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ON UNIQUENESS OF MEROMORPHIC FUNCTION THAT SHARE ONE SMALL FUNCTION WITH THEIR DERIVATIVES

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Abstract: In this paper, we study the uniqueness theorem of meromorphic function that share one small function with their derivatives and obtained one theorem which improves the result of Jin-Dong Li and Guang-Xin Huang [10].

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1. Introduction and Main Results

Let f be a non-constant meromorphic function defined in the whole complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the notations of the Nevanlinna theory such as $T(r, f), N(r, f)$ and so on, that can be found, for instance in [1,2].

Let f and g be two non-constant meromorphic functions. Let a be a finite complex number. We say that f and g share the value a CM(counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicites and we say that f and g share the value a IM(ignoring multiplicities) if we do not consider the multiplicities. When f and g share 1 IM, let z_0 be a 1-points of f of order p, a 1-points of g of order q, we denote by $N_{11}(r, \frac{1}{f-1})$ the counting function of those 1-points of f and g where $p = q = 1$; and $N_E^{(2}(r, \frac{1}{f-1})$ the counting function of those 1-points of f and g where $p = q \geq 2$.$\overline{N}_L(r, \frac{1}{f-1})$ is the counting function

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of those 1-points of both f and g where $p > q$. In the same way, we can define $N_{11}(r, \frac{1}{g-1})$, $N_E^{(2}(r, \frac{1}{g-1})$ and $\overline{N}_L(r, \frac{1}{g-1})$. If f and g share 1 IM, it is easy to see that

$$\overline{N}(r, \frac{1}{f-1}) = N_{11}(r, \frac{1}{f-1}) + \overline{N}_L(r, \frac{1}{f-1}) + \overline{N}_L(r, \frac{1}{g-1}) + N_E^{(2}(r, \frac{1}{g-1}) \\ = \overline{N}(r, \frac{1}{g-1}).$$

Let f be a non-constant meromorphic function. Let a be a finite complex number, and k be a positive integer, we denote by $N_{k)}(r, \frac{1}{f-a})(or\overline{N}_{k)}(r, \frac{1}{f-a}))$ the counting function for zeros of $f-a$ with multiplicity $\leq k$ (ignoring multiplicities), and by $N_{(k}(r, \frac{1}{f-a})(or\overline{N}_{(k}(r, \frac{1}{f-a}))$ the counting function for zeros of $f-a$ with multiplicity atleast k(ignoring multiplicities). Set

$$N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_{(2}(r, \frac{1}{f-a}) + ... + \overline{N}_{(k}(r, \frac{1}{f-a})$$

$$\Theta(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, \frac{1}{f-a})}{T(r, f)}, \quad \delta(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N(r, \frac{1}{f-a})}{T(r, f)}.$$

We further define

$$\delta_k(a, f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N_k(r, \frac{1}{f-a})}{T(r, f)}.$$

Clearly

$$0 \leq \delta(a, f) \leq \delta_k(a, f) \leq \delta_{k-1}(a, f)... \leq \delta_2(a, f) \leq \delta_1(a, f) = \Theta(a, f).$$

Definition 1.1. (see [3]) Let k be a nonnegative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k, clearly if f, g share (a, k), then f, g share (a, p) for all integers p with $0 \leq p \leq k$. Also, we note that f, g share a value a IM or CM if and only if they share $(a, 0)$ or (a, ∞), respectively.

A meromorphic function a is said to be a small function of f where $T(r, a) = S(r, f)$, that is $T(r, a) = o(T(r, f))$ as $r \to \infty$, outside of a possible exceptional set of finite linear measure. Similarly, we can define that f and g share a small function a IM or CM or with weight k.

R.Bruck [4] first considered the uniqueess problems of an entire function sharing one value with its derivative and proved the following result.

Theorem A. *Let f be a non-constant entire function satisfying $N(r, \frac{1}{f'}) = S(r, f)$. If f and f' share the value 1 CM, then $\frac{f'-1}{f-1} \equiv c$ for some nonzero constant c.*

Bruck [4] further posed the following conjecture.

