International Electronic Journal of Pure and Applied Mathematics

Volume 9 No. 3 2015, 167-179
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v9i3.6

SOME PROPERTIES OF SEMI-OPEN SETS WITH RESPECT TO AN IDEAL

M.E. Abd El-Monsef[1], A.A. Nasef[2 §], A.E. Radwan[3]
F.A. Ibrahem[4], R.B. Esmaeel[5]

[1]Department of Mathematics
Faculty of Seience
Tanta University, Tanta, EGYPT

[2]Department of Physics and Engineering Mathematics
Faculty of Engineering
Kafr El-Sheikh University
Kafr El-Sheikh, EGYPT

[3]Department of Mathematics
Faculty of Science
Ain Shams University
Cairo, EGYPT

[4]Department of Mathematics
Faculty of Science
Ain Shams University, EGYPT

[5]Department of Mathematics
Ibn-Al-Haitham college of Education
Baghdad University
IRAQ

Abstract: The aim of this paper is to introduce some types of compactness and connectedness in ideal topological spaces. We further study some properties of these types. Finally, the relationship between various types of compactness and connectedness was summarized.

Received: December 19, 2014

[§]Correspondence author

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

AMS Subject Classification: 54A05, 54C10, 54 D05, 54D30
Key Words: I-semi-open set, I-compact, SI-compact, IS-compact, countably IS-compact, IS-separated sets, $I*$-hyperconnected, $IS*$-connected, IS-separated connected, ideal

1. Introduction and Preliminaries

Throughout the present paper, $(X,\tau),(Y,\sigma)$ and (Z,μ) will denote topological spaces with no separation properties assumed. For a subset A of a topological space $(X,\tau), cl(A)$ and $int(A)$ will denote the closure and interior of A in (X,τ), respectively.

The subject of ideals in topological spaces has been studied by Kuratowski [8] and Vaidyanathaswamy [14]. An ideal I on a set X is a nonempty collection of subsets of X which satisfies the conditions: (1) $A \in I$ and $B \subseteq A$ implies $B \in I$, (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X,τ) with an ideal I on X and $\mathcal{P}(\mathcal{X})$ is the set of all subsets of X. A set operator $(\ )^* : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is called a local function [8] of A with respect to τ and I iff for $A \subseteq X, A^*(I,\tau) = \{x \in X : A \cap U \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(\ )$ [14] for a topology $\tau^*(I,\tau)$, called the $*$-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(I,\tau)$. When there is no chance for confusion, we will simply write A^* for $A^*(I,\tau)$ and τ^* for $\tau^*(I,\tau)$. If I is an ideal on X, then (X,τ,I) is called an ideal topological space.

In an ideal topological space (X,τ,I), if $A \subseteq X, int^*(A)$ will denote the interior of A in (X,τ^*). Closed subsets in (X,τ^*) are called τ^*-closed sets. A subset A of an ideal topological space (X,τ,I) is τ^*-closed [7] if and only if $A^* \subseteq A$.

For any ideal topological space (X,τ,I), the collection $\{V - J : V \in \tau$ and $J \in I\}$ is a basis for τ^*. The elements of τ^* are called τ^*-open sets. A subset A of an ideal topological space (X,τ,I) is said to be τ^*-dense if $cl^*(A) = X$. It is clear that, in a space (X,τ,I) if $I = \{\varnothing\}$, then $\tau = \tau^*$ [3].

Recall that if (X,τ,I) is an ideal topological space and A is a subset of X, then (A,τ_A,I_A) is an ideal topological space where τ_A is the relative topology on A and $I_A = \{A \cap J : J \in I\}$.

It is interesting to note that if I is a proper ideal (i.e. $X \notin I$), then the collection of the complements of the members of I form a filter on X. This is why sometimes ideals are called dual filters.

The following collections form important ideals [7,14] on a topological space (X,τ):

(i) $\{\varnothing\}$ or $I_{\{\varnothing\}}$: the trivial ideal.

(ii) $\mathcal{P}(\mathcal{X})$: the improper ideal.

(iii) $\mathcal{F} : (\mathcal{I}_{\{})$ the ideal of all finite sets.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

(iv) $\mathcal{C}: (\mathcal{I}_\rfloor)$ the ideal of all countable subsets of X.

A subset A of a topological space (X, τ) is said to be semi-open [9] if $A \subseteq cl(int(A))$. The complement of a semi-open set in a space (X, τ) is said to be semi-closed [1]. The smallest semi-closed set containing A is said to be semi-closure [2] of A and denoted by $scl(A)$. If $A \subseteq Y$ and Y is a subset of a topological space (X, τ), then $scl(A) \cap Y = scl_Y(A)$. A topological space (X, τ) is said to be semi-compact [4] if every cover of X by semi-open sets has a finite subcover for X.

A subset A of a topological space (X, τ) is said to be α-open [11] if $A \subseteq int(cl(int(A)))$. The complement of an α-open set in a space (X, τ) is said to be α-closed.

2. I-Semi-Compactness

In this section, we will introduce new type of compactness in an ideal topological space by using the concept of I-semi- open set. Some properties of this type are studied.

