International Electronic Journal of Pure and Applied Mathematics

Volume 10 No. 1 2016, 9-21
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v10i1.2

SIGNED EDGE TOTAL DOMINATION NUMBER OF GRAPHS

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Abstract: Let $\gamma'_{st}(G)$ be the signed edge total domination number of a graph G and let us define $g(n) = \min\{\gamma'_{st}(G)|$ G is a cubic graph of order n$\}$. The lower sharp bound of the signed edge total domination number of a cubic graph G is given. We determine the value of $g(n)$ and the signed edge total domination number of complete bipartite graphs.

AMS Subject Classification: 05C88, 05C89
Key Words: graph, complete bipartite graph, signed total edge domination number

1. Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order n of G denotes the number of vertices of G and the size m of G denotes the number of edges of G. For any edge $e \in E(G)$, $N_G(e)$ will denote the open neighborhood of e in G and $N_G[v] = N_G(e) \bigcup \{e\}$ will denote the closed neighborhood. For a subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced by S. For two disjoint subjects X and Y of $V(G)$, we write $E(X, Y) = \{uv \in E(G)|\ u \in X, v \in Y\}$. If $e = uv \in E(G)$, then we put $N_G(e) = \{u'v' \in E(G) \mid u' = u \text{ or } v' = v\} \backslash \{uv\}$.

Let G be a graph of order n($n \geq 3$). A signed edge total dominating function(SETDF) is a function $f : E(G) \to \{-1, +1\}$ satisfying $\sum\limits_{e' \in N_G(e)} f(e') \geq 1$ for every $e \in E(G)$. Let the value of a signed edge total dominating function be

Received: June 30, 2015

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$f(E(G)) = \sum_{e\in E(G)} f(e)$. The signed edge total domination number $\gamma_{st}^{'}(G)$ is the minimum value of a signed edge total dominating function on G, then the function f that has the minimum value is called $\gamma_{st}^{'}(G)$-function.

The signed edge total domination number was introduced by S.Akbari[5]. The signed edge total domination number has been studied by several authors. Let $g(n) = \min\{\gamma_{st}^{'}(G)|$ G is a cubic graph of order n $\}$ is the minimal value of a cubic graph. Here we give a sharp lower bound of cubic graphs and determine the exact value of $g(n)$ for cubic graphs and $\gamma_{st}^{'}(K_{m,n})$ for any complete bipartite graph with m and n.

2. Signed edge total domination number in cubic graph

By the definition of the signed edge total dominating function, it is not hard to see that two following lemmas are correct.

Lemma 1. *For any two vertex-disjoint graphs G_1 and G_2, $\gamma_{st}^{'}(G_1 \cup G_2) = \gamma_{st}^{'}(G_1) + \gamma_{st}^{'}(G_2)$.*

Lemma 2. *For any graph G, $\gamma_{st}^{'}(G) \equiv |E(G)| (\mathrm{mod}\, 2)$.*

Theorem 3. *If G is a cubic graph of order n, then $\gamma_{st}^{'}(G) \geq \lceil 9n/10 \rceil$.*

Proof. Let f be a $\gamma'_{st}(G)$-function, let $E_1 = \{e \in E(G) \mid f(e) = +1\}$ and $E_2 = \{e \in E(G) \mid f(e) = -1\}$, then $E(G) = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. We define two subgraphs G_1 and G_2 where $V(G_1) = V(G_2) = V(G)$, $E(G_1) = E_1$, $E(G_2) = E_2$. For every $u \in V(G)$, we denote $d^*(u) = d_{G_1}(u) - d_{G_2}(u)$. Since G is a cubic graph, so $d^*(u) \in \{1, 3\}$. Define $A = \{u \in V(G) \mid d^*(u) = 3\}$, $B = \{u \in V(G) \mid d^*(u) = 1\}$ and $G_A = G[A]$, $G_B = G[B]$ where $|E(G_A)| = s$, $|E(G_B)| = t$ and $|A| = p$.

Claim 1. $E(A, B) \cap E_2 = \emptyset$, $E(G_A) \cap E_2 = \emptyset$ and $E(G_B) \cap E_1 = \emptyset$.