Conjecture 1.1. *Let f be a non-constant entire function, $\rho_1(f)$ be the first iterated order of f. If $\rho_1(f)$ is not a positive integer or infinite, f and f' share the value 1 CM, then $\frac{f'-1}{f-1} \equiv c$ for some nonzero constant c.*

Yang [5] proved that the conjecture is true if f is an entire function of finite order. Yu [6] considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

Theorem B. *Let f be a non-constant entire function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.*

Theorem C. *Let f be a non-constant non-entire meromorphic function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If*

(i) f and a have no common poles.

(ii) $f - a$ and $f^{(k)} - a$ share 0 CM.

(iii) $4\delta(0, f) + 2(8 + k)\Theta(\infty, f) > 19 + 2k$,
then $f \equiv f^{(k)}$ where k is a positive integer.

In the same paper, Yu [6] posed the following open questions.

(i) Can a CM shared be replaced by an IM shared value?

(ii) Can the condition $\delta(0, f) > \frac{3}{4}$ of Theorem B be further relaxed?

(iii) Can the condition (iii) in Theorem C be further relaxed?

(iv) Can in general the condition (i) of Theorem C be dropped?

In 2004, Liu and Gu [7] improved Theorem B and obtained the following results.

Theorem D. *Let f be a non-constant entire function and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If $f - a$ and $f^{(k)} - a$ share 0 CM and $\delta(0, f) > \frac{1}{2}$, then $f \equiv f^{(k)}$.*

Lahiri and Sarkar [8] gave some affirmative answers to the first three questions imposing some restrictions on the zeros and poles of a. They obtained the following results.

Theorem E. *Let f be a non-constant meromorphic function, k be a positive integer, and $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. If:*

(i) a has no zero(pole) which is also a zero(pole) of f or $f^{(k)}$ with the same multiplicity.

(ii) $f - a$ and $f^{(k)} - a$ share (0,2)

(iii) $2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k$.

Then $f \equiv f^{(k)}$.

In 2005, Zhang [9] improved the above results and proved the following theorem.

Theorem F. *Let f be a non-constant meromorphic function, $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If: $l \geq 2$ and*

$$(3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4, \tag{1.1}$$

or $l = 1$ and

$$(4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6, \tag{1.2}$$

or $l = 0$ and

$$(6 + 2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10. \tag{1.3}$$

Then $f \equiv f^{(k)}$.

In 2015, Jin-Dong Li and Guang-Xiu Huang proved the following Theorem.

Theorem G. *Let f be a non-constant meromorphic function, $k(\geq 1), l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$.*

If $l \geq 2$ and

$$(3 + k)\Theta(\infty, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 4, \tag{1.4}$$

$l = 1$ and

$$(\frac{7}{2} + k)\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \delta_2(0, f) + \delta_{2+k}(0, f) > k + 5, \tag{1.5}$$

or $l = 0$ and

$$(6 + 2k)\Theta(\infty, f) + 2\Theta(\infty, f) + \delta_2(0, f) + \delta_{1+k}(0, f) + \delta_{2+k}(0, f) > 2k + 10. \tag{1.6}$$

Then $f \equiv f^{(k)}$.

In this paper we pay our attention to the uniqueness of more generalised form of a function namely f^n and $(f^{(k)})^m$ sharing a small function for two arbitrary positive integer n and m.

Theorem 1.1. *Let f be a non-constant meromorphic function, $k(\geq 1)$, $n(\geq 1)$, $m(\geq 2)$, $l(\geq 0)$ be integers. Also let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic small function. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$. If*
$l \geq 2$ and

$$(2k+3)\Theta(\infty, f) + \delta_2(0, f) + 2\delta_{1+k}(0, f) > 6 + 2k - n, \tag{1.7}$$

$l = 1$ and

$$(2k + \frac{7}{2})\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \delta_2(0, f) + 2\delta_{1+k}(0, f) > 2k + 7 - n, \tag{1.8}$$

or $l = 0$ and

$$(3k+6)\Theta(\infty, f) + 2\Theta(0, f) + \delta_2(0, f) + 3\delta_{1+k}(0, f) > 3k + 12 - n. \tag{1.9}$$

Then $f^n \equiv (f^{(k)})^m$.

