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UNIQUENESS OF MEROMORPHIC FUNCTIONS OF A CERTAIN NONLINEAR DIFFERENTIAL POLYNOMIALS

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Abstract: In this paper by introducing the notion of multiplicity, we study the uniqueness of meromorphic functions whose certain nonlinear differential polynomials share a polynomial.The results of the paper improve and generalize some results due to Fang [2], S.S.B. Math, R.S. Dyavanal [1], and X-M. Li, H.X. Yi [8].

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1. Introduction, Definitons and Results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [3,4,9 and 10]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$, as $r \to \infty$, $r \notin E$.

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Let f and g be two nonconstant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that f and g share $\infty\, CM$, if $\frac{1}{f}$ and $\frac{1}{g}$ share $0\, CM$, and we say that f and g share ∞IM, if $\frac{1}{f}$ and $\frac{1}{g}$ share $0\, IM$ (see [9]). We say that a is a small function of f , if a is a meromorphic function satisfying $T(r,a) = S(r,f)$ (see [9]). In addition, we need the following three definitions.

Definition 1.1. (see [5, Definition 1]). Let p be a positive integer and $a \in C \cup \{\infty\}$. Then by $N_{p)}(r, \frac{1}{f-a})$ we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}_{p)}(r, \frac{1}{f-a})$ we denote the corresponding reduced counting function (ignoring multiplicities). By $N_{(p}(r, \frac{1}{f-a})$ we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r, \frac{1}{f-a})$ we denote the corresponding reduced counting function (ignoring multiplicities), where and what follows, $N_{p)}(r, \frac{1}{f-a})$, $\overline{N}_{p)}(r, \frac{1}{f-a})$, $N_{(p}(r, \frac{1}{f-a})$, $\overline{N}_{(p}(r, \frac{1}{f-a})$ mean $N_{p)}(r,f)$, $\overline{N}_{p)}(r,f)$, $N_{(p}(r,f)$ and $\overline{N}_{(p}(r,f)$ respectively, if $a=\infty$.

Definition 1.2. Let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$\delta_k(a,f) = 1 - \limsup_{r\to\infty} \frac{N_k\left(r, \frac{1}{f-a}\right)}{T(r,f)}, \tag{1.1}$$

where

$$N_k\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2}\left(r, \frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r, \frac{1}{f-a}\right). \tag{1.2}$$

Remark 1.1. From (1.1) and (1.2) we have

$$0 \le \delta_k(a,f) \le \delta_{k-1}(a,f) \le \delta_1(a,f) \le \Theta(a,f) \le 1.$$

Definition 1.3. Let a be any value in the extended complex plane, and let k be an arbitrary nonnegative integer. We define

$$\Theta_{k)}(a,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_{k)}\left(r, \frac{1}{f-a}\right)}{T(r,f)}. \tag{1.3}$$

Remark 1.2. From (1.3) we have $0 \le \Theta(a,f) \le \Theta_{k)}(a,f) \le \Theta_{k-1)}(a,f) \le \Theta_{1)}(a,f) \le 1$.

In 1997, Lahiri posed the following question.

Question 1.1. (see [6]). What can be said about the relationship between two meromorphic functions f, g when two differential polynomials, generated by f and g respectively, share certain values?

In 2002, Fang proved the following result, which dealt with Question 1.1.

Theorem A (see [2, Theorem 2]) *Let f and g be two nonconstant entire functions, and let n, k be two positive integers satisfying $n \geq 2k+8$. If $\{f^n(f-1)\}^{(k)}$ and $\{g^n(g-1)\}^{(k)}$ share 1 CM, then $f = g$.*

Extending Theorem A, Bhoosnurmath and Dyavanal proved the following theorem in 2007.

Theorem B (see [1, Theorem 4]) *Let f and g be two nonconstant meromorphic functions such that $\Theta(\infty, f) > \frac{3}{n+1}$, and let n, k be two positive integers satisfying $n \geq 3k+13$. If $\{f^n(f-1)\}^{(k)}$ and $\{g^n(g-1)\}^{(k)}$ share 1 CM, then $f = g$.*

Regarding Theorem B, it is natural to ask the following question.

Question 1.2. What can be said about the relationship between two meromorphic functions f, g when two differential polynomials, generated by f and g respectively, have the same fixed points?

In 2011, X-M. Li and H. X. Yi proved the following theorems, which is an IM-analog of Theorem A and Theorem B.

Theorem C (see [8, Theorem 1.1]) *Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers satisfying $n > 9k+20$ and $max\{\chi_1, \chi_2\} < 0$, where*

$$\chi_1 = \frac{2}{n-2k+1} + \frac{2}{n+2k+1} + \frac{2k+1}{n+k+1} + 1 - \Theta_{k)}(1, f) - \Theta_{k-1)}(1, f) \quad (1.4)$$

and

$$\chi_2 = \frac{2}{n-2k+1} + \frac{2}{n+2k+1} + \frac{2k+1}{n+k+1} + 1 - \Theta_{k)}(1, g) - \Theta_{k-1)}(1, g). \quad (1.5)$$

If $\Theta(\infty, f) > 2/n$ and if $\{f^n(f-1)\}^{(k)} - P$ and $\{g^n(g-1)\}^{(k)} - P$ share 0 IM, where P is a nonzero polynomial, then $f = g$.

Theorem D (see [8, Theorem 1.2]) *Let f and g be two transcendental meromorphic functions, and let n, k be two positive integers satisfying $n > 3k+11$ and $max\{\chi_1, \chi_2\} < 0$, where χ_1, and χ_2 are defined as in (1.4) and (1.5) respectively. If $\Theta(\infty, f) > 2/n$ and if $\{f^n(f-1)\}^{(k)} - P$ and $\{g^n(g-1)\}^{(k)} - P$ share 0 CM, where P is a nonzero polynomial, then $f = g$.*

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Theorem 1.1. *Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers and $max\{\chi_1, \chi_2\} < 0$, where*

$$\chi_1 = \frac{2m}{n+m-2k} + \frac{2}{(n+m)s+2k} + \frac{2k+1}{(n+m)s+k} + 2 - \Theta_{k)}(1, f) - \Theta_{k-1)}(1, f) \quad (1.6)$$

and

$$\chi_2 = \frac{2m}{n+m-2k} + \frac{2}{(n+m)s+2k} + \frac{2k+1}{(n+m)s+k} + 2 - \Theta_{k)}(1, g) - \Theta_{k-1)}(1, g). \quad (1.7)$$

Let $\{f^nP(f)\}^{(k)} - P$ and $\{g^nP(g)\}^{(k)} - P$ share 0 IM, where P is a nonzero polynomial, then one of the following holds:

(i) When $m = 0$, $f(z) \neq \infty$, $g(z) \neq \infty$ and $sn > 9k + 14$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$;

(ii) When $m = 1$, $(n + 1)s > 9k + 20$ and $\Theta(\infty, f) > 2/n$ then $f \equiv g$.

(iii) When $m \geq 2$ and $(n + m)s \geq 9k + 5m + 2m^* + 14$ then either $[f^n(f - 1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f \equiv g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(x, y) = x^n(x - 1)^m - y^n(y - 1)^m$.

Corollary 1. *Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers and $max\{\chi_1, \chi_2\} < 0$, where χ_1 and χ_2 are defined as in (1.4) and (1.5) respectively. Let $\{f^n(f - 1)\}^{(k)}$ and $\{g^n(g - 1)\}^{(k)}$ have the same fixed points ignoring multiplicities, then one of the following holds:*

(i) When $m = 0$, $f(z) \neq \infty$, $g(z) \neq \infty$ and $sn > 9k + 14$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$;

(ii) When $m = 1$, $(n + 1)s > 9k + 20$ and $\Theta(\infty, f) > 2/n$ then $f \equiv g$.

(iii) When $m \geq 2$ and $(n + m)s \geq 9k + 5m + 2m^* + 14$ then either $[f^n(f - 1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f \equiv g$ or f and g satisfy the algebraic equation $R(f, g) = 0$, where $R(x, y) = x^n(x - 1)^m - y^n(y - 1)^m$.

Theorem 1.2. *Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers and $max\{\chi_1, \chi_2\} < 0$, where χ_1 and χ_2 are defined as in* (1.6) *and* (1.7) *respectively. Let $\{f^nP(f)\}^{(k)} - P$ and $\{g^nP(g)\}^{(k)} - P$ share 0 CM, where P is a nonzero polynomial, then one of the following holds:*

(i) When $m = 0$, $f(z) \neq \infty$, $g(z) \neq \infty$ and $sn > 3k+8$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f \equiv tg$ for a constant t such that $t^n = 1$;

(ii) When $m = 1$, $(n + 1)s > 3k + 12$ and $\Theta(\infty, f) > 2/n$ then $f \equiv g$.

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(iii) When $m \geq 2$ and $(n+m)s \geq 3k+2m+2m^*+8$ then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f \equiv g$ or f and g satisfy the algebraic equation $R(f,g)=0$, where $R(x,y)=x^n(x-1)^m-y^n(y-1)^m$.

Corollary 2. *Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers and $max\{\chi_1,\chi_2\}<0$, where χ_1 and χ_2 are defined as in (1.4) and (1.5) respectively. Let $\{f^n(f-1)\}^{(k)}$ and $\{g^n(g-1)\}^{(k)}$ have the same fixed points counting multiplicities, then one of the following holds:*

(i) When $m=0$, $f(z)\neq\infty$, $g(z)\neq\infty$ and $sn>3k+8$, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$ or $f\equiv tg$ for a constant t such that $t^n=1$;

(ii) When $m=1$, $(n+1)s>3k+12$ and $\Theta(\infty,f)>2/n$ then $f\equiv g$.

(iii) When $m \geq 2$ and $(n+m)s \geq 3k+2m+2m^*+8$ then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f \equiv g$ or f and g satisfy the algebraic equation $R(f,g)=0$, where $R(x,y)=x^n(x-1)^m-y^n(y-1)^m$.

Theorem 1.3. *Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers and $max\{\chi_1,\chi_2\}<0$, where*

$$\chi_1=\frac{2m}{n+m-2k}+\frac{2}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}+2-\Theta_{k)}(1,f)-\Theta_{k-1)}(1,f) \quad (1.8)$$

and

$$\chi_2=\frac{2m}{n+m-2k}+\frac{2}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}+2-\Theta_{k)}(1,g)-\Theta_{k-1)}(1,g). \quad (1.9)$$

Let $\{f^n(f-1)\}^{(k)}-P$ and $\{g^n(g-1)\}^{(k)}-P$ share 0 IM, where P is a nonzero polynomial, then one of the following holds:

(i) When $m=0$, $f(z)\neq\infty$, $g(z)\neq\infty$ and $sn>9k+14$, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=1$ or $f\equiv tg$ for a constant t such that $t^n=1$;

(ii) When $m=1$, $(n+1)s>9k+20$ and $\Theta(\infty,f)>2/n$ then $f\equiv g$.

(iii) When $m \geq 2$ and $(n+m)s \geq 9k+5m+2m^*+14$ then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ or $f \equiv g$ or f and g satisfy the algebraic equation $R(f,g)=0$, where $R(x,y)=x^n(x-1)^m-y^n(y-1)^m$.

Theorem 1.4. *Let f and g be two transcendental meromorphic functions, and let n, k and m be three positive integers and $max\{\chi_1,\chi_2\}<0$, where χ_1 and χ_2 are*

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defined as in (1.8) *and* (1.9) *respectively. Let* $\{f^n(f-1)\}^{(k)}-P$ *and* $\{g^n(g-1)\}^{(k)}-P$ *share 0 CM, where P is a nonzero polynomial, then one of the following holds:*

(i) When $m = 0$, $f(z) \neq \infty$, $g(z) \neq \infty$ *and* $sn > 3k+8$, *then either* $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, *where* c_1, c_2 *and* c *are three constants satisfying* $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ *or* $f \equiv tg$ *for a constant* t *such that* $t^n = 1$;

(ii) When $m = 1$, $(n+1)s > 3k+12$ *and* $\Theta(\infty, f) > 2/n$ *then* $f \equiv g$.

(iii) When $m \geq 2$ *and* $(n+m)s \geq 3k+2m+2m^*+8$ *then either* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1$ *or* $f \equiv g$ *or* f *and* g *satisfy the algebraic equation* $R(f,g) = 0$, *where* $R(x,y) = x^n(x-1)^m - y^n(y-1)^m$.

Remark 1.3. In Theorem 1.1 giving specific values for s,when $m \geq 2$ we get the following interesting cases:

(i) If s = 1, then $n > 9k + 4m + 2m^* + 14$.

(ii) If s = 2, then $n > \frac{9k+5m+2m^*+14}{2} - m$.

(iii) If s = 3, then $n > \frac{9k+5m+2m^*+14}{3} - m$.

We conclude that if f and g have zeros and poles of higher order multiplicity, then we can reduce the value of n. This holds for all cases in Theorem 1.1 and Theorem 1.2.

Remark 1.4. Under the assumptions of Theorems 1.1 and 1.2, from Definition 1.3 we can see that the notation χ_1 defined as (1.4) measures the relative number of those 1-points of f with multiplicities $\leq k-1$ and with multiplicities $\leq k$ in $|z| < r$, where if z_1 is a 1-point of f in $|z| < r$ such that its multiplicity $\leq k-1$, then the multiplicity of z_1 is counted 2, if z_1 is a 1-point of f in $|z| < r$ such that its multiplicity is k, then the multiplicity of z_1 is counted 1. The notation χ_2 defined as (1.5) has similar meanings. From Remark 1.2 we have

$$\frac{2m}{n+m-2k} + \frac{2}{(n+m)s+2k} + \frac{k-n}{(n+m)s+k} \leq \chi_j \leq \frac{2m}{n+m-2k} + \frac{2}{(n+m)s+2k} + \frac{n+3k+2}{(n+m)s+k},$$

for j = 1, 2. Moreover, χ_1 becomes smaller if and only if the relative number of those 1-points of f with multiplicities $\leq k-1$ and with multiplicities $\leq k$ in $|z| < r$ becomes smaller, so does χ_2. Under the assumptions of Theorems 1.1 and 1.2, if $max\,\{\chi_2,\, \chi_2\}$ becomes small enough, then the relative number of those 1-points of f with multiplicities $\leq k-1$ and with multiplicities $\leq k$ in $|z| < r$, and the relative number of those 1-points of g with multiplicities $\leq k-1$ and with multiplicities $\leq k$

in $|z| < r$ become small enough, and so the conclusions of Theorems 1.1 and 1.2 can be valid. In this paper, under the assumptions of Theorems 1.1 and 1.2, we will prove that the conclusions of Theorems 1.1 and 1.2 hold if $max\,\{\chi_2,\,\chi_2\} < 0$.

2. Some Lemmas

Lemma 2.1. *([8]). Let f and g be two transcendental meromorphic functions such that $f^{(k)} - P$ and $g^{(k)} - P$ share 0 IM, where $k(\geq 1)$ is a positive integer, P is a nonzero polynomial. If*

$$\Delta_1 = (2k+3)\Theta(\infty, f) + (2k+4)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + 3\delta_{k+1}(0, g) > 4k + 13 \quad (2.1)$$

and

$$\Delta_2 = (2k+3)\Theta(\infty, g) + (2k+4)\Theta(\infty, f) + \Theta(0, g) + \Theta(0, f) + 2\delta_{k+1}(0, g) + 3\delta_{k+1}(0, f) > 4k + 13, \quad (2.2)$$

then either $f^{(k)}g^{(k)} = P^2$ or $f \equiv g$.

Lemma 2.2. *([8]). Let f and g be two transcendental meromorphic functions such that $f^{(k)} - P$ and $g^{(k)} - P$ share 0 CM, where $k(\geq 1)$ is a positive integer, P is a nonzero polynomial. If*

$$\Delta_1 = (k+2)\Theta(\infty, f) + 2\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > k + 7 \quad (2.3)$$

and

$$\Delta_2 = (k+2)\Theta(\infty, g) + 2\Theta(\infty, f) + \Theta(0, g) + \Theta(0, f) + \delta_{k+1}(0, g) + \delta_{k+1}(0, f) > k + 7, \quad (2.4)$$

then either $f^{(k)}g^{(k)} = P^2$ or $f \equiv g$.

Lemma 2.3. *([8]). Let h be a nonconstant meromorphic function that is not a polynomial with its degree $\leq k - 1$. Then*

$$N_o\left(r, \frac{1}{h^{(k)}}\right) \leq N_k\left(r, \frac{1}{h}\right) + k\overline{N}(r, h) + S(r, h),$$

where $k(\geq 1)$ is a positive integer, and $N_o\left(r, \frac{1}{h^{(k)}}\right)$ denotes the counting function of those zeros of $h^{(k)}$ that are not the zeros of h.

Lemma 2.4. *([12]). Let $s > 0$ and t be relatively prime integers, and let c be a finite complex number such that $c^s = 1$, then there exists one and only one common zero of $w^s - 1$ and $w^t - c$.*

3. Proof of Theorems

Proof. (Theorem 1.1) Let

$$F_1 = f^n P(f) \ and \ G_1 = g^n P(g). \tag{3.1}$$

Consider

$$\begin{aligned}\overline{N}\left(r, \frac{1}{F_1}\right) = \overline{N}\left(r, \frac{1}{f^n P(f)}\right) &\leq \frac{1}{s(n+m)} N\left(r, \frac{1}{F_1}\right) \\ &\leq \frac{1}{s(n+m)}[T(r, F_1) + o(1)],\end{aligned}$$

then we have

$$\Theta(0, F_1) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{F_1}\right)}{T(r, F_1)} \geq 1 - \frac{2}{s(n+m)}, \tag{3.2}$$

and

$$\Theta(\infty, F_1) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, F_1\right)}{T(r, F_1)} \geq 1 - \frac{1}{s(n+m)}. \tag{3.3}$$

Similarly

$$\Theta(0, G_1) \geq 1 - \frac{2}{s(n+m)}, \quad \Theta(\infty, G_1) \geq 1 - \frac{1}{s(n+m)}. \tag{3.4}$$

Consider

$$\begin{aligned}N_{k+1}\left(r, \frac{1}{F_1}\right) = N_{k+1}\left(r, \frac{1}{f^n P(f)}\right) &= (k+m+1)\overline{N}\left(r, \frac{1}{f^n P(f)}\right) \\ &\leq \frac{k+m+1}{s(n+m)}[T(r, F_1) + o(1)].\end{aligned}$$

Next, we have

$$\delta_{k+1}(0, F_1) = 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, \frac{1}{F_1})}{T(r, F_1)} \geq 1 - \frac{k+m+1}{s(n+m)}. \tag{3.5}$$

Similarly

$$\delta_{k+1}(0, G_1) \geq 1 - \frac{k+m+1}{s(n+m)}. \tag{3.6}$$

From (2.1), (3.1)-(3.6) and the condition $s(n+m) > 9k + 5m + 16$, we get

$$\Delta_1 > 4k + 13. \tag{3.7}$$

Similarly, from (2.2), and (3.1)-(3.6) we get

$$\Delta_2 > 4k + 13. \tag{3.8}$$

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From (3.7), (3.8), Lemma 2.1 and the condition that $F_1^{(k)} - P$ and $G_1^{(k)} - P$ share 0 IM we know that F_1 and G_1 are transcendental meromorphic functions such that $F_1^{(k)} G_1^{(k)} = P^2$ or $F_1 = G_1$.

We discuss the following two cases.

Case 1. Suppose that $F_1^{(k)} G_1^{(k)} = P^2$. Then it follows from (3.1) that

$$\{f^n P(f)\}^{(k)} \{g^n P(g)\}^{(k)} = P^2. \tag{3.9}$$

Let $z_0 \notin \{z : P(z) = 0\}$ be a zero of f of order p. Then it follows from (3.9) that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q, then we have $np - k = (n+m)q + k$, i.e., $n(p-q) = mq + 2k$, which implies that $p \geq q + 1$ and $q \geq \frac{n-2k}{m}$, so we have

$$p \geq \frac{n+m-2k}{m}. \tag{3.10}$$

Let $z_1 \notin \{z : P(z) = 0\}$ be a zero of $P(f)$ of order $p_1 \geq k+1$, then it follows from (3.9) that z_1 is a pole of g. Suppose that z_1 is a pole of g of order q_1. Then from (3.9) we have $p_1 - k = (n+m)q_1 + k$. From this we get

$$p_1 \geq (n+m)s + 2k. \tag{3.11}$$

Let $z_2 \notin \{z : P(z) = 0\}$ be a zero of $\{f^n P(f)\}^{(k)}$ of order p_2 that is not a zero of $fP(f)$. Then from (3.9) we see that z_2 is a pole of g. Suppose that z_2 is a pole of g of order q_2, then $p_2 = (n+m)q_2 + k$. Thus

$$p_2 \geq (n+m)s + k. \tag{3.12}$$

Let $z_3 \notin \{z : P(z) = 0\} \bigcup \{z : f(z)P(f) = 0\}$ be a zero of $\{f^n P(f)\}^{(k)}$ of multiplicity p_3. Then, from (3.9) we deduce that z_3 is a pole of g of multiplicity q_3, say. Hence $p_3 = (n+m)q_3 + k \geq (n+m)s + k$. This together with (3.10)-(3.12) and Lemma 2.3 gives

$$\begin{aligned}
\overline{N}(r,f) &\leq \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}_{k-1)}\left(r,\frac{1}{g-1}\right) + \frac{1}{(n+m)s+2k} N\left(r,\frac{1}{g-1}\right) \\
&+ \frac{1}{(n+m)s+k} N_0\left(r,\frac{1}{\{g^n P(g)\}^{(k)}}\right) + O(\log r). \\
&\leq \frac{1}{\frac{n+m-2k}{m}} N\left(r,\frac{1}{g}\right) + \overline{N}_{k-1)}\left(r,\frac{1}{g-1}\right) + \frac{1}{(n+m)s+2k} \overline{N}\left(r,\frac{1}{g-1}\right) \\
&+ \frac{1}{(n+m)s+k}\left\{k\overline{N}(r,g) + k\overline{N}\left(r,\frac{1}{g}\right) + N_k\left(r,\frac{1}{g-1}\right)\right\} \\
&+ O(\log r) + S(r,g).
\end{aligned}$$

$$\overline{N}(r,f) \leq \left\{\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} + \frac{2k+1}{(n+m)s+k} + 1 - \Theta_{k-1)}(1,g) + \epsilon\right\} T(r,g) + S(r,g). \tag{3.13}$$

By (3.13), the above analysis and the second fundamental theorem we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + S(r,f) \\ &\leq \left\{\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} + \frac{2k+1}{(n+m)s+k} + 1 - \Theta_{k-1)}(1,g) + \epsilon\right\} \\ &\quad T(r,g) \\ &+ \frac{m}{n+m-2k} N\left(r,\frac{1}{f}\right) + \overline{N}_{k)}\left(r,\frac{1}{f-1}\right) + \frac{1}{(n+m)s+2k} N\left(r,\frac{1}{f-1}\right) \\ &+ S(r,f) + S(r,g). \end{aligned}$$

$$\begin{aligned} T(r,f) \leq &\left\{\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} + \frac{2k+1}{(n+m)s+k} + 1 - \Theta_{k-1)}(1,g) + \epsilon\right\} \\ &T(r,g) \\ &+ \left\{\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} + 1 - \Theta_{k)}(1,f) + \epsilon\right\} T(r,f) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{3.14}$$