(i) Assume, to the contrary, that there exists an edge $e = uv \in E(A, B) \cap E_2$ for $u \in A, v \in B$, and $f(e) = -1$, we have $d^*(u) = d^*(v) = 1$, which is a contradiction.

(ii) Analogously, if there exists an edge $e = uv \in E(G_A) \cap E_2$ and $f(e) = -1$, so that $d^*(u) = d^*(v) = 1$, which is a contradiction.

(iii) Assume that there exists an edge $e = uv \in E(G_B) \cap E_1$ and $f(e) = +1$, so that $\sum_{e' \in N_G(e)} f(e') = d^*(u) + d^*(v) - 2f(e) = 1 + 1 - 2 = 0$, which is a contradiction again.

Claim 2. $E(G_B) \neq \emptyset$, $\Delta(G_B) = \delta(G_B) = 1$(and hence $G_B = qK_2$ for some $q \geq 1$).

Assume that $E(G_B) = \emptyset$. For every $u \in V(G_B)$, according to Claim 1, we have that the number of negative edges that are incident with u is zero, which implies $d^*(u) = 3$, which is a contradiction. Thus, $\delta(G_B) \geq 1$.

Next, we prove that $\Delta(G_B) \leq 1$.

Assume, to the contrary that there exists $v_0 \in B$ such that $d_{G_B}(v_0) = \Delta(G_B) \geq 2$. By Claim 1, we have v_0 are incident with more than two negative edges. By the definition of $d^*(u)$, $d^*(u) = -1$ or $d^*(u) = -3$,which is a contradiction. Hence, $\Delta(G_B) = \delta(G_B) = 1$.

According to Claim 1 and Claim 2, we have that $|E(A, B)| = 4t$, so $3n/2 = s+5t$. Thus

$$\begin{aligned} \gamma'_{st}(G) &= s + 4t - t = s + 3t = s + \frac{3}{5}(\frac{3}{2}n - s) = \frac{9}{10}n + \frac{2}{5}s \\ &\geq \lceil 9n/10 \rceil. \end{aligned}$$

This completes the proof of Theorem 3. □

By the following theorem,we have that the lower bound is sharp.

Theorem 4. *Let $g(n)$ be the minimal signed edge total domination number of any cubic graph with order n, then*

(I) $g(4) = 4,\ g(6) = 7,\ g(8) = 8$;

$$(II) \qquad g(n) = \begin{cases} \frac{9n}{10} & n \equiv 0 (\mathrm{mod}\, 10), \\ \frac{9n+12}{10} & n \equiv 2 (\mathrm{mod}\, 10), \\ \frac{9n+4}{10} & n \equiv 4 (\mathrm{mod}\, 10), \\ \frac{9n+16}{10} & n \equiv 6 (\mathrm{mod}\, 10), \\ \frac{9n+8}{10} & n \equiv 8 (\mathrm{mod}\, 10). \end{cases}$$

Proof. (I) It is not hard to find that it is right.

(II) Let f_i be defined in G_i $(i = 1, 2, 3, 4, 5)$ as follows: if $e \in E(G_i)$ is bold, then $f_i(e) = -1$, otherwise $f_i(e) = +1$. And it is not hard to find that f_i is an SETDF of G_i $(i = 1, 2, 3, 4, 5)$.

(i) By Theorem 3, $g(10) \geq 9$. $\gamma'_{st}(G_1) \leq 9$, thus $g(10) = \gamma'_{st}(G_1) = 9$.

(ii) By Theorem 3, $g(12) \geq \lceil \frac{9}{10} \times 12 \rceil = 11$, and by Lemma 2, $g(12) \geq 12$. $\gamma'_{st}(G_2) \leq 12$, thus $g(12) = \gamma'_{st}(G_2) = 12$.

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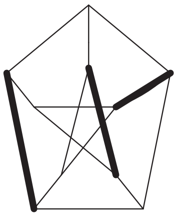

Figure1:G1

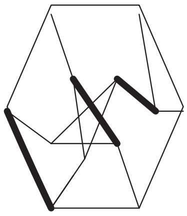

Figure2:G2

Figure3:G3

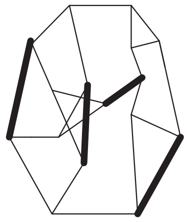

Figure4:G4

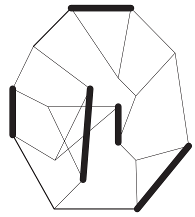

Figure5:G5

(iii) Analogously, we can know that $g(14) = \gamma_{st}^{'}(G_3) = 13,\ g(16) = \gamma_{st}^{'}(G_4) = 16,\ g(18) = \gamma_{st}^{'}(G_5) = 18$.

By Lemma 2 and Theorem 3, it is easy to see that $g(n)$ is no less than the right part of equation (II). The next we will split discussion into ten parts, in which we construct a cubic graph respectively such that the signed edge total domination number of the cubic graph is equal to the right part of equation (II).

(1) $n = 20k$. Let $G = 2k \times G_1$, then

$$\gamma_{st}^{'}(G) \leq 2k \times \gamma_{st}^{'}(G_1) = 18k = \frac{9n}{10}.$$

(2) $n = 20k + 2$. Let $G = (2k-1) \times G_1 \cup G_2$, then

$$\gamma_{st}^{'}(G) \leq (2k-1)\gamma_{st}^{'}(G_1) + \gamma_{st}^{'}(G_2) = (2k-1) \times 9 + 12 = 18k + 3 = \frac{9n+12}{10}.$$

(3) $n = 20k + 4$. Let $G = (2k-1) \times G_1 \cup G_3$, then

$$\gamma_{st}^{'}(G) \leq (2k-1)\gamma_{st}^{'}(G_1) + \gamma_{st}^{'}(G_3) = (2k-1) \times 9 + 13 = 18k + 4 = \frac{9n+4}{10}.$$

(4) $n = 20k + 6$. Let $G = (2k-2) \times G_1 \cup G_2 \cup G_3$, then

$$\begin{aligned}\gamma_{st}^{'}(G) &\leq (2k-2)\gamma_{st}^{'}(G_1) + \gamma_{st}^{'}(G_2) + \gamma_{st}^{'}(G_3)\\ &= (2k-2) \times 9 + 12 + 13 = 18k + 7 = \frac{9n+16}{10}.\end{aligned}$$

(5) $n = 20k + 8$. Let $G = (2k-2) \times G_1 \cup (2 \times G_3)$, then

$$\gamma_{st}^{'}(G) \leq (2k-2) \times 9 + 2 \times 13 = 18k + 8 = \frac{9n+8}{10}.$$

(6) $n = 20k + 10$. Let $G = (2k+1) \times G_1$, then

$$\gamma_{st}^{'}(G) \leq (2k+1)\gamma_{st}^{'}(G_1) = (2k+1) \times 9 = 18k + 9 = \frac{9n}{10}.$$

(7) $n = 20k + 12$. Let $G = 2k \times G_1 \cup G_2$, then

$$\gamma'_{st}(G) \leq 2k\gamma'_{st}(G_1) + \gamma'_{st}(G_2) = 2k \times 9 + 12 = 18k + 12 = \frac{9n + 12}{10}.$$

(8) $n = 20k + 14$. Let $G = 2k \times G_1 \cup G_3$, then

$$\gamma'_{st}(G) \leq 2k\gamma'_{st}(G_1) + \gamma'_{st}(G_3) = 2k \times 9 + 13 = 18k + 13 = \frac{9n + 4}{10}.$$

(9) $n = 20k + 16$. Let $G = (2k - 1) \times G_1 \cup G_2 \cup G_3$, then

$$\gamma'_{st}(G) \leq (2k - 1)\gamma'_{st}(G_1) + \gamma'_{st}(G_2) + \gamma'_{st}(G_3)$$
$$= (2k - 1) \times 9 + 12 + 13 = \frac{9n + 16}{10}.$$

(10) $n = 20k + 18$. Let $G = (2k - 1) \times G_1 \cup (2 \times G_3)$, then

$$\gamma'_{st}(G) \leq (2k - 1)\gamma'_{st}(G_1) + 2\gamma'_{st}(G_3)$$
$$= (2k - 1) \times 9 + 2 \times 13 = 18k + 17 = \frac{9n + 8}{10}. \quad \square$$