Corollary. *Let f be a non-constant meromorphic function, $k(\geq 1), l(\geq 0), n(\geq 1)$ be integers. Also let $a \equiv a(z)(\neq 0, \infty)$ be a meromorphic small function. Suppose that $f^n - a$ and $(f^{(k)})^m - a$ share $(0, l)$.*
If $l \geq 2$ and $\delta_{1+k}(0, f) > \frac{1}{3}$
or $l = 1$ and $\delta_{1+k}(0, f) > \frac{5}{7}$
or $l = 0$ and $\delta_{1+k}(0, f) > \frac{5}{6} - \frac{1}{6}[\delta_2(0, f) + 2\Theta(0, f) - 3\delta_{1+k}(0, f)]$
then $f^n \equiv (f^{(k)})^m$.

From Theorem 1.1 we have the following corollary.

2. Lemmas

Lemma 2.1. (see [10]) *Let f be a non-constant meromorphic function, k, p be two positive integers, then*

$$N_p(r, \frac{1}{f^{(k)}}) \leq N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f)$$

clearly $\overline{N}(r, \frac{1}{f^{(k)}}) = N_1(r, \frac{1}{f^{(k)}})$.

Lemma 2.2. (see [10]) *Let*

$$H = (\frac{F''}{F'} - \frac{2F'}{F-1}) - (\frac{G''}{G'} - \frac{2G'}{G-1}) \tag{2.1}$$

where F and G are two non constant meromorphic functions. If F and G share 1 IM and $H \not\equiv 0$, then

$$N_{11}(r, \frac{1}{F-1}) \leq N(r, H) + S(r, F) + S(r, G)$$

Lemma 2.3. (see [12]) *Let f be a non-constant meromorphic function and p and k two positive integers. Then*

$$N_p(r, \frac{1}{f^{(k)}}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, \frac{1}{f}) + S(r, f)$$
$$N_p(r, \frac{1}{f^{(k)}}) \leq N_{p+k}(r, \frac{1}{f}) + k\overline{N}(r, f) + S(r, f)$$

Lemma 2.4. (see [11]) *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\Sigma_{k=0}^{n} a_k f^k}{\Sigma_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$ where $a_n \neq 0$ and $b_m \neq 0$. Then

$$T(r, R(f)) = dT(r, f) + S(r, f),$$

where $d = max\{n, m\}$.

3. Proof of Theorem 1.1

Let $F = \frac{f^n}{a}$ and $G = \frac{(f^{(k)})^m}{a}$. Then F and G share $(1, l)$, except the zeros and poles of $a(z)$. Let H be defined by (2.1).

Case 1. Let $H \not\equiv 0$.. By our assumptions, H have poles only at zeros of F' and G' and poles of F and G, and those 1-points of F and G whose multiplicities are distinct from the multiplicities of corresponding 1-points of G and F respectively. Thus, we deduce from (2.1) that

$$\begin{aligned} N(r, H) \leq\ & \overline{N}_{(2}(r, \frac{1}{F}) + \overline{N}_{(2}(r, \frac{1}{G}) + \overline{N}(r, F) \\ & + N_0(r, \frac{1}{F'}) + N_0(r, \frac{1}{G'}) + \overline{N}_L(r, \frac{1}{F-1}) \\ & + \overline{N}_L(r, \frac{1}{G-1}) \end{aligned} \tag{3.1}$$

here $N_0(r, \frac{1}{F'})$ is the counting function which only counts those points such that $F' = 0$ but $F(F-1) \neq 0$.

Because F and G share 1 IM, it is easy to see that

$$\begin{aligned} \overline{N}(r, \frac{1}{F-1}) &= N_{11}(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{F-1}) + \overline{N}_L(r, \frac{1}{G-1}) + N_E^{(2}(r, \frac{1}{G-1}) \\ &= \overline{N}(r, \frac{1}{G-1}) \end{aligned} \tag{3.2}$$

By the second fundamental theorem, we see that

$$\begin{aligned} T(r,F)+T(r,G) \le{}& \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}(r,\frac{1}{F}) \\ &+\overline{N}(r,\frac{1}{G})+\overline{N}(r,\frac{1}{F-1})+\overline{N}(r,\frac{1}{G-1}) \\ &-N_0(r,\frac{1}{F'})-N_0(r,\frac{1}{G'})+S(r,F)+S(r,G) \end{aligned} \tag{3.3}$$

Using Lemma 2.2 and (3.1), (3.2) and (3.3) We get

$$\begin{aligned} T(r,F)+T(r,G) \le{}& 3\overline{N}(r,F)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G}) \\ &+N_{11}(r,\frac{1}{F-1})+2N_E^{(2}(r,\frac{1}{G-1}) \\ &+3\overline{N}_L(r,\frac{1}{F-1})+3\overline{N}_L(r,\frac{1}{G-1})+S(r,F)+S(r,G) \end{aligned} \tag{3.4}$$

We discuss the following three sub-cases.

Sub-case 1.1. $l \ge 2$. Obviously

$$\begin{aligned} N_{11}(r,\frac{1}{F-1})+2N_E^{(2}(r,\frac{1}{G-1})+3\overline{N}_L(r,\frac{1}{F-1})+3\overline{N}_L(r,\frac{1}{G-1}) \\ \le N(r,\frac{1}{G-1})+S(r,F) \\ \le T(r,G)+S(r,F)+S(r,G) \end{aligned} \tag{3.5}$$

Combining (3.4) and (3.5), we get

$$T(r,F) \le 3\overline{N}(r,F)+N_2(r,\frac{1}{F})+N_2(r,\frac{1}{G})+S(r,F) \tag{3.6}$$

that is

$$nT(r,f) \le 3\overline{N}(r,f^n)+N_2(r,\frac{1}{f^n})+N_2(r,\frac{1}{(f^{(k)})^m})+S(r,f)$$

By Lemma 2.1 for p=2, we get

$$nT(r,f) \le (2k+3)\overline{N}(r,f)+2N_{1+k}(r,\frac{1}{f})+N_2(r,\frac{1}{f})+S(r,f)$$

So

$$(2k+3)\Theta(\infty,f)+2\delta_{1+k}(0,f)+\delta_2(0,f) \le 2k+6-n.$$

Which contradicts with (1.7).

Sub-case 1.2. $l = 1$. It is easy to see that

$$\begin{aligned} N_{11}(r, \frac{1}{F-1}) + 2N_E^{(2}(r, \frac{1}{G-1}) + 2\overline{N}_L(r, \frac{1}{F-1}) + 3\overline{N}_L(r, \frac{1}{G-1}) \\ \leq N(r, \frac{1}{G-1}) + S(r, F) \\ \leq T(r, G) + S(r, F) + S(r, G) \end{aligned} \tag{3.7}$$

$$\begin{aligned} \overline{N}_L(r, \frac{1}{F-1}) \leq \frac{1}{2}N(r, \frac{F}{F'}) \leq \frac{1}{2}N(r, \frac{F'}{F}) + S(r, F) \\ \leq \frac{1}{2}[\overline{N}(r, \frac{1}{F}) + \overline{N}(r, F)] + S(r, F) \end{aligned} \tag{3.8}$$

Combining (3.4) and (3.7) and (3.8), we get

$$T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + \frac{7}{2}\bar{N}(r, F) + \frac{1}{2}\bar{N}(r, \frac{1}{f}) + S(r, f) \tag{3.9}$$

that is

$$nT(r, f) \leq N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{(f^{(k)})^m}) + \frac{7}{2}\bar{N}(r, f) + \frac{1}{2}\bar{N}(r, \frac{1}{f}) + S(r, f).$$

By Lemma 2.1 for $p = 2$, we get

$$nT(r, f) \leq (2k + \frac{7}{2})\overline{N}(r, f) + \frac{1}{2}\bar{N}(r, \frac{1}{f}) + N_2(r, \frac{1}{f}) + 2N_{k+1}(r, \frac{1}{f}) + S(r, f)$$

So

$$(2k + \frac{7}{2})\Theta(\infty, f) + \frac{1}{2}\Theta(0, f) + \delta_2(0, f) + 2\delta_{1+k}(0, f) \leq 2k + 7 - n$$

Which contradicts with (1.8).

