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SYMMETRIZING OPERATIONS ON SOME ORTHOGONAL POLYNOMIALS

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Abstract: In this paper we show how the action of the symmetrizing endomorphism operators $L^k_{e_1e_2}$ to the series $\sum_{j=0}^{+\infty} a_j z^j$ allows the obtention of an alternative approach for the determination of Fibonacci numbers and Tchebychev polynomials of the first and second kind.

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1. Introduction

By studying the Fibonacci sequence $F_n = \frac{1}{\sqrt{5}}(\Phi_1^n - \Phi_2^n)$, we note its close connection with the equation $x^2 - x - 1 = 0$, whose roots are the golden numbers $\Phi_1 = \frac{1+\sqrt{5}}{2}$ and $\Phi_2 = \frac{1-\sqrt{5}}{2}$. It is also noticed that the eigenvalues of the symmetric matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.1}$$

represent the two golden numbers Φ_1 and Φ_2 of Fibonacci sequence [2]. Consequently,

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we obtain the following Vieta's formulas

$$\sigma_1 = \lambda_1 + \lambda_2 = 1 \text{ and } \sigma_2 = \lambda_1\lambda_2 = -1, \tag{1.2}$$

where σ_1, σ_2 are called elementary symmetric functions of real roots λ_1, λ_2, respectively. So, the eigenvectors of matrix M are multiples of

$$v_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}. \tag{1.3}$$

If we assume that $|\lambda_1| > |\lambda_2|$, then for any positive integer n, we have

$$M^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}. \tag{1.4}$$

The rest of the paper is organized as follows. In Section 2, we present preliminary notions on symetric functions and divided differences. In Section 3, the proposed theorem is detailed and demontrated. Applications of the proposed theorem on Fibonacci numbers and Tchebychev polynomials of the first and Second kind are given in Section 4. Finally, conclusion is given in Section 5.

2. Preliminaries and Notations

In this section, we recall some basic definitions and theorems that are needed in the sequel.

Definition 1. [1] Given two sets of indeterminate A and B (called alphabets), we define $S_j(A-B)$ as follows

$$\frac{\Pi_{b\epsilon B}(1-zb)}{\Pi_{a\epsilon A}(1-za)} = \sum_{j=0}^{\infty} S_j(A-B)z^j, \tag{2.1}$$

with $S_j(A-B) = 0$ for $j < 0$.

Remark 2. By taking $A = \Phi$ in (2.1), we obtain

$$\Pi_{b\epsilon B}(1-zb) = \sum_{j=0}^{\infty} S_j(-B)z^j. \tag{2.2}$$

Proposition 3. *[1] Considering successively the case of $A = \Phi$, $B = \Phi$, we can derive the following factorization*

$$\sum_{j=0}^{\infty} S_j(A-B)z^j = \sum_{j=0}^{\infty} S_j(A)z^j \sum_{j=0}^{\infty} S_j(-B)z^j. \tag{2.3}$$

Thus,

$$S_n(A-B) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-B). \tag{2.4}$$

The summation is in fact limited to a finite number of nonzero terms. In particular, we have [3]

$$\Pi_{b\epsilon B}(x-b) = S_n(x-B) = x^n S_0(-B) + x^{n-1}S_1(-B) + x^{n-2}S_2(-B) + \cdots, \tag{2.5}$$

where $S_j(-B)$ are the coefficients of polynomials $S_n(x-B)$ for $0 < j < n$. We note that $S_j(-B) = 0$ for $j > n$.

For $B = \{b, b, b...\}$ (we note $B = nb$), we have

$$S_n(x - nb) = (x-b)^n. \tag{2.6}$$

Thus, the special case of $B = \{1, 1, 1, ..., 1\}$ gives the two binomial coefficients

$$S_j(-n) = (-1)^j \binom{n}{j} \text{ and } S_j(n) = \binom{n+j-1}{j}. \tag{2.7}$$

By combining (2.4) and (2.7), we obtain the following expression

$$S_j(A - jx) = S_j(A) - \binom{j}{1}x^1 S_{j-1}(A) + \cdots \pm \binom{j}{j}x^j. \tag{2.8}$$