3. Signed Edge Total Domination of Complete Bipartite Graphs

Theorem 5. *Let $K_{m,n}$ be a complete bipartite graph. If m and n are even where $m \leq n$, then:*

(I) If $m = 2$, then

$$\gamma'_{st}(K_{2,n}) = 4.$$

(II) If $m \geq 4$, then

$$\gamma'_{st}(K_{m,n}) = \begin{cases} 2m & \text{if} \quad m = n, \\ 2n & \text{if} \quad m + 2 \leq n \leq \frac{3}{2}m, \\ 3m & \text{if} \quad \frac{3}{2}m + 2 \leq n \leq 3m, \\ n + 2 & \text{if} \quad 3m + 2 \leq n \leq 4m - 6, \\ 4m - 4 & \text{if} \quad n \geq 4m - 4. \end{cases}$$

Proof. Let f be an $\gamma'_{st}(K_{m,n})$-function where $K_{m,n} = (X, Y)$ is a complete bipartite graph with $X = (x_1, x_2, x_3, \cdots, x_m)$ and $Y = (y_1, y_2, y_3 \cdots, y_n)$. Let the value of an edge $e \in E(K_{m,n})$ be positive if $f(e) = 1$, otherwise the value of

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the edge is negative. We can write $E(K_{m,n})$ as the disjoint union of two sets $E_1 = \{e \in E(K_{m,n}) \mid f(e) = +1\}$ and $E_2 = \{e \in E(K_{m,n}) \mid f(e) = -1\}$ where $|E_2| = s$. Denote the number of negative edges that are incident with vertex $x_i(y_j)$ by $a_i(b_j)$, respectively. For any edge $e \in E(K_{m,n})$,

$$\sum_{e' \in N_{K_{m,n}}(e)} f(e') = f(N_{K_{m,n}}(e)) \in \{2, 4, 6, \cdots, n\}$$

is true because m, n are even. There are two cases for this theorem.

(I) Let $m = 2$ and n is even. We will prove that $\gamma'_{st}(K_{2,n}) = 4$ by induction on n.

When $n = 2$, it's obvious that the value of every edge in $K_{2,2}$ is positive, so $\gamma'_{st}(K_{2,2}) = 4$.

When $n = 4$, for any edge $e \in E(K_{2,4})$, it's open neighborhood is $N_{K_{2,4}}(e)$ and $|N_{K_{2,4}}(e)| = 4$.

Assume that g_4 is an SETDF of $K_{2,4}$, then there are at most one negative edge by the definition of SETDF. We have

$$\sum_{e' \in N_{K_{2,4}}(e)} g_4(e') \in \{2, 4\},$$

then

$$g_4(E(K_{2,4})) = \frac{\sum\limits_{e \in E(K_{2,4})} \sum\limits_{e' \in N_{K_{2,4}}(e)} g_4(e')}{|N_{K_{2,4}}(e)|} \geq \frac{2 \times 8}{2} = 4,$$

so $\gamma'_{st}(K_{2,4}) \geq 4$.

Assume that g'_4 is a function of $K_{2,4}$, and

$$g'_4(x_i y_j) = \begin{cases} -1 & \text{if} \quad i = 1, 2;\ j = 1, \\ +1 & \text{if} \quad i = 1, 2;\ j = 2, 3, 4. \end{cases}$$

It is not hard to find that g'_4 is an SETDF of $K_{2,4}$ and $g'_4(E(K_{2,4})) = 4$, so $\gamma'_{st}(K_{2,4}) \leq 4$. Hence, we have $\gamma'_{st}(K_{2,4}) = 4$.

Assume that the theorem is true for $n = 2k (k \geq 3)$, that is, $\gamma'_{st}(K_{2,2k}) = 4$. Then there exists $\gamma'_{st}(K_{2,2k})$-function of $K_{2,2k}$, say f_{2k}, satisfies $f_{2k}(E(K_{2,2k})) = 4$. For any edge $e \in E(K_{2,2k})$, we all have $f_{2k}(N_{K_{2,2k}}(e)) \in \{2, 4, 6, \ldots, 2k\}$. Hence,

$$\gamma'_{st}(K_{2,2k}) = \frac{\sum\limits_{e \in E(K_{2,2k})} \sum\limits_{e' \in N_{K_{2,2k}}(e)} f_{2k}(e')}{|N_{K_{2,2k}}(e)|} \geq \frac{2 \times 2k \times 2}{2k} = 4.$$

So for every $e \in E(K_{2,2k})$, $\sum\limits_{e' \in N_{K_{2,2k}}(e)} f_{2k}(e') = 2$.