Definition 2.1. A subset A of an ideal topological space (X, τ, I) is said to be I-semi-open set [6] if there exists an open set U such that $U - A \in I$ and $A - cl(U) \in I$.

It is clear that, every semi-open set is I-semi- open, but not conversely. If $I = \{\varnothing\}$, then the two concepts are the same and if $A \in I$, then A is an I-semi-open set.

Proposition 2.2. *If a subset A of an ideal topological space (X, τ^*, I) is an I-semi-open set, then it is I-semi-open in the ideal topological space (X, τ, I).*

Proof. Let A be an I-semi-open subset of an ideal topological space (X, τ^*, I), then there exists a τ^*-open set U such that $U - A \in I$ and $A - cl^*(U) \in I$. Implies, there exists a τ-open set V and $J \in I$ such that $U = V - J$. So, $U - A = (V - J) - A \in I$ and $V - A \subseteq ((V - J) - A) \cup J \in I$. In the other side, $A - cl(V) \subseteq A - cl^*(V) \subseteq A - cl^*(V - J) \in I$. Then, the proof is over. □

In the following example, we will show that the implication in Proposition 2.2 is not reversible.

Example 2.3. Consider the ideal topological space $(\mathbb{N}, \tau, I)$, where $\mathbb{N}$ is the set of all natural numbers, $\tau = \{\mathbb{N}, \varnothing, \{1\}\}$ and $I = \mathcal{P}(\mathcal{O}^+)$ is the set of all subsets of the odd numbers. Then for every element $n \in \mathbb{N}$, the set $\{1, n\}$ is an I-semi-open in an ideal topological space $(\mathbb{N}, \tau, I)$ but it is not I-semi-open in an ideal topological space $(\mathbb{N}, \tau^*, I)$ when $\tau^* = \{A \subseteq \mathbb{N} : E^+ \subseteq A\} \cup \{\varnothing, \{1\}\}$ and E^+ is the set of all even numbers.

Remark 2.4. In an ideal topological space (X, τ, I), the following fundamental relationships hold:

$$\tau \subseteq \tau^* \subseteq ISO(X, \tau^*, I) \subseteq ISO(X, \tau, I).$$

Definition 2.5. A subset A of an ideal topological space (X, τ, I) is said to be I-compact [13] (resp. SI-compact [10]) if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of A by open (resp. semi-open) sets of X, there exists a finite subset Λ_0 of Λ such that $A - \cup\{U_\lambda : \lambda \in \Lambda_0\} \in I$. An ideal topological space (X, τ, I) is said to be I-compact (resp. SI-compact) if X is I-compact (resp. SI-compact) as a subset.

Definition 2.6. A subset A of an ideal topological space (X, τ, I) is said to be IS-compact if for every cover $\{U_\lambda : \lambda \in \Lambda\}$ of A by I-semi- open sets of X, there exists a finite subset Λ_0 of Λ such that $A - \cup\{U_\lambda : \lambda \in \Lambda_0\} \in I$. An ideal topological space (X, τ, I) is said to be IS-compact if X is IS-compact as a subset.

Proposition 2.7. *The following statements are equivalent;*

(i) (X, τ) *is semi- compact.*

(ii) $(X, \tau, \{\varnothing\})$ *is* IS*-compact.*

(iii) $(X, \tau, \{\varnothing\})$ *is* SI*-compact.*

Proposition 2.8. *In an ideal topological space* (X, τ, I)*, the following statements are equivalent;*

(i) (X, τ, I) *is* IS*-compact.*

(ii) For any family $\{F_\lambda : \lambda \in \Lambda\}$ *of* I*-semi-closed sets of* X *such that* $\cap\{F_\lambda : \lambda \in \Lambda\} = \varnothing$*, there exists a finite subset* Λ_0 *of* Λ *such that* $\cap\{F_\lambda : \lambda \in \Lambda_0\} \in I$*.*

Proposition 2.9. *If* (X, τ, I) *is* IS*-compact, then* (X, τ^*, I) *is* IS*-compact.*

Proof. Follows from Proposition 2.2. □

The reverse implication of Proposition 2.9 is not true, in general as shown in the following example:

Example 2.10. Consider the ideal topological space $(\mathbb{N}, \tau, I)$ as an Example 2.3. Then $(\mathbb{N}, \tau^*, I)$ is IS-compact space, since every IS-open cover in $(\mathbb{N}, \tau^*, I)$ for

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

$\mathbb{N}$ has at least one element A such that $E^+ \subseteq A$, where E^+ is the set of all even numbers, so $X - A \in I$. But $\{\{1, n\} : n \in \mathbb{N}\}$ is IS-open cover in $(\mathbb{N}, \tau, I)$ for $\mathbb{N}$ which has no finite subcover $\{\{1, n\} : n = 1, 2, \ldots, r\}$ such that $\mathbb{N} - \cup\{\{1, n\} : n = 1, 2, \ldots, r\} \in I$.

The following diagram shows the relationships between some types of compactness:

$$\begin{array}{ccccc} & & & & \text{Compactness} \\ & & & & \downarrow \\ IS\text{-compactness} & \longrightarrow & SI\text{-compactