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UNIVALENCE CRITERION, STARLIKENESS AND CONVEXITY FOR A NEW INTEGRAL OPERATOR

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Abstract: For analytic functions f and g in the open unit disk U, a new integral operator $I(f, g)$ is introduced. The main objective of this paper is to obtain univalence, starlikeness and convexity conditions for the integral operator $I(f, g)$. Also, some other properties in the class $N(\beta)$ are given. Our main results contain some interesting corollaries as special cases.

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1. Introduction

Let $\mathcal{A}$ denote the class of normalized functions given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$.
Consider $S = \{f \in \mathcal{A} : f \text{ is univalent in } U\}$.

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By $S^*(\alpha)$ we denote a subclass of $\mathcal{A}$ consisting of starlike univalent functions of order $\alpha\,(0 \le \alpha < 1)$ which satisfies

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)}\right] > \alpha \quad (z \in U).$$

By $K(\alpha)$ we denote a subclass of $\mathcal{A}$ consisting of convex univalent functions of order $\alpha\,(0 \le \alpha < 1)$ which satisfies

$$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] > \alpha \quad (z \in U).$$

It is well known that:
(i) $S^*(0) \equiv S^*$ the class of all starlike functions with respect to the origin;
(ii)$K(0) \equiv K$ the class of all convex functions;
(iii) $K \subset S^* \subset S, \quad K(\alpha) \subset S^*(\alpha), \quad K(\alpha) \subset K$ and $S^*(\alpha) \subset S^*$.
Further, a function $f \in \mathcal{A}$ is said to be in the class $N(\beta)$ if

$$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)} + 1\right] < \beta \quad (\beta > 1, z \in U).$$

The class $N(\beta)$ was studied in [2, 10, 13].

Also, we say that a function $f \in \mathcal{A}$ is said to be in the class R_λ if and only if $\operatorname{Re}[f'(z)] > \lambda$, for all $z \in U$ and $0 \le \lambda < 1$.

Frasin and Jahangiri [6] introduced the family $B(\mu,\lambda), \mu \ge 0, 0 \le \lambda < 1$ consisting of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z) \left[\frac{z}{f(z)}\right]^{\mu} - 1 \right| < 1 - \lambda, \quad z \in U.$$

It is obvious that $B(1,\lambda) = S^*(\lambda)$, $B(2,\lambda) = B(\lambda)$ (see [7]) and $B(2,0) = S$ (see [4]).

For the functions $f, g \in \mathcal{A}$, we introduce a new integral operator defined by:

$$I(f,g)(z) = \int_0^z \left[\frac{te^{f(t)}}{g(t)}\right]^{\alpha} dt, \tag{2}$$

where parameter α is a complex number.

In this paper our purpose is to obtain univalence conditions, starlikeness properties and the order of convexity for the integral operator abovementioned. Recently, various types of integral operators were studied by different authors (see [5, 9, 11]), and some of them motivated us to come up with the integral operator defined in (2).

In the proof of our main results, we need to recall here the following:

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Lemma 1.1. *[3] Let $M_0 = 1,5936...$, the positive solution of equation*

$$(2 - M)e^M = 2. \tag{3}$$

If $f \in \mathcal{A}$ and

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M_0 \quad (z \in U) \tag{4}$$

then

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad (z \in U). \tag{5}$$

The edge M_0 is sharp.

Theorem 1.2. *[1] If the function f given by (1) is regular in the unit disk U and*

$$(1 - |z|^2) \cdot \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \quad (z \in U),$$

then the function f is univalent in U.

Lemma 1.3. *[8] Let f be regular function in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$ with $|f(z)| < M$, M fixed. If f has in z=0 one zero with multiply bigger than m, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \quad (z \in U_R).$$

The equality case hold only if $f(z) = e^{i\theta} \cdot \frac{M}{R^m} z^m$, where θ is constant.

Lemma 1.4. *[12] Let the functions p and q be analytic in U with*

$$p(0) = q(0) = 0,$$

and let δ be a real number. If the function q maps the unit disk U onto a region which is starlike with respect to the origin, the inequality

$$Re\left[\frac{p'(z)}{q'(z)}\right] > \delta \quad (z \in U)$$

implies that

$$Re\left[\frac{p(z)}{q(z)}\right] > \delta \quad (z \in U).$$

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2. Main Results

The univalence condition for the operator $I(f,g)$ defined in (2) is proved in the next theorem, by using the Becker univalence criterion.

Theorem 2.1. *Let α be a complex number, $f \in B(\mu,\lambda)$ and $g \in \mathcal{A}$. Suppose also that*

$$|f(z)| < M \text{ and } \left|\frac{g''(z)}{g'(z)}\right| \leq M_0, \quad z \in U, \tag{6}$$

where M is a positive real number, $M \geq 1$ and M_0 is the positive solution of the equation (3). If

$$|\alpha| \leq \frac{27b^2}{2[9b^2 - 1 + (3b^2+1)\sqrt{3b^2+1}]}, \quad b = (2-\lambda)M^{\mu}, \tag{7}$$

then the function $I(f,g)$ is in the class S.