Assume that $n = 2k + 2 (k \geq 3)$. Let

$$V(K_{2,2k+2}) = V(K_{2,2k}) \cup \{y_{2k+1}, y_{2k+2}\}$$

and

$$E(K_{2,2k+2}) = E(K_{2,2k}) \cup \{x_1y_{2k+1}, x_1y_{2k+2}, x_2y_{2k+1}, x_2y_{2k+2}\}.$$

We assume that g_{2k+2} is an SETDF of $K_{2,2k+2}$, then for every edge e, we have

$$g_{2k+2}(N_{K_{2,2k+2}}(e)) \in \{2, 4, 6, \ldots, 2k+2\},$$

then

$$g_{2k+2}(E(K_{2,2k+2})) = \frac{\sum\limits_{e \in E(K_{2,2k+2})} \sum\limits_{e' \in N_{K_{2,2k+2}}(e)} g_{2k+2}(e')}{|N_{K_{2,2k+2}}(e)|} \geq \frac{2 \times 2 \times (2k+2)}{2k+2} = 4,$$

so $\gamma'_{st}(K_{2,2k+2}) \geq 4$. Assume that g'_{2k+2} is a function of $K_{2,2k+2}$, and

$$g'_{2k+2}(e) = \begin{cases} f_{2k}(e) & \text{if} \quad e \in \{x_iy_j \mid i = 1, 2;\ j = 1, 2, \ldots, 2k\}, \\ -1 & \text{if} \quad e \in \{x_iy_j \mid i = 1, 2;\ j = 2k+1\}, \\ +1 & \text{if} \quad e \in \{x_iy_j \mid i = 1, 2;\ j = 2k+2\}. \end{cases}$$

It is not hard to find that g'_{2k+2} is an SETDF of $K_{2,2k+2}$ and $g'_{2k+2}(E(K_{2,2k+2})) = 4$, so $\gamma'_{st}(K_{2,2k+2}) \leq 4$.

Hence, we have $\gamma'_{st}(K_{2,2k+2}) = 4$.

(II) There are five cases for the signed edge total domination number of a complete bipartite graph $K_{m,n}$ when $m \geq 4$ and m, n are even.

Case 1. Let $n = m$, $\gamma'_{st}(K_{m,n}) = 2m$.

We prove that $\gamma'_{st}(K_{m,n}) \geq 2m$ by contradiction.

Assume that $\gamma'_{st}(K_{m,n}) = mn - 2s \leq 2m - 2$, according to Lemma 2, we can deduce that $s \geq (mn - 2m + 2)/2$, then

$$\begin{cases} s/m \geq \lceil (mn - 2m + 2)/2m \rceil = n/2, \\ s/n \geq \lceil (mn - 2m + 2)/2n \rceil = m/2. \end{cases}$$

Denote the number of negative edges that are incident with vertex $x_i(y_j)$ by $a_i(b_j)$ and there exist some $i, j (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$, such that $a_i \geq n/2$ and $b_j \geq m/2$. Then there exists an edge $x_iy_j \in E(K_{m,n})$,

$$\sum_{e \in N_{K_{m,n}}(x_iy_j)} f(e) = m + n - 2 - 2(a_i + b_j) \leq 0,$$

this is inconsistent with definition of f. Hence $\gamma_{st}^{'}(K_{m,n}) \geq 2m$.

