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SOME CHARACTERIZATIONS OF SELF-SYMMETRIC RINGS WITH INVOLUTION

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Abstract: I.N. Herstein as well as Susan Montgomery have defined and made many contributions to the algebraic structures of a ring R with involution *. An intrinsic algebraic subset of such a ring is the set of symmetric elements $S = \{a|a = a^*\}$. A ring R with involution * is self-symmetric provided all its nontrivial ideals are symmetric. This paper will show that all self-symmetric rings with involution * are MP3 (Moore-Penrose 3) rings. In addition, some attempts will be made to characterize the $n \times n$ matrix ring, $M_n(R)$ over a self-symmetric MP3 ring.

This paper is dedicated to the late Dr. Cleon Russell Yohe,
algebraist at Washington University in St. Louis.

Key Words: involution, self-symmetric, automorphism, nilpotent

1. Ring With Involution and Self-Symmetric Ring

Definition 1. Let R be a ring with involution *. Call an ideal A of R self-symmetric if $A^* = \{a^*|a \in A\} = A$.

Note. More representations of rings with involution can be studied in the classical work by Herstein which has motivated some extensive algebraic research [1].

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Example 1. $0^* = 0$ (here 0 represents the zero ideal), and $R^* = R$ are trivial self-symmetric ideals of R.

Example 2. Let

$$R = \left\{ \begin{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \end{bmatrix} \middle| \{0, a, b, c, d\} \in F \right\} \quad (1)$$

with F a field and let * be matrix transpose on elements of R, i.e.,

$$\begin{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \end{bmatrix}^* = \begin{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}^T & 0 \\ 0 & \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}^T \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \end{bmatrix} \quad (2)$$

Obviously, any ideal of R is self-symmetric.

Definition 2. If R is a ring with involution *, then R is MP3 if for each nonzero $a \in R$, $\exists r \in R$ such that $ra \in S$, the set of symmetric elements in R. More succinctly, R is MP3 if for each $a \in R$, $a \neq 0$ then $Ra \cap S \neq 0$ where 0 represents the zero ideal.

Example 3. Let $C =$ the field of complex numbers with involution "*" equal to conjugation represented by $^-$, i. e., if $z = a + bi$, then $z^* = \bar{z} = a - bi$. Note that complex conjugation is a field automorphism of order 2. In fact, any field with an automorphism of order 2 may be classified as a ring with involution with involution being the particular automorphism. Note that in the case of C, the set of symmetric elements $S = R$ the set of real numbers.

Example 4. Now consider the Galois group $G(Q(\sqrt{2})/Q) \approx Z_2 = \{i_{Q(\sqrt{2})}, \phi\}$, where $i_{Q(\sqrt{2})}$ is the identity automorphism on

$$Q(\sqrt{2})\{a + b\sqrt{2} | a, bQ, \text{the field of rational numbers}\}$$

and $\phi : Q(\sqrt{2}) \to Q(\sqrt{2})$ is given by $\phi(a + b\sqrt{2}) = a - b\sqrt{2}$. Set $* = \phi$; then $Q(\sqrt{2})$ in a ring with involution * and its set of symmetric elements $S = Q$ (fixed field) under ϕ.

Note. Susan Montgomery has done representation of algebras [2] with involutions arising from field automorphisms of order 2.

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2. Self-Symmetric Rings Generalized To Matrix Rings

Theorem 1. *Let R be a ring with identity and involution *. If R is self-symmetric and R has no nonzero nilpotent left ideals then R is MP3.*

Proof. Let $a \in R, a \neq 0$. Will show that $Ra \cap S \neq 0$ where S is the set of symmetric elements of R. Since $a \neq 0$, the two-sided ideal generated by a, namely $RaR \neq 0$. Of course $RaR = \{as + ra + \sum_\alpha r_\alpha a s_\alpha | s, r, r_\alpha, s_\alpha \in R\}$. Since R is self-symmetric, $(RaR)^* = RaR$; thereby $(RaRa)^* = a^*Ra^*R = a^*RaR$; also, $(aRaR)^* = Ra^*Ra^* = RaRa^*$. If for no $r \in R, RaRa \neq 0$, then Ra is a nilpotent left ideal of R. This contradicts the hypothesis.

Claim. *$a^*RaRa \in Ra \cap S$ and $a^*RaRa \neq 0$.*

Proof. First, a^*RaRa lies in $Ra \cap S$ since a^*rasa is symmetric and nonzero since if $\forall r, s \in R, a^*rasa = 0$ then $(a^*(ras)^*a)^* = 0$ if and only if $a^*RaRa = 0$ if and only if $Ra^*RaRa = 0$ if and only if $RaRaRa = 0$ (since $RaR = Ra^*R$) contradicting hypothesis that R has no nonzero nilpotent left ideals. Therefore $a^*RaRa \neq 0$ and $Ra \cap S \neq 0$. Consequently, R is MP3.

Under the same condition imposed on R, the following generalization can be made:

Theorem 2. *Let R be a ring with identity and involution *. If R is self-symmetric then the $n \times n$ matrix ring $M_n(R)$ is also self-symmetric.*

Note. if $A = (a_{ij})$ is an element of $M_n(R)$ then set $A^* = (a^*_{ij})$.