Lemma 4. *[1] Given two alphabets $A = \{x\}$ and $B = \{b_1, b_2, ...\}$, we have*

$$S_{j+k}(x-B) = x^k S_j(x-B) \textit{ for all } k \geq 0.$$

Definition 5. [1] Given a function $g(x_1, x_2, ...)$, the divided difference operator is defined by the following formula:

$$g(x_1, ..., x_i, x_{i+1}, ...) \to \partial_{x_i x_{i+1}}(g) = \frac{g - g^\sigma}{x_i - x_{i+1}},$$

where g^σ is the image of g by the transposition (x_i, x_{i+1}), i.e.,

$$g^\sigma = g(x_1, ..., x_{i+1,} x_i, x_{i+2} ...).$$

Definition 6. Let f and g be two polynomials, the implementation of x_i and x_{i+1} is given by the Leibniz formula:

$$\partial_i(fg) = \partial_i(g)f^\sigma + \partial_i(f)g.$$

In the particular case of $f = f^\sigma$, we have

$$\begin{cases} \partial_i(f) = 0, \\ \partial_i(fg) = \partial_i(g)f. \end{cases}$$

The divided difference operator ∂_{xy} commutes with symmetric functions at x, y and is compatible with the function S_n [1, 4].

3. The Main Results

Proposition 7. *[3] Let E be an alphabet such that $E = \{e_1, e_2\}$. The operator $L^k_{e_1e_2}$ is defined as follows*

$$L^k_{e_1e_2} f(e_1) = S_{k-1}(e_1+e_2)f(e_1) + e_2^k \partial_{e_1e_2} f(e_1), \quad \text{for all } k \in \mathbb{N}.$$

Theorem 8. *Given an alphabet $E = \{e_1, e_2\}$, and two series $\sum_{j=0}^{\infty} a_j z^j$, $\sum_{j=0}^{\infty} b_j z^j$ such that $(\sum_{j=0}^{\infty} a_j z^j)(\sum_{j=0}^{\infty} b_j z^j) = 1$, we have*

$$\sum_{j=0}^{\infty} a_j S_{k+j-1}(e_1+e_2) z^j = \frac{\sum\limits_{j=0}^{k-1} b_j e_1^j e_2^j S_{k-j-1}(e_1+e_2) z^j - e_1^k e_2^k z^{k+1} \sum\limits_{j=0}^{\infty} b_{j+k+1} S_j(e_1+e_2) z^j}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j \sum\limits_{j=0}^{\infty} b_j e_2^j z^j}. \quad (3.1)$$

Proof. Let $\sum\limits_{j=0}^{\infty} a_j z^j$ and $\sum\limits_{j=0}^{\infty} b_j z^j$ be two sequences such that $\sum\limits_{j=0}^{\infty} a_j z^j = \frac{1}{\sum_{j=0}^{\infty} b_j z^j}$, the left hand side of the formula (3.1) can be written as

$$L^k_{e_1e_2} f\ (e_1) = L^k_{e_1e_2}\left(\sum_{j=0}^{\infty} a_j e_1^j z^j\right) = \sum_{j=0}^{\infty} a_j S_{k+j-1}(e_1+e_2) z^j,$$

while the right hand side can be expressed as :

$$\begin{aligned} & S_{k-1}(e_1+e_2) f\ (e_1) + e_2^k \partial_{e_1 e_2} f(e_1) \\ = \ & \frac{S_{k-1}(e_1+e_2)}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} + e_2^k \partial_{e_1e_2} \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} \\ = \ & \frac{S_{k-1}(e_1+e_2)}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} - \frac{\sum\limits_{j=0}^{\infty} b_j S_{j-1}(e_1+e_2) z^j}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j \sum\limits_{j=0}^{\infty} b_j e_2^j z^j} \end{aligned}$$

$$= \frac{\sum_{j=0}^{\infty} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2)\right] z^j}{\sum_{j=0}^{\infty} b_j e_1^j z^j \sum_{j=0}^{\infty} b_j e_2^j z^j}$$

$$= \frac{\sum_{j=0}^{k-1} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2)\right] z^j}{\sum_{j=0}^{\infty} b_j e_1^j z^j \sum_{j=0}^{\infty} b_j e_2^j z^j}$$

$$+ \frac{\sum_{j=k+1}^{\infty} b_j \left[e_2^j S_{k-1}(e_1+e_2) - e_2^k S_{j-1}(e_1+e_2)\right] z^j}{\sum_{j=0}^{\infty} b_j e_1^j z^j \sum_{j=0}^{\infty} b_j e_2^j z^j}$$

$$= \frac{\sum_{j=0}^{k-1} b_j e_1^j e_2^j S_{k-j-1}(e_1+e_2) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} b_{j+k+1} S_j(e_1+e_2) z^j}{\left(\sum_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum_{j=0}^{\infty} b_j e_2^j z^j\right)}.$$

The two quantities are equal; proof completed. □

4. Applications to the Generating Functions

4.1. The Case $\sum_{j=0}^{\infty} z^j = \frac{1}{1-z}$

Corollary 9. *Given an alphabet* $E = \{e_1, e_2\}$, *we have*

$$\sum_{j=0}^{\infty} S_{k+j-1}(e_1+e_2) z^j = \frac{S_{k-1}(e_1+e_2) - e_1 e_2 S_{k-2}(e_1+e_2) z}{(1-ze_1)(1-ze_2)}, \quad \textit{for all } k \in \mathbb{N}.$$

In the case $k = 1$ Corollary 8 can be written as follows:

$$\sum_{j=0}^{\infty} S_j(e_1+e_2) z^j = \frac{1}{(1-ze_1)(1-ze_2)}. \tag{4.1}$$

Replacing e_2 by $(-e_2)$ in (4.1), we obtain

$$\sum_{j=0}^{\infty} S_j(e_1+[-e_2]) z^j = \frac{1}{(1-ze_1)(1+ze_2)}. \tag{4.2}$$

Choosing e_1 and e_2 such that

$$\begin{cases} e_1 e_2 = 1 \\ e_1 - e_2 = 1 \end{cases}$$

and substituting in (4.2) we end up with

$$\sum_{j=0}^{\infty} S_j(e_1 + [-e_2]) z^j = \frac{1}{1 - z - z^2},$$

which represents a generating function for Fibonacci numbers [5], such that $F_j = S_j(e_1 + [-e_2])$.

On the other hand, when replacing e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (4.2), and under the condition $4e_1e_2 = -1$, we obtain for $x = e_1 - e_2$

$$\sum_{j=0}^{\infty} S_j(2e_1 + [-2e_2]) z^j = \frac{1}{1 - 2xz + z^2}, \tag{4.3}$$

which represents a generating function for Tchebychev polynomials of the second kind [5], such that $U_j(x) = S_j(2e_1 + [-2e_2])$.

Moreover, from (4.3), we deduce that

$$\sum_{j=0}^{\infty} \left[S_j(2e_1 + [-2e_2]) - x S_{j-1}(2e_1 + [-2e_2])\right] z^j = \frac{1 - xz}{1 - 2xz + z^2},$$

which represents a generating function for Tchebychev polynomials of the first kind , such that

$$T_j(x) = \left[S_j(2e_1 + [-2e_2]) - x S_{j-1}(2e_1 + [-2e_2])\right].$$

In the case $k = 2$ Corollary 8 can be written as follows:

$$\sum_{j=0}^{\infty} S_{j+1}(e_1 + e_2) z^j = \frac{e_1 + e_2 - e_1 e_2 z}{(1 - ze_1)(1 - ze_2)}. \tag{4.4}$$

Replacing e_2 by $(-e_2)$ in (4.4), we obtain

$$\sum_{j=0}^{\infty} S_{j+1}(e_1 + [-e_2]) z^j = \frac{e_1 - e_2 + e_1 e_2 z}{(1 - ze_1)(1 + ze_2)}. \tag{4.5}$$

Choosing e_1 and e_2 such that

$$\begin{cases} e_1 e_2 = 1 \\ e_1 - e_2 = 1 \end{cases}$$

and substituting in (4.5) we end up with

$$\sum_{j=0}^{\infty} S_{j+1}(e_1 + [-e_2])z^j = \frac{1+z}{1-z-z^2},$$

which represents a new generating function, such that $S_{j+1}(e_1 + [-e_2]) = (1+z)F_j$.

