$$v(s,t) = c_1 e^{\frac{t}{a}} + c_2 e^{\frac{-t}{a}} - a.$$

From (3.4) and (3.7) we have

$$w(s,t) = ct$$

where c and $c_1$are constant.

From (3.3)

$$v(t) = a(\frac{1 \pm \sqrt{1+c^2}}{2} e^{\frac{t}{a}} + \frac{1 \mp \sqrt{1+c^2}}{2} e^{\frac{-t}{a}} - 1).$$

Thus we have the timelike minimal surface family as

$$P_2(s,t;|c|) = (ct, a((\frac{1 \pm \sqrt{1+c^2}}{2}) e^{\frac{t}{a}} + \frac{1 \mp \sqrt{1+c^2}}{2} e^{\frac{-t}{a}}))(\sinh\frac{s}{a}, \cosh\frac{s}{a})).$$

For $c = 0$, the spacelike curve α is an asymptotic curve on the timelike plane $P_2(s,t;0)$. For a=4, we show two members of the timelike minimal surface family $P_2(s,t;|c|)$ in Figure (2).

4. Conclusion

We derive necessary and sufficient condition for timelike minimal surface family prescribed a given spacelike curve as differential equation systems. We point out that condition of a given spacelike curve is a geodesic curve or asymptotic curve on timelike minimal surface.

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