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NORMAL Γ^*-RING ON SEMIPRIME Γ-RING M WITH INVOLUTION

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Abstract: Let M be a 2-torsion free semiprime Γ-ring with involution satisfying the condition that $a\alpha b\beta c = a\beta b\alpha c$ ($a,\ b,\ c \in M$ and $\alpha, \beta \in \Gamma$). In this paper we will give the relation between normal and commutative Γ^*-ring on semiprime Γ-ring with involution.

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Key Words: Γ-ring M with involution, normal Γ-ring, commutative Γ-ring

1. Introduction

Let M and Γ be additive abelian groups. Define a mapping $M \times \Gamma \times M \longrightarrow M$ by $(x, \alpha, y) \longrightarrow (x\alpha y)$ which satisfies the conditions

(i) $x\alpha y \in M$.

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z,\ x(\alpha + \beta)y = x\alpha y + x\beta y,\ x\alpha(y + z) = x\alpha y + x\alpha z$.

(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

then M is called a Γ-ring [8]. Every ring M is a Γ-ring with $M = \Gamma$. However a Γ-ring need not be a ring. Γ-rings, more general than rings, were introduced by

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Nobusawa [6]. Bernes [8] slightly weakened the conditions in the definition of Γ-ring in the sense of Nobusawa. Let M be a Γ-ring, then an additive subgroup U of M is called a left (right) ideal of M if $M\Gamma U \subset U (U\Gamma M \subset U)$. If U is both a left and a right ideal, then we say U is an ideal of M. Suppose again that M is a Γ-ring, then M is said to be 2-torsion free if $2x = 0$ implies $x = 0$ for all $x \in M$. An ideal P_1 of a Γ-ring M is said to be prime if for some ideals A and B of M, $A\Gamma B \subseteq P_1$ implies $A \subseteq P_1$ or $B \subseteq P_1$. An ideal P_2 of a Γ-ring M is said to be semiprime if for any ideal U of M, $U\Gamma U \subseteq P_2$ implies $U \subseteq P_2$. A Γ-ring M is said to be prime if $a\Gamma M\Gamma b = (0)$ with $a, b \in M$, implies $a = 0$ or $b = 0$ and semiprime if $a\Gamma M\Gamma a = (0)$ with $a \in M$ implies $a = 0$. Furthermore, M is said to be a commutative Γ-ring if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$. Moreover, the set $Z(M) = \{x \in M : x\alpha y = y\alpha x$ for all $\alpha \in \Gamma,\ y \in M\}$ is called the center of the Γ-ring M. If M is a Γ-ring, then $[x, y]_\alpha = x\alpha y - y\alpha x$ is known as the commutator of x and y with respect to α, where $x,\ y \in M$ and $\alpha \in \Gamma$. We make the following basic commutator identities:

$$[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta \quad (1)$$

$$[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta \quad (2)$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Now,we consider the following assumption:

(A) $x\alpha y\beta z = x\beta y\alpha z$, for all $x,\ y,\ z \in M$ and $\alpha,\ \beta \in \Gamma$. According to assumption (A), the above commutator identities reduce to
$[x\alpha y, z]_\beta = [x, z]_\beta\alpha y + x\alpha[y, z]_\beta$ and $[x, y\alpha z]_\beta = [x, y]_\beta\alpha z + y\alpha[x, z]_\beta$.
which we will extensively used.

Bernes [8], Luh [1], Kyuno [7], Hoque and Paul [2, 3, 5] studied the structure of Γ-ring and obtained various generalizations of corresponding parts in ring theory.

Definition 1.1. An additive mapping $(x\alpha x) \to (x\alpha x)^*$ on a Γ-ring M is called an involution if $(x\alpha y)^* = y^*\alpha x^*$ and $(x\alpha x)^{**} = (x\alpha x)$ for all $x, y \in M$ and $\alpha \in \Gamma$. A Γ-ring M equipped with an involution is called a Γ-ring M with involution.

Definition 1.2. An element x in a Γ-ring M with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of M will be denoted by $H(M)$ and $S(M)$, respectively.

Definition 1.3. Let M be a Γ-ring with involution, an element $x \in M$ is called normal element if $x\alpha x^* = x^*\alpha x$ and if all elements of M are normal then M is normal ring.