Proof. It is easily seen that $I(f,g)$ is regular in U. Since

$$I'(f,g)(z) = \left[\frac{ze^{f(z)}}{g(z)}\right]^{\alpha}$$

and

$$I''(f,g)(z) = \alpha\left[\frac{ze^{f(z)}}{g(z)}\right]^{\alpha-1}\left[\frac{ze^{f(z)}}{g(z)} + \frac{ze^{f(z)}f'(z)}{g(z)} - \frac{ze^{f(z)}g'(z)}{g^2(z)}\right],$$

we obtain

$$\frac{zI''(f,g)(z)}{I'(f,g)(z)} = \alpha\left[1 + zf'(z) - \frac{zg'(z)}{g(z)}\right]. \tag{8}$$

Therefore, we get

$$(1-|z|^2)\left|\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right| \leq (1-|z|^2)|\alpha|\left[\left|zf'(z)\right| + \left|\frac{zg'(z)}{g(z)} - 1\right|\right].$$

From (6) and applying Lemma 1.1, we have

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < 1, \quad z \in U. \tag{9}$$

Also, since $|f(z)| < M$, $z \in U$, by applying Lemma 1.3, we obtain

$$|f(z)| \leq M|z|, \quad z \in U.$$

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Thus, we find that

$$
\begin{aligned}
(1-|z|^2)\left|\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right| &\le (1-|z|^2)|\alpha|\left[\left(\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|+1\right)\frac{|f(z)|^{\mu}}{|z|^{\mu-1}}+1\right] \\
&\le (1-|z|^2)|\alpha|\left[\left(\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|+1\right)M^{\mu}|z|+1\right]. \\
&\le (1-|z|^2)|\alpha|\left[(2-\lambda)M^{\mu}|z|+1\right]. \qquad (10)
\end{aligned}
$$

Let consider the function $h:[0,1)\to\mathbb{R}$,

$$h(x)=(1-x^2)(bx+1),$$

where $x=|z|$ and $b=(2-\lambda)M^{\mu}$.
Since the maximum point of h is $x=\frac{\sqrt{1+3b^2}-1}{3b}<1$, we find that

$$h(x)\le\frac{2[9b^2-1+(3b^2+1)\sqrt{3b^2+1}]}{27b^2},\quad x\in[0,1). \qquad (11)$$

Finally, by using (11) in (10) and applying Theorem 1.2 we yield that the function $I(f,g)$ is in the class S. □

In the following theorem we give sufficient conditions such that the integral operator $I(f,g)\in S^*$.

Theorem 2.2. *Let α be a complex number, $f\in B(\mu,\lambda)$ and $g\in B(\nu,\eta)$. Suppose also that positive real numbers M_1 and M_2 ($M_1,M_2\ge 1$) are so constrained that*

$$|f(z)|<M_1 \text{ and } |g(z)|<M_2,\quad z\in U. \qquad (12)$$

If

$$|\alpha|\le\frac{1}{1+(2-\lambda)M_1^{\mu}+(2-\eta)M_2^{\nu-1}}, \qquad (13)$$

then the function $I(f,g)$ is in the class S^*.

Proof. We consider

$$\frac{zI'(f,g)(z)}{I(f,g)(z)}=\frac{p(z)}{q(z)},$$

where

$$p(z)=\frac{z^{\alpha+1}e^{\alpha f(z)}}{[g(z)]^{\alpha}} \text{ and } q(z)=I(f,g)(z),\quad z\in U.$$

Obviously, $p(0)=q(0)=0$, and q satisfies the starlikeness condition of Lemma 1.4. Since

$$\frac{p'(z)}{q'(z)}=1+\alpha\left[1+zf'(z)-\frac{zg'(z)}{g(z)}\right],$$

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we obtain

$$\begin{aligned}\left|\frac{p'(z)}{q'(z)}-1\right| &\leq |\alpha|+|\alpha||zf'(z)|+|\alpha|\left|\frac{zg'(z)}{g(z)}\right| \\ &\leq |\alpha|+|\alpha|\left(\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|+1\right)\frac{|f(z)|^{\mu}}{|z|^{\mu-1}} \\ &\quad+|\alpha|\left(\left|g'(z)\left(\frac{z}{g(z)}\right)^{\nu}-1\right|+1\right)\left|\frac{g(z)}{z}\right|^{\nu-1}. \end{aligned} \tag{14}$$

Also, using the Schwarz Lemma, we have

$$\left|\frac{f(z)}{z}\right| \leq M_1 \text{ and } \left|\frac{g(z)}{z}\right| \leq M_2, \text{ for all } z \in U. \tag{15}$$

Next, replacing (15) and (13) in inequation (14), we obtain

$$\begin{aligned}\left|\frac{p'(z)}{q'(z)}-1\right| &\leq |\alpha|+|\alpha|\left(\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|+1\right)M_1^{\mu}|z| \\ &\quad+|\alpha|\left(\left|g'(z)\left(\frac{z}{g(z)}\right)^{\nu}-1\right|+1\right)M_2^{\nu-1} \\ &\leq |\alpha|[1+(2-\lambda)\cdot M_1^{\mu}+(2-\eta)M_2^{\nu-1}] \leq 1. \end{aligned}$$