Let f be defined as follows: Let $m = n = 2p$,

$$f'(x_iy_j) = \begin{cases} +1 & \text{if} \quad i = 1, \ldots, p;\ j = p+1, \ldots, 2p, \\ +1 & \text{if} \quad i = p+1, \ldots, 2p;\ j = 1, \ldots, p, \\ +1 & \text{if} \quad i = j = p+1, \ldots, 2p, \\ +1 & \text{if} \quad j = i+1;\ i = p+1, \ldots, 2p, \\ +1 & \text{if} \quad i = 2p;\ j = p+1, \\ -1 & \text{if} \quad others. \end{cases}$$

It is not hard to find that f' is an SETDF of $K_{m,n}$, then it implies that

$$\gamma_{st}^{'}(K_{m,n}) \leq mn - 2[\frac{m}{2} \cdot \frac{n}{2} + \frac{m}{2}(\frac{n}{2} - 2)] = 2m.$$

Hence, $\gamma_{st}^{'}(K_{m,n}) = 2m$.

Case 2. Let $m + 2 \leq n \leq \frac{3}{2}m$, $\gamma_{st}^{'}(K_{m,n}) = 2n$.

We prove that $\gamma_{st}^{'}(K_{m,n}) \geq 2n$ by contradiction. We assume that $\gamma_{st}^{'}(K_{m,n}) = mn - 2s < 2n$. According to Lemma 2, we have $\gamma_{st}^{'}(K_{m,n}) = mn - 2s \leq 2n - 2$. We can deduce that $s \geq (mn - 2n + 2)/2$. Then

$$\begin{cases} s/m \geq \lceil mn - 2n + 2)/2m \rceil = n/2 - 1, \\ s/n \geq \lceil (mn - 2n + 2)/2n \rceil = m/2. \end{cases}$$

Denote the number of negative edges that are incident with vertex $x_i(y_j)$ by $a_i(b_j)$ and there exist some $i, j(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$, such that $a_i \geq n/2 - 1$ and $b_j \geq m/2$. There exists an edge $x_iy_j \in E(K_{m,n})$,

$$\sum_{e \in N_{K_{m,n}}(x_iy_j)} f(e) = m + n - 2 - 2(a_i + b_j) \leq 0,$$

this is inconsistent with definition of f. Hence $\gamma_{st}^{'}(K_{m,n}) \geq 2n$.

Let f' be defined as follows: Let $m = 2p, n = 2q$,

$$f'(x_iy_j) = \begin{cases} +1 & \text{if} \quad i = 1, \dots, p;\ j = q+1, \dots, 2q, \\ +1 & \text{if} \quad i = p+1, \dots, 2p;\ j = 1, \dots, q, \\ +1 & \text{if} \quad j \equiv q+1 (\bmod 2)\ \textit{and}\ j \ \geq p+1; \\ & \quad i = j+p-q, j+p-q+1, \\ +1 & \text{if} \quad j \equiv q+2 (\bmod 2)\ \textit{and}\ j \ \geq p+1; \\ & \quad i = j+p-q-1, j+p-q, \\ -1 & \text{if} \quad \textit{others}. \end{cases}$$

It is not hard to find that f' is an SETDF of $K_{m,n}$, then it implies

$$\gamma_{st}^{'}(K_{m,n}) \leq mn - 2[\frac{m}{2} \cdot \frac{n}{2} + \frac{n}{2}(\frac{m}{2} - 2)] = 2n.$$

Hence, $\gamma_{st}^{'}(K_{m,n}) = 2n$.

Case 3. $\frac{3}{2}m + 2 \leq n \leq 3m$, $\gamma_{st}^{'}(K_{m,n}) = 3m$.

We prove that $\gamma_{st}^{'}(K_{m,n}) \geq 3m$ by contradiction. We assume that $\gamma_{st}^{'}(K_{m,n}) = mn - 2s < 3m$. According to Lemma 2, then

$$\gamma_{st}^{'}(K_{m,n}) = mn - 2s \leq 3m - 2$$

and we can deduce that $s \geq (mn - 3m + 2)/2$.

Then

$$\begin{cases} s/m \geq \lceil (mn - 3n + 2)/2m \rceil = n/2 - 2, \\ s/n \geq \lceil (mn - 3n + 2)/2n \rceil = m/2 - 1. \end{cases}$$

Denote the number of negative edges that are incident with $x_i(y_j)$ by $a_i(b_j)$ and there exist some $i, j (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$, such that $a_i \geq n/2 - 2$ and $b_j \geq m/2 - 1$. There exists an edge $x_iy_j \in E(K_{m,n})$,

$$\sum_{e \in N_{K_{m,n}}(x_iy_j)} f(e) = m + n - 2 - 2(a_i + b_j) \leq m + n - 2 - m - n + 3 = 1.$$

For any edge $e \in E(K_{m,n})$,

$$\sum_{e' \in N_{K_{m,n}}(e)} = f(N_{K_{m,n}}(e)) \in \{2, 4, 6, \cdots, n\},$$

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then this is inconsistent with definition of f.