Sub-case 1.3. $l = 0$. It is easy to see that

$$\begin{aligned} N_{11}(r, \frac{1}{F-1}) + 2N_E^{(2}(r, \frac{1}{G-1}) + \overline{N}_L(r, \frac{1}{F-1}) + 2\overline{N}_L(r, \frac{1}{G-1}) \\ \leq N(r, \frac{1}{G-1}) + S(r, F) \\ \leq T(r, G) + S(r, F) + S(r, G) \end{aligned} \tag{3.10}$$

$$\begin{aligned} \overline{N}_L(r, \frac{1}{F-1}) \leq N(r, \frac{1}{F-1}) - \overline{N}(r, \frac{1}{F-1}) \\ \leq N(r, \frac{F}{F'}) \leq N(r, \frac{F'}{F}) + S(r, F) \\ \leq \overline{N}(r, \frac{1}{F}) + \overline{N}(r, F) + S(r, F) \end{aligned} \tag{3.11}$$

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Similarly, we have

$$\begin{aligned}\overline{N}_L(r, \frac{1}{G-1}) &\leq \overline{N}(r, \frac{1}{G}) + \overline{N}(r, G) + S(r, F) \\ &\leq N_1(r, \frac{1}{G}) + \overline{N}(r, F) + S(r, G)\end{aligned} \tag{3.12}$$

Combining (3.4) and (3.10) – (3.12), we get

$$\begin{aligned}T(r, F) \leq\; & N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 2\overline{N}(r, \frac{1}{F}) \\ & + 6\overline{N}(r, F) + N_1(r, \frac{1}{G}) + S(r, F)\end{aligned} \tag{3.13}$$

that is

$$\begin{aligned}nT(r, f) \leq\; & N_2(r, \frac{1}{f}) + N_2(r, \frac{1}{(f^{(k)})^m}) + 2\overline{N}(r, \frac{1}{f}) \\ & + 6\overline{N}(r, f) + N_1(r, \frac{1}{(f^{(k)})^m}) + S(r, f).\end{aligned}$$

By Lemma 2.1 for p=2 and for p=1 respectively, we get

$$nT(r, f) \leq (3k+6)\overline{N}(r, f) + N_2(r, \frac{1}{f}) + 3N_{1+k}(r, \frac{1}{f}) + 2\overline{N}(r, \frac{1}{f})$$

So

$$(3k+6)\Theta(\infty, f) + \delta_2(0, f) + 3\delta_{1+k}(0, f) + 2\Theta(0, f) \leq 3k + 12 - n$$

which contradicts with (1.9).

Case 2. Let $H \equiv 0$.. By integration we get from

$$\frac{1}{F-1} \equiv \frac{C}{G-1} + D, \tag{3.14}$$

where C,D are constants and $C \neq 0$.

We first show that $D = 0$. Suppose that there exist a pole z_0 of f with multiplicity p which is not a pole or a zero of a(z). Then z_0 is the pole of F with multiplicity np and the pole of G with multiplicity $m(p+k)$.We assume that $np \neq m(p+k)$, Since other wise we know from (3.14) that $D = 0$ and we are done.

Sub-case 2.1. Suppose $D \neq 0$. Since $np \neq m(p+k)$, we get a contradiction from (3.14). So, $N(r, f) = S(r, f)$ and hence $\Theta(\infty, f) = 1$. Also it is clear that $\overline{N}(r, F) = \overline{N}(r, G) = S(r, f)$. From (1.7)-(1.9) we know respectively

$$\delta_2(0, f) + 2\delta_{1+k}(0, f) > 2 - n \tag{3.15}$$

$$\frac{1}{2}\Theta(0,f)+\delta_2(0,f)+2\delta_{1+k}(0,f)>\frac{7}{2}-n \tag{3.16}$$

and

$$2\Theta(0,f)+\delta_2(0,f)+3\delta_{1+k}(0,f)>6-n \tag{3.17}$$

Since $D\neq 0$, from (3.14) we get

$$\frac{-D(F-1-\frac{1}{D})}{F-1}\equiv C\frac{1}{G-1}$$

So,

$$\overline{N}\left(r,\frac{1}{F-(1+\frac{1}{D})}\right)=\overline{N}(r,G)=S(r,f)$$

Suppose $D\neq -1$.

Using the second fundamental theorem for F we get

$$\begin{aligned}T(r,F)&\leq \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\overline{N}\left(r,\frac{1}{F-(1+\frac{1}{D})}\right)\\&\leq \overline{N}(r,\frac{1}{F})+S(r,f)\\ \textit{that is}&\\ nT(r,f)&\leq \overline{N}(r,\frac{1}{f})+S(r,f).\end{aligned}$$

If $n>1$ we have a contradiction from above. So, we have $n=1$ and so $\Theta(0,f)=0$ which contradicts (3.15) – (3.17).

If $D=-1$, Then

$$\frac{F}{F-1}\equiv C\frac{1}{G-1} \tag{3.18}$$

Clearly we know from above that $\overline{N}(r,\frac{1}{F})=\overline{N}(r,G)=S(r,f)$ and hence, $\overline{N}(r,\frac{1}{F})=S(r,f)$. If $C\neq -1$, We know from (3.18) that

$$\overline{N}\left(r,\frac{1}{G-(1+C)}\right)=\overline{N}(r,F)=S(r,f)$$

So from Lemma 2.1 and the Second fundamental theorem we get

$$\begin{aligned}mT(r,f^{(k)})&\leq T(r,G)+S(r,f)\\&\leq \overline{N}(r,G)+\overline{N}(r,\frac{1}{G})+\overline{N}\left(r,\frac{1}{G-1-C}\right)+S(r,f)\\&\leq \overline{N}\left(r,\frac{1}{f^{(k)}}\right)+S(r,f)\end{aligned}$$

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that is

$$\begin{aligned}(m-1)T(r,f^{(k)})+T(r,f) &\leq (k+1)\overline{N}(r,\frac{1}{f})+S(r,f)\\ &= S(r,f)\end{aligned}$$

which is absurd.

So $C=-1$ and we get from (3.18) that $FG\equiv 1$, which ultimately yields

$$\left[\frac{f^{(k)}}{f}\right]^m=\frac{a^2}{f^{n+m}}.$$

In view of the First fundamental theorem, we get from above

$$(n+m)T(r,f)\leq mk[\overline{N}(r,f)+\overline{N}(r,\frac{1}{f})]+S(r,f)=S(r,f),$$

which is impossible.

Sub-case 2.2. $D=0$ and so from (3.14) we get

$$G-1\equiv C(F-1).$$

If $C\neq 1$, then

$$\begin{aligned}G &\equiv C(F-1+\frac{1}{C})\\ \textit{and}\quad \overline{N}(r,\frac{1}{G}) &= \overline{N}\left(r,\frac{1}{F-(1-\frac{1}{C})}\right).\end{aligned}$$

By the second fundamental theorem and Lemma 2.3 for $p=1$ and Lemma 2.4 we have

$$\begin{aligned}nT(r,f)+S(r,f) &= T(r,F)\\ &\leq \overline{N}(r,F)+\overline{N}(r,\frac{1}{F})+\left(r,\frac{1}{F-(1-\frac{1}{C})}\right)+S(r,G)\\ &\leq \overline{N}(r,f)+\overline{N}(r,\frac{1}{F})+\overline{N}\left(r,\frac{1}{G}\right)+S(r,f)\\ &\leq \overline{N}(r,\frac{1}{f})+N_{1+k}(r,\frac{1}{f})+(1+k)\overline{N}(r,f)+S(r,f)\end{aligned}$$

Hence

$$(k+1)\Theta(\infty,f)+\Theta(0,f)+\delta_{1+k}(0,f)\leq 3+k-n$$

So, it follows that

$$\begin{aligned}(2k+3)\Theta(\infty,f)+&2\delta_{1+k}(0,f)+\delta_2(0,f)=(2k+3)\Theta(\infty,f)+2\delta_{1+k}(0,f)\\&+2\Theta(0,f)-1\\&\leq(k+1)\Theta(\infty,f)+\Theta(0,f)+\delta_{1+k}(0,f)\\&+(k+2)\Theta(\infty,f)+\Theta(0,f)+\delta_{1+k}(0,f)-1\\&\leq 2k+6-n\end{aligned}$$

and

$$(2k+\frac{7}{2})\Theta(\infty,f)+\frac{1}{2}\Theta(0,f)+\delta_2(0,f)+2\delta_{1+k}(0,f)\leq 2k+7-n,$$

and

$$(3k+6)\Theta(\infty,f)+2\Theta(0,f)+\delta_2(0,f)+3\delta_{1+k}(0,f)\leq 3k+12-n.$$

This contradicts (1.7) – (1.9). Hence $C=1$ and so $F\equiv G$, that is $f^n\equiv(f^{(k)})^m$. This completes the proof of the theorem.

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