Similarly

$$\begin{aligned} T(r,g) \leq &\left\{\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} + \frac{2k+1}{(n+m)s+k} + 1 - \Theta_{k-1)}(1,f) + \epsilon\right\} \\ &T(r,f) \\ &+ \left\{\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} + 1 - \Theta_{k)}(1,g) + \epsilon\right\} T(r,g) \\ &+ S(r,f) + S(r,g). \end{aligned} \tag{3.15}$$

From (3.14) and (3.15) we get

$$(-\chi_1 - 2\epsilon)T(r,f) + (-\chi_2 - 2\epsilon)T(r,g) \leq S(r,f) + S(r,g), \tag{3.16}$$

where χ_1 and χ_2 are defined as in (1.6) and (1.7) respectively. From (3.16) and the condition $max\{\chi_1, \chi_2\} < 0$ we get a contradiction.

Case 2. Suppose that $F_1 \equiv G_1$. i.e. Then from (3.1) we get

$$f^n P(f) \equiv g^n P(g).$$

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i.e.,

$$f^n(a_m f^m + ... + a_0) = g^n(a_m g^m + ... + a_0). \tag{3.17}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ into (3.17) we deduce

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1} g^{n+m-1}(h^{n+m-1} - 1) + ... + a_0 g^n(h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = (n+m, ...n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$. Thus $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = (n+m, ...n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$. Thus $f(z) = tg(z)$ for a constant t such that $t^d = 1$, where $d = (n+m, ...n+m-i, ..., n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$. If h is not a constant, then we know by (3.17) that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(w_1, w_2) = w_1^m(a_m w_2^m + a_{m-1} w_2^{m-1} + ... + a_0) - w_2^n(a_m w_2^m + a_{m-1} w_2^{m-1} + ... + a_0).$$

This proves the Theorem 1.1. □

Proof. From (2.3), (3.1)-(3.6) and the condition $s(n+m) > 3k + 2m + 10$, we get

$$\Delta_1 > 4k + 13. \tag{3.18}$$

Similarly, from (2.4), and (3.1)-(3.6) we get

$$\Delta_2 > 4k + 13. \tag{3.19}$$

From (3.18), (3.19), Lemma 2.2 and the condition that $F_1^{(k)} - P$ and $G_1^{(k)} - P$ share 0 CM we know that F_1 and G_1 are transcendental meromorphic functions such that $F_1^{(k)} G_1^{(k)} = P^2$ or $F_1 = G_1$.

Proceeding as in the proof of Theorem 1.1 we can prove Theorem 1.2 □

Proof. Let

$$F_1 = f^n(f-1)^m \ and \ G_1 = g^n(g-1)^m. \tag{3.20}$$

Consider

$$\begin{aligned} \overline{N}\left(r, \frac{1}{F_1}\right) = \overline{N}\left(r, \frac{1}{f^n(f-1)^m}\right) &\leq \frac{1}{s(n+m)} N\left(r, \frac{1}{F_1}\right) \\ &\leq \frac{1}{s(n+m)}[T(r, F_1) + o(1)], \end{aligned}$$

then we have

$$\Theta(0, F_1) = 1 - \limsup_{r \to \infty} \frac{\overline{N}\left(r, \frac{1}{F_1}\right)}{T(r, F_1)} \geq 1 - \frac{1 + m^*}{s(n+m)}, \tag{3.21}$$

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and

$$\Theta(\infty, F_1) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, F_1)}{T(r, F_1)} \geq 1 - \frac{1}{s(n+m)}. \tag{3.22}$$

Similarly

$$\Theta(0, G_1) \geq 1 - \frac{1+m^*}{s(n+m)}, \quad \Theta(\infty, G_1) \geq 1 - \frac{1}{s(n+m)}, \tag{3.23}$$

where

$$m^* = \begin{cases} 0 & \text{if } m = 0 \\ 1 & \text{if } m \geq 1 \end{cases}$$

Consider

$$\begin{aligned} N_{k+1}\left(r, \frac{1}{F_1}\right) = N_{k+1}\left(r, \frac{1}{f^n(f-1)^m}\right) &= (k+m+1)\overline{N}\left(r, \frac{1}{f^n(f-1)^m}\right) \\ &\leq \frac{k+m+1}{s(n+m)}[T(r, F_1) + O(1)] \end{aligned}$$

Next, we have

$$\delta_{k+1}(0, F_1) = 1 - \limsup_{r\to\infty} \frac{N_{k+1}(r, \frac{1}{F_1})}{T(r, F_1)} \geq 1 - \frac{k+m+1}{s(n+m)}. \tag{3.24}$$

Similarly

$$\delta_{k+1}(0, G_1) \geq 1 - \frac{k+m+1}{s(n+m)}. \tag{3.25}$$

From (2.1), (3.20)-(3.25) and the condition $s(n+m) > 9k + 5m + 2m^* + 14$, we get

$$\Delta_1 > 4k + 13. \tag{3.26}$$

Similarly, from (2.2), (3.20)-(3.25) we get

$$\Delta_2 > 4k + 13. \tag{3.27}$$

From (3.26), (3.27), Lemma 2.1 and the condition that $F_1^{(k)} - P$ and $G_1^{(k)} - P$ share 0 IM we know that F_1 and G_1 are transcendental meromorphic functions such that $F_1^{(k)}G_1^{(k)} = P^2$ or $F_1 = G_1$. We discuss the following two cases.