Proof. Follows from the result due to Dennis S. Keeler [3] in his paper "Ideals of Matrix Rings" which asserts "Let R be a ring and let $M_n(R)$ be the ring of $n \times n$ matrices over R. If I is a (two sided) ideal of $M_n(R)$, then $I = M_n(A)$ for some uniquely determined ideal A of R. Conversely, if A is an ideal of R, then $M_n(A)$ is an ideal of $M_n(R)$. So the ideals of R are in 1-1 correspondence with the ideals of $M_n(R)$." In other words, given I an ideal of $M_n(R)$, $I \neq 0$, then $I = M_n(A)$ and $I^* = M_n(A)^* = M_n(A^*)$ for ideal A in R. But R is self-symmetric. Hence, $M_n(A^*) = M_n(A) = I$. Thereby $I = I^*$ and $M_n(R)$ is self-symmetric.

Theorem 3. *Let R be a ring with identity and involution *. If R is self-symmetric and R has nonzero nilpotent elements then $M_n(R)$ is MP3.*

The proof of the previous Theorem involves the proof of the following assertion offered itself as a major theorem:

Theorem 4. *If R is a ring with identity and involution * and R contains no nonzero nilpotent elements and R is MP3, then $M_n(R)$ is MP3.*

Proof. Since R is MP3 and let $A \in M_n(R)$, $A \neq 0$. Let a_{ij} be the first nonzero entry of A. Without loss of generality, assume $i = 1$ since A is row equivalent to the matrix resulting from switching the ith and 1st rows of A. Hence,

$$A = \begin{bmatrix} 0 & \dots & a_{1j} & a_{1,j+1} & \dots & a_{1n} \\ & & & & & \\ & & & & & \end{bmatrix}$$

and set

$$E_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \\ 0 & & \dots & 0 \end{bmatrix}$$

Then

$$E_1A = \begin{bmatrix} 0 & \dots & a_{1j} & a_{1,j+1} & \dots & a_{1n} \\ 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & \dots & & & & 0 \end{bmatrix}.$$

Since R is MP3, $\exists r_j \dots r_n \in R$ such that $r_j a_{1j} = s_j, \dots r_n a_{1n} = s_n$ where $s_j, s_{j+1}, \dots, s_n$ are all nonzero symmetric elements in S. Now set

$$P = \begin{bmatrix} r_j & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & \dots & 0 \end{bmatrix}.$$

Then

$$PE_1A = \begin{bmatrix} 0 & \dots & 0 & s_j & r_j a_{1,j+1} & \dots & r_j a_{1n} \\ 0 & \dots & 0 & & & & 0 \\ \vdots & & & & & & \vdots \\ 0 & \dots & 0 & & & & 0 \end{bmatrix}.$$

Now set $C = PE_1A$; then C^*C has $s_j^2 \neq 0$ in the (j,j) position since R has no nonzero nilpotent elements and C^* is the involution transpose of C. Hence, set $X = C^*PE_1$. Now $X \neq 0$ since otherwise $XA = C^*C = 0 \implies s_j^2 = 0$ which is a contradiction. Therefore, $M_n(R)$ is MP3.

Proof of Theorem 3. Since R is self-symmetric and R has no nonzero nilpotent elements, then R has no nonzero left ideals. Hence R by Theorem 1 is MP3. Since R is MP3 then under the hypothesis of Theorem 4, $M_n(R)$ is MP3.

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3. Deduction From MP3 Matrix Rings

Theorem 5. *If $M_n(R)$ is MP3 then R is MP3.*

Proof. Let $a \in R$, $a \neq 0$. Put $A = \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \neq 0$ in $M_n(R)$. Then $\exists X \in M_n(R)$, $X \neq 0$ such that $(XA)^* = XA \neq 0$. If $X = (x_{ij})$, then

$$XA = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ \vdots & & \\ x_{n1} & \dots & x_{nn} \end{bmatrix} \begin{bmatrix} a & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} = \begin{bmatrix} x_{11}a & 0 & \dots & 0 \\ x_{21}a & 0 & \dots & 0 \\ \vdots & & & \\ x_{n1}a & 0 & \dots & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} x_{11}a & 0 & \dots & 0 \\ x_{21}a & 0 & \dots & 0 \\ \vdots & & & \vdots \\ x_{n1}a & 0 & \dots & 0 \end{bmatrix} = XA = (XA)^* = \begin{bmatrix} (x_{11}a)^* & (x_{11}a)^* & \dots & (x_{n1}a)^* \\ 0 & & \dots & 0 \\ \vdots & & & \\ 0 & & \dots & 0 \end{bmatrix}.$$

Hence, $x_{i1}a = 0, \forall i > 1$. Since $XA \neq 0$, must have $x_{11}a \neq 0$. Thus, $x_{11}a = (x_{11}a)^*$ and thereby R is MP3.

Corollary. *If $M_n(R)$ is self-symmetric and MP3 then R is self-symmetric and MP3.*

Proof. Any nonzero ideal A of R becomes becomes an ideal $I = M_n(A)$ of $M_n(R)$. Then $I^* = M_n(A)^* = M_n(A^*) = I$ since $M_n(R)$ is self-symmetric. Thereby, $A = A^*$ and R is self-symmetric. Of course R is MP3 by previous Theorem.

References

[1] I.N. Herstein, *Rings With Involution*, University of Chicago Press, 1976.

[2] Susan Montgomery, Algebraic algebras with involution, *Proc. AMS*, **31**, No. 2 (1972).

[3] Dennis S. Keeler, *Ideals of Matrix Rings*, Miami University, Oxford, Ohio, 2001.