Hence $\gamma'_{st}(K_{m,n}) \geq 3m$.

Let f' be defined as follows: Let $m = 2p, n = 2q$,

$$f'(x_iy_j) = \begin{cases} +1 & \text{if} \quad i = 1, \ldots, p;\ j = q+1, \ldots, 2q, \\ +1 & \text{if} \quad i = p+1, \ldots, 2p;\ j = 1, \ldots, q, \\ +1 & \text{if} \quad j \equiv q+1 (\bmod 3)\ \mathit{and}\ j \geq p+1; \\ & \quad 3i = j + 3p - q + 2, \\ +1 & \text{if} \quad j \equiv q+2 (\bmod 3)\ \mathit{and}\ j \geq p+1; \\ & \quad 3i = j + 3p - q + 1, \\ +1 & \text{if} \quad j \equiv q+3 (\bmod 3)\ \mathit{and}\ j \geq p+1; \\ & \quad 3i = j + 3p - q, \\ -1 & \text{if} \quad \mathit{others}. \end{cases}$$

It is not hard to find that f' is an SETDF of $K_{m,n}$, then it implies

$$\gamma'_{st}(K_{m,n}) \leq mn - 2[\frac{m}{2}(\frac{n}{2} - 1) + \frac{m}{2}(\frac{n}{2} - 2)] = 3m.$$

Hence, $\gamma'_{st}(K_{m,n}) = 3m$.

Case 4. Let $3m + 2 \leq n \leq 4m - 6$, $\gamma'_{st}(K_{m,n}) = n + 2$.

We prove that $\gamma'_{st}(K_{m,n}) \geq n + 2$ by contradiction. Assume that $\gamma'_{st}(K_{m,n}) = mn - 2s \leq n$. According to Lemma 2, we deduce that $s \geq (mn - n)/2$,

$$\begin{cases} s/m \geq \lceil (mn-n)/2m \rceil = n/2 - 3, \\ s/n \geq \lceil (mn-n)/2n \rceil = m/2. \end{cases}$$

Denote the number of negative edges that are incident with $x_i(y_j)$ by $a_i(b_j)$ and there exist some $i, j (i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$, such that $a_i \geq n/2 - 3$ and $b_j \geq m/2$. There exists an edge $x_iy_j \in E(K_{m,n})$,

$$\sum_{e \in N_{K_{m,n}}(x_iy_j)} f(e) = m + n - 2 - 2(a_i + b_j) \leq m + n - 2 - 2((m + n - 3)/2) = 1.$$

For any edge $e \in E(K_{m,n})$,

$$\sum_{e' \in N_{K_{m,n}}(e)} f(e') = f(N_{K_{m,n}}(e)) \in \{2, 4, 6, \ldots, n\},$$

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then this is inconsistent with definition of f. Hence $\gamma_{st}^{'}(K_{m,n}) \geq n+2$.

Let f' be defined as follows: Let $m = 2p, n = 2q$,

$$f'(x_iy_j) = \begin{cases} +1 & \text{if} \quad i = 1,\dots,p;\ j = q,\dots,2q, \\ +1 & \text{if} \quad i = p+1,\dots,2p;\ j = 1,\dots,q-1, \\ +1 & \text{if} \quad j \equiv q (\operatorname{mod} 3) \ \mathit{and}\ j \geq p+1; \\ & \quad 3i = j + 3p - q + 3, \\ +1 & \text{if} \quad j \equiv q+1 (\operatorname{mod} 3) \ \mathit{and}\ j \geq p+1; \\ & \quad 3i = j + 3p - q + 2, \\ +1 & \text{if} \quad j \equiv q+2 (\operatorname{mod} 3) \ \mathit{and}\ j \geq p+1; \\ & \quad 3i = j + 3p - q + 1, \\ -1 & \text{if} \quad \mathit{others}. \end{cases}$$

It is not hard to find that f' is an SETDF of $K_{m,n}$, then it implies

$$\gamma_{st}^{'}(K_{m,n}) \leq mn - 2[\frac{m}{2}(\frac{n}{2}-1) + (\frac{n}{2}+1)(\frac{m}{2}-1)] = n+2.$$

So $\gamma_{st}^{'}(K_{m,n}) = n+2$.