Example 1.4. Let R be a commutative ring.
Define

$$M = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in R \right\}$$

and

$$\Gamma = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} : \alpha \in \Gamma \right\},$$

then M is a Γ-ring under addition and multiplication of matrices.

Define

$$x = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}, \quad x^* = \begin{bmatrix} b & 0 \\ 0 & a \end{bmatrix}, \quad y = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \quad y^* = \begin{bmatrix} d & 0 \\ 0 & c \end{bmatrix}.$$

Since $(x\alpha y)^* = y^*\alpha x^*$ and $(x\alpha x)^{**} = (x\alpha x)$ then a Γ-ring M is involution, also $x\alpha x^* = x^*\alpha x$, then M is normal Γ-ring

In this paper we will prove that if M is a semiprime normal Γ^*-ring satisfying $[s, s_1]_\alpha \in Z(M)$, then M is commutative Γ^*-ring .

2. Normal and Commutative Γ^*-Ring on a Semiprime Γ-Ring with Involution

To prove our main results we need the following lemmas.

Lemma 2.1. *Let M be a 2-torsion free Γ-ring with involution, then M is a normal Γ^*-ring if and only if $[h, s]_\alpha = 0$ for all $h \in H(M)$ and $s \in S(M)$.*

Proof. If M is a normal Γ^*-ring, then we have

$$[x, x^*]_\alpha = 0 \tag{3}$$

for all $x \in M$ and $\alpha \in \Gamma$. Replace x by $h + s$ in (3), we get,

$$[h + s, (h + s)^*]_\alpha = 0$$

then

$$[h + s, h - s]_\alpha = 0. \tag{4}$$

Hence

$$[h + s, h - s]_\alpha = [h, h]_\alpha - [h, s]_\alpha + [s, h]_\alpha - [s, s]_\alpha = 2[h, s]_\alpha = 0$$

for all $h \in H(M)$, $s \in S(M)$ and $\alpha \in \Gamma$. Since M is 2-torsion free, we get

$$[h, s]_\alpha = 0. \tag{5}$$

Conversely, assume $[h, s]_\alpha = 0$ for all $h \in H(M)$,$s \in S(M)$ and $\alpha \in \Gamma$, then similar as above we get M is a normal Γ^*-ring.

Lemma 2.2. *Let M be a 2-torsion free semiprime Γ-ring satisfying assumption A and suppose that $a \in M$ centralizes all $[a, x]_\alpha$ for all $x \in M$ and $\alpha \in \Gamma$. Then $a \in Z(M)$.*

Proof. The proof of this lemma can be found in [4].

Lemma 2.3. *Let M be a 2-torsion free semiprime normal Γ^*-ring, then $H(M) \subseteq Z(M)$, and if M is a non-commutative prime normal Γ^*-ring, then $H(M) = Z(M)$.*

Proof. By linearization of the relation (3), we obtain

$$[x, y^*]_\alpha + [y, x^*]_\alpha = 0 \tag{6}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Putting $y\gamma z$ for y in (6), we get

$$[x, z^*\gamma y^*]_\alpha + [y\gamma z, x^*]_\alpha = 0$$

for all $x, y, z \in M$ and $\alpha, \gamma \in \Gamma$. Therefore from above relation we get

$$[x, z^*]_\alpha\gamma y^* + z^*\gamma[x, y^*]_\alpha + y\gamma[z, x^*]_\alpha + [y, x^*]_\alpha\gamma z = 0 \tag{7}$$

for all $x, y, z \in M$ and $\alpha, \gamma \in \Gamma$. From (6) , we get

$$[x, y^*]_\alpha = -[y, x^*]_\alpha = [x^*, y]_\alpha$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Substituting $[x^*, y]_\alpha$ for $[x, y^*]_\alpha$ in (7) , we get

$$[x, z^*]_\alpha\gamma y^* + z^*\gamma[x^*, y]_\alpha + y\gamma[z, x^*]_\alpha + [y, x^*]_\alpha\gamma z = 0 \tag{8}$$

for all $x, y, z \in M$ and $\alpha, \gamma \in \Gamma$. Setting $x = z \in H(M)$ in the relation (8) and by definition of $H(M)$, we obtain

$$[y, x]_\alpha\gamma x = x\gamma[y, x]_\alpha \tag{9}$$

for all $x \in H(M)$,$y \in M$ and $\alpha, \gamma \in \Gamma$. By using Lemma 2.2, we get $H(M) \subseteq Z(M)$. Now assume M is non-commutative prime Γ^*-ring, for $z \in Z(M)$, then from (8), we get