Case 1. Suppose that $F_1^{(k)}G_1^{(k)} = P^2$. Then it follows from (3.20) that

$$\{f^n(f-1)^m\}^{(k)}\{g^n(g-1)^m\}^{(k)} = P^2. \tag{3.28}$$

Let $z_0 \notin \{z : P(z) = 0\}$ be a zero of f of order p. Then it follows from (3.28) that z_0 is a pole of g. Suppose that z_0 is a pole of g of order q, then we have

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$np-k=(n+m)q+k$, i.e., $n(p-q)=mq+2k$, which implies that $p\geq q+1$ and $q\geq \frac{n-2k}{m}$, so we have

$$p\geq \frac{n+m-2k}{m}, \tag{3.29}$$

Let $z_1\notin\{z: P(z)=0\}$ be a zero of $(f-1)^m$ of order $p_1\geq k+1$, then it follows from (3.28) that z_1 is a pole of g. Suppose that z_1 is a pole of g of order q_1. Then from (3.28) we have $p_1-k=(n+m)q_1+k$. From this we get

$$p_1\geq (n+m)s+2k. \tag{3.30}$$

Let $z_2\notin\{z: P(z)=0\}$ be a zero of $\{f^n(f-1)^m\}^{(k)}$ of order p_2 that is not a zero of $f(f-1)$. Then from (3.28) we see that z_2 is a pole of g. Suppose that z_2 is a pole of g of order q_2, then $p_2=(n+m)q_2+k$. Thus

$$p_2\geq (n+m)s+k. \tag{3.31}$$

Let $z_3\notin\{z: P(z)=0\}\bigcup\{z: f(z)(f(z)-1)=0\}$ be a zero of $\{f^n(f-1)^m\}^{(k)}$ of multiplicity p_3. Then, from (3.28) we deduce that z_3 is a pole of g of multiplicity q_3, say. Hence $p_3=(n+m)q_3+k\geq(n+m)s+k$. This together with (3.29)-(3.31) and Lemma 2.3 gives

$$\begin{aligned}
\overline{N}(r,f) &\leq \overline{N}\left(r,\frac{1}{g}\right)+\overline{N}_{k-1)}\left(r,\frac{1}{g-1}\right)+\frac{1}{(n+m)s+2k}N\left(r,\frac{1}{g-1}\right)\\
&+ \frac{1}{(n+m)s+k}N_0\left(r,\frac{1}{\{g^n(g-1)^m\}^{(k)}}\right)+O(log\, r)\\
&\leq \frac{1}{\frac{n+m-2k}{m}}N\left(r,\frac{1}{g}\right)+\overline{N}_{k-1)}\left(r,\frac{1}{g-1}\right)+\frac{1}{(n+m)s+2k}\overline{N}\left(r,\frac{1}{g-1}\right)\\
&+ \frac{1}{(n+m)s+k}\left\{k\overline{N}(r,g)+k\overline{N}\left(r,\frac{1}{g}\right)+N_k\left(r,\frac{1}{g-1}\right)\right\}\\
&+ O(log\, r)+S(r,g).
\end{aligned}$$

$$\begin{aligned}
\overline{N}(r,f)\leq &\left\{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}+1-\Theta_{k-1)}(1,g)+\epsilon\right\}*\\
&T(r,g)+S(r,g).
\end{aligned} \tag{3.32}$$

By the above analysis and the second fundamental theorem we get

$$\begin{aligned}
T(r,f) &\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)+S(r,f)\\
&\leq \left\{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}+1-\Theta_{k-1)}(1,g)+\epsilon\right\}\\
&\quad T(r,g)
\end{aligned}$$

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$$\begin{aligned}
&+ \quad \frac{m}{n+m-2k}\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}_{k)}\left(r,\frac{1}{f-1}\right)+\frac{1}{(n+m)s+2k}N\left(r,\frac{1}{f-1}\right)\\
&+ \quad S(r,f)+S(r,g).
\end{aligned}$$

$$\begin{aligned}
T(r,f) \le &\left\{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}+1-\Theta_{k-1)}(1,g)+\epsilon\right\}\\
&T(r,g)\\
&+\left\{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+1-\Theta_{k)}(1,f)+\epsilon\right\}T(r,f)\\
&+S(r,f)+S(r,g).
\end{aligned} \tag{3.33}$$