On the other hand, when replacing e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (4.4), and under the condition $4e_1e_2 = -1$, we obtain for $x = e_1 - e_2$

$$\sum_{j=0}^{\infty} S_{j+1}(2e_1 + [-2e_2])z^j = \frac{2x-z}{1-2xz+z^2},$$

which represents a new generating function, such that $S_{j+1}(2e_1 + [-2e_2]) = (2x - z)U_j(x)$.

4.2. The Case $\sum_{j=0}^{\infty}(-1)^j z^j = \frac{1}{1+z}$

Corollary 10. *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\sum_{j=0}^{\infty}(-1)^j S_{k+j-1}(e_1+e_2)z^j = \frac{S_{k-1}(e_1+e_2) + e_1e_2S_{k-2}(e_1+e_2)z}{(1+ze_1)(1+ze_2)}, \quad \textit{for all } k \in \mathbb{N}.$$

In the case $k = 1$ Corollary 9 can be written as follows:

$$\sum_{j=0}^{\infty}(-1)^j S_j(e_1+e_2)z^j = \frac{1}{(1+ze_1)(1+ze_2)}. \tag{4.6}$$

Replacing e_2 by $(-e_2)$ in (4.6), we obtain

$$\sum_{j=0}^{\infty}(-1)^j S_j(e_1+[-e_2])z^j = \frac{1}{(1+ze_1)(1-ze_2)}. \tag{4.7}$$

Choosing e_1 and e_2 such that

$$\begin{cases} e_1e_2 = 1 \\ e_1 - e_2 = 1 \end{cases}$$

and substituting in (4.7) we end up with

$$\sum_{j=0}^{\infty}(-1)^j F_j z^j = \frac{1}{1+z-z^2},$$

which represents a new generating function, such that $F_j = S_j(e_1 + [-e_2])$.

On the other hand, when replacing e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (4.7), and under the condition $4e_1e_2 = -1$, we obtain for $x = e_1 - e_2$

$$\sum_{j=0}^{\infty}(-1)^j U_j(x)z^j = \frac{1}{1+2xz+z^2}, \tag{4.8}$$

which represents a new generating function, such that $U_j(x) = S_j(2e_1 + [-2e_2])$.

Moreover, from (4.8), we deduce that

$$\sum_{j=0}^{\infty}(-1)^j T_j(x)z^j = \frac{1-xz}{1+2xz+z^2},$$

which represents a new generating function, such that

$$T_j(x) = [S_j(2e_1 + [-2e_2]) - xS_{j-1}(2e_1 + [-2e_2])].$$

In the case $k = 2$ Corollary 9 can be written as follows:

$$\sum_{j=0}^{\infty}(-1)^j S_{j+1}(e_1 + e_2)z^j = \frac{e_1 + e_2 + e_1e_2z}{(1+ze_1)(1+ze_2)}. \tag{4.9}$$

Replacing e_2 by $(-e_2)$ in (4.9), we obtain

$$\sum_{j=0}^{\infty}(-1)^j S_{j+1}(e_1 + [-e_2])z^j = \frac{e_1 - e_2 - e_1e_2z}{(1+ze_1)(1-ze_2)}. \tag{4.10}$$

Choosing e_1 and e_2 such that

$$\begin{cases} e_1e_2 = 1 \\ e_1 - e_2 = 1 \end{cases}$$

and substituting in (4.5) we end up with

$$\sum_{j=0}^{\infty}(-1)^j S_{j+1}(e_1 + [-e_2])z^j = \frac{1-z}{1+z-z^2},$$

which represents a new generating function.

On the other hand, when replacing e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (4.9), and under the condition $4e_1e_2 = -1$, we obtain for $x = e_1 - e_2$

$$\sum_{j=0}^{\infty}(-1)^j S_{j+1}(2e_1 + [-2e_2])z^j = \frac{2x+z}{1+2xz+z^2},$$

which represents a new generating function.

5. Conclusion

In this paper, a new theorem has been proposed in order to determine the generating functions. The proposed theorem is based on the symmetric fonctions.The obtained results agree with the results obtained in some previous works.

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