Case 5. Let $n \geq 4m-4$, $\gamma_{st}^{'}(K_{m,n}) = 4m-4$.

We prove that $\gamma_{st}^{'}(K_{m,n}) \geq 4m-4$ by contradiction. We assume that $\gamma_{st}^{'}(K_{m,n}) = mn - 2s \leq 4m-6$. According to Lemma 2, we deduce that $s \geq (mn-4m+6)/2$,

$$\begin{cases} s/m \geq \lceil (mn-4m+6)/2m \rceil = n/2 - 1, \\ s/n \geq \lceil (mn-4m+6)/2n \rceil = m/2. \end{cases}$$

Denote the number of negative edges that are incident with $x_i(y_j)$ by $a_i(b_j)$ and there exist some $i, j (i = 1, 2, \dots, m; j = 1, 2, \dots, n)$, such that $a_i \geq n/2 - 1$ and $b_j \geq m/2$. There exists an edge $x_iy_j \in E(K_{m,n})$,

$$\sum_{e \in N_{K_{m,n}}(x_iy_j)} f(e) = m + n - 2 - 2(a_i + b_j) - 2 = -1.$$

For any edge $e \in E(K_{m,n})$,

$$\sum_{e' \in N_{K_{m,n}}(e)} = f(N_{K_{m,n}}(e)) \in \{2, 4, 6, \dots, n\},$$

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then this is inconsistent with definition of f. Hence $\gamma_{st}^{'}(K_{m,n}) \geq 4m-4$.

Let f' be defined as follows: Let $m = 2p, n = 2q$,

$$f'(x_iy_j) = \begin{cases} +1 & \text{if} \quad i = 1, \ldots, p;\ j = q, \ldots, 2q, \\ +1 & \text{if} \quad i = p+1, \ldots, 2p;\ j = 1, \ldots, q-1, \\ +1 & \text{if} \quad j \equiv q (\bmod 3) \ \textit{and} \ j \geq p+1; \\ & \qquad 3i = j + 3p - q + 3, \\ +1 & \text{if} \quad j \equiv q+1 (\bmod 3) \ \textit{and} \ j \geq p+1; \\ & \qquad 3i = j + 3p - q + 2, \\ +1 & \text{if} \quad j \equiv q+2 (\bmod 3) \ \textit{and} \ j \geq p+1; \\ & \qquad 3i = j + 3p - q + 1, \\ -1 & \text{if} \quad \textit{others}. \end{cases}$$

It is not hard to find that f' is an SETDF of $K_{m,n}$, then it implies

$$\gamma_{st}^{'}(K_{m,n}) \leq mn - 2[(\frac{m}{2}+1)(\frac{n}{2}-1) + (\frac{m}{2}-1)(\frac{n}{2}-3)] = 4m-4.$$

Therefore $\gamma_{st}^{'}(K_{m,n}) = 4m-4$. □

Theorem 6. *Let $K_{m,n}$ be a complete bipartite graph. If m and n are odd where $m \leq n$, then*

(I) If $m = 1$, then

$$\gamma_{st}^{'}(K_{1,n}) = 3$$

(II) If $m \geq 3$, then

$$\gamma_{st}^{'}(K_{m,n}) = \begin{cases} 2m+1 & \textit{if} \quad m = n, \\ 3m-1 & \textit{if} \quad m+2 \leq n \leq 3m-4, \\ n+2 & \textit{if} \quad 3m-2 \leq n \leq 4m-3, \\ 4m-1 & \textit{if} \quad n \geq 4m-1. \end{cases}$$

We can prove this theorem is true in same way of proof of Theorem 5.

Acknowledgments

This work is supported by the National Natural Science Foundation of P.R. China (61170302).

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