Similarly

$$\begin{aligned}
T(r,g) \le &\left\{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2k+1}{(n+m)s+k}+1-\Theta_{k-1)}(1,f)+\epsilon\right\}\\
&T(r,f)\\
&+\left\{\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+1-\Theta_{k)}(1,g)+\epsilon\right\}T(r,g)\\
&+S(r,f)+S(r,g).
\end{aligned} \tag{3.34}$$

From (3.33) and (3.34) we get

$$(-\chi_1-2\epsilon)T(r,f)+(-\chi_2-2\epsilon)T(r,g)\le S(r,f)+S(r,g), \tag{3.35}$$

where χ_1 and χ_2 are defined as in (1.8) and (1.9) respectively. From (3.35) and the condition $max\{\chi_1,\, \chi_2\}<0$ we get a contradiction.

Case 2. Suppose that $F_1 \equiv G_1$. i.e. Then from (3.20) we get

$$f^n(f-1)^m \equiv g^n(g-1)^m. \tag{3.36}$$

Now we consider the following three cases.

Case i. Let $m=0$. Then from (3.36) we get $f=tg$ for a constant t such that $t^n=1$.

Case ii. Let $m=1$. Then from (3.36), we have $f^n(f-1)\equiv g^n(g-1)$. Let

$$h=\frac{f}{g} \tag{3.37}$$

We discuss the following two subcases.

Subcase i. Suppose that h is a nonconstant meromorphic function. Then from (3.36) and (3.37) we get

$$g = \frac{1-h^n}{1-h^{n+1}}. \tag{3.38}$$

Noting that n and $n+1$ are two relatively prime integers, from (3.37), (3.38), Lemma 2.4 and the standard Valiron-Mokhon'ko lemma we get

$$T(r,f) = T(r,hg) = (n+1)T(r,h) + O(1). \tag{3.39}$$

From (3.37) - (3.39) and the second fundamental theorem we get

$$\overline{N}(r,f) = \sum_{j=1}^{n} \overline{N}\left(r, \frac{1}{h-\lambda_j}\right) \geq (n-2)T(r,h) + S(r,f), \tag{3.40}$$

where λ_1, $\lambda_2, ..., \lambda_n$ are finite complex numbers satisfying $\lambda_j \neq 1$ and $\lambda_j^{n+1} = 1$ $(1 \leq j \leq n)$. From (3.39) and (3.40) we get

$$\Theta(\infty,f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{T(r,f)} \leq \frac{2}{n}. \tag{3.41}$$

which contradicts the assumption $\Theta(\infty,f) > \frac{2}{n}$.

Subcase ii. Suppose that h is a constant. If $h^{n+1} \neq 1$. From (3.36) and (3.37) we get (3.38). From (3.38) we know that g is a constant, which is impossible. Thus $h^{n+1} = 1$. From (3.36) and (3.37) we get

$$(h^{n+1}-1)g = h^n - 1. \tag{3.42}$$

From (3.42) and $h^{n+1} = 1$ we get $h^{n+1} = h^n = 1$, which implies $h = 1$. This together with (3.37) implies $f \equiv g$.

Case iii. Let $m \geq 2$. Then from (3.36) we obtain

$$\begin{aligned} &f^n(f^m + ... + (-1)^i m_{c_{m-i}} f^{m-i} + ... + (-1)^m) \\ &\qquad = g^n(g^m + ... + (-1)^i m_{c_{m-i}} g^{m-i} + ... + (-1)^m). \end{aligned} \tag{3.43}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = hg$ into (3.43), we obtain

$$g^{n+m}(h^{n+m}-1) + ... + (-1)^i m_{c_{m-i}} g^{m+n-i}(h^{n+m-i}-1) + ... + g^n(h^n-1) = 0$$

which implies $h = 1$. Thus $f(z) \equiv g(z)$. If h is not a constant, then we know by (3.43) that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(x,y) = x^n(x-1)^m - y^n(y-1)^m$. This completes the proof. □

Proof. From (2.3), (3.20)-(3.25) and the condition $s(n+m) > 3k+2m+2m^*+8$, we get

$$\Delta_1 > 4k + 13. \tag{3.44}$$

Similarly, from (2.4), and (3.20)-(3.25) we get

$$\Delta_2 > 4k + 13. \tag{3.45}$$

From (3.44), (3.45), Lemma 2.2 and the condition that $F_1^{(k)} - P$ and $G_1^{(k)} - P$ share 0 CM we know that F_1 and G_1 are transcendental meromorphic functions such that $F_1^{(k)} G_1^{(k)} = P^2$ or $F_1 = G_1$.

Proceeding as in the proof of Theorem 1.3 we can prove Theorem 1.4 □

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