Thus, we have

$$Re\left[\frac{p'(z)}{q'(z)}\right]>0, \quad z\in U,$$

and, applying Lemma 1.4, we yield that

$$Re\left[\frac{p(z)}{q(z)}\right]>0, \quad z\in U.$$

This completes the proof. of the theorem. □

Corollary 2.3. *Let α be a complex number and f, $g \in S^*$. Suppose also that positive real number M, $M \geq 1$ is so constrained that*

$$|f(z)|<M, \quad z\in U.$$

If

$$|\alpha| \leq \frac{1}{2M+3},$$

then the function $I(f,g)$ is in the class S^*.

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Theorem 2.4. *Let α be a complex number, $f \in B(\mu, \lambda)$ and $g \in \mathcal{A}$. Suppose also that*

$$|f(z)| < M \text{ and } \left|\frac{g''(z)}{g'(z)}\right| \leq M_0, \quad z \in U$$

where M is a positive real number, $M \geq 1$ and M_0 is the positive solution of the equation (3). Then the function $I(f,g)$ is in the class $K(\delta)$, where

$$\delta = 1 - |\alpha| \left[(2-\lambda)M^{\mu} + 1\right] \text{ and } 0 < |\alpha| \left[(2-\lambda)M^{\mu} + 1\right] \leq 1.$$

Proof. Similarly as in the proof of Theorem 2.1, we get

$$\left|\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right| \leq |\alpha| \left[\left(\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| + 1\right)\frac{|f(z)|^{\mu}}{|z|^{\mu-1}} + \left|\frac{zg'(z)}{g(z)} - 1\right|\right]. \quad (16)$$

From the hypothesis and applying the Schwarz Lemma in inequation (16), we obtain

$$\begin{aligned}\left|\frac{zI''(f,g)(z)}{I'(f,g)(z)}\right| &\leq |\alpha| \left[\left(\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| + 1\right) M^{\mu}|z| + 1\right] \\ &\leq |\alpha| \left[(2-\lambda)M^{\mu}|z| + 1\right] \leq |\alpha| \left[(2-\lambda)M^{\mu} + 1\right] = 1 - \delta.\end{aligned}$$

This evidently completes the proof. □

Letting $\mu = 1$, $\delta = \lambda = 0$ in Theorem 2.4, we have

Corollary 2.5. *Let α be a complex number, $f \in S^*$ and $g \in \mathcal{A}$. Suppose also that*

$$|f(z)| < M \text{ and } \left|\frac{g''(z)}{g'(z)}\right| \leq M_0, \quad z \in U,$$

where M is a positive real number, $M \geq 1$ and M_0 is the positive solution of the equation (3). If

$$|\alpha| = \frac{1}{1+2M},$$

then the function $I(f,g)$ is convex in U.

Further, if we consider $g \in S^*(\beta)$, we obtain

Theorem 2.6. *Let α be a positive real number and $g \in S^*(\beta)$, $0 \leq \beta < 1$. If $f \in B(\mu, \lambda)$ satisfies*

$$|f(z)| < M \quad (M \geq 1, z \in U)$$

then the function $I(f,g)$ is in the class $N(\delta)$, where $\delta = 1 + \alpha[1 - \beta + (2-\lambda)M^{\mu}]$.

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Proof. For $g \in S^*(\beta)$, from (8) we obtain

$$\mathrm{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)}+1\right)$$
$$=\alpha+1+\alpha\left[\mathrm{Re}(zf'(z))-\mathrm{Re}\frac{zg'(z)}{g(z)}\right]<1+\alpha(1-\beta)+\alpha\left[\mathrm{Re}zf'(z)\right]. \quad (17)$$

Since f is in the class $B(\mu,\lambda)$, $|f(z)|<M$, from General Schwarz Lemma, we get

$$\mathrm{Re}(zf'(z))\le|zf'(z)|\le\left(\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|+1\right)\frac{|f(z)|^{\mu}}{|z|^{\mu-1}}$$
$$\le(2-\lambda)M^{\mu}|z|<(2-\lambda)M^{\mu}. \quad (18)$$

Using (18) in (17), we obtain

$$\mathrm{Re}\left(\frac{zI''(f,g)(z)}{I'(f,g)(z)}+1\right)<1+\alpha[1-\beta+(2-\lambda)M^{\mu}]. \quad (19)$$

Because $\alpha[1-\beta+(2-\lambda)M^{\mu}]>0$, we yield that $I(f,g)\in N(\delta)$, where $\delta=1+\alpha[1-\beta+(2-\lambda)M^{\mu}]$. □

Letting $\mu=1$ and $\beta=\lambda=0$ in Theorem 2.6, we obtain

Corollary 2.7. *If α is a positive real number and $f,g\in S^*$ with $|f(z)|<M(M\ge1, z\in U)$ then the function $I(f,g)$ is in the class $N(\delta)$, where* $\delta=1+\alpha(1+2M)$.

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