$$z^*\gamma[x^*, y]_\alpha + [y, x^*]_\alpha\gamma z = 0.$$

Hence

$$[y, x^*]_\alpha\gamma z = z^*\gamma[y, x^*]_\alpha,$$

then we obtain

$$[y, x^*]_\alpha\gamma(z - z^*) = 0 \tag{10}$$

for all $x, y, z \in M$ and $\alpha, \gamma \in \Gamma$. Replace x by $w^*\gamma x^*$ in relation (10), we get

$$[y, x\gamma w]_\alpha\gamma(z - z^*) = x\gamma[y, w]_\alpha\gamma(z - z^*) + [y, x]_\alpha\gamma w\gamma(z - z^*) = 0$$

for all $x, y, z, w \in M$ and $\alpha, \gamma \in \Gamma$. Hence

$$[y, x]_\alpha\gamma w\gamma(z - z^*) = 0 \tag{11}$$

for all $x, y, z, w \in M$ and $\alpha, \gamma \in \Gamma$. Since M is a non-commutative prime Γ^*-ring, then $[y, x]_\alpha \neq 0$ and $z = z^*$, therefore we get $H(M) = Z(M)$.

Note 1. Let M be a 2-torsion free Γ-ring with involution, then every $x \in M$ can be uniquely represented in the form $2x = h + k$ where $h \in H(M)$ and $k \in K(M)$.

Theorem 2.4. *Let M be a 2-torsion free semiprime normal Γ^*-ring, which satisfies $[s, s_1]_\alpha \in Z(M)$ for all $s, s_1 \in S(M)$ and satisfying assumption (A), then M is commutative Γ^*-ring.*

Proof. We have
$$[s, s_1]_\alpha \in Z(M) \tag{12}$$
for all $s, s_1 \in S(M)$ and $\alpha \in \Gamma$. By using Lemma 2.1, we get
$$[h, s_1]_\alpha = 0$$
for all $h \in H(M)$, $s_1 \in S(M)$ and $\alpha \in \Gamma$. Since $s\beta s \in H(M)$ for all $s \in S(M)$ and $\beta \in \Gamma$, then from above relation, we get
$$[s\beta s, s_1]_\alpha = 0$$
for all $s, s_1 \in S(M)$ and $\alpha, \beta \in \Gamma$. Therefore
$$2s\beta[s, s_1]_\alpha = 0$$
for all $s, s_1 \in S(M)$ and $\alpha, \beta \in \Gamma$. Since M is 2-torsion free, we get
$$s\beta[s, s_1]_\alpha = 0 \tag{13}$$
for all $s, s_1 \in S(M)$ and $\alpha, \beta \in \Gamma$. Since $[s, s_1] \in Z(M)$, then from relation (13) and using assumption (A), we obtain
$$[s\beta[s, s_1]_\alpha, s_1]_\beta = [s, s_1]_\alpha\gamma[s, s_1]_\alpha = 0 \tag{14}$$
for all $s, s_1 \in S(M)$ and $\alpha, \beta, \gamma \in \Gamma$. Right multiplication (14) by z, we get
$$[s, s_1]_\alpha\gamma z\gamma[s, s_1]_\alpha = 0$$
for all $s, s_1 \in S(M)$ and $\alpha, \gamma \in \Gamma$. By the semiprimness of M, we have
$$[s, s_1]_\alpha = 0 \tag{15}$$
for all $s, s_1 \in S(M)$ and $\alpha \in \Gamma$. Since M is 2-torsion free,to prove M is commutative we only show
$$4[x, y]_\alpha = 0$$
for all $x, y \in M$ and $\alpha \in \Gamma$. By using Note 1, we get
$$4[x, y]_\alpha = [2x, 2y]_\alpha = [h + s, h_1 + s_1]_\alpha = [h, h_1]_\alpha + [h, s_1]_\alpha + [s, h_1]_\alpha + [s, s_1]_\alpha = 0$$
for all $x, y \in M$, $h, h_1 \in H(M)$, $s, s_1 \in S(M)$ and $\alpha \in \Gamma$. By using Lemma 2.3 and relation (15), we get M is commutative Γ^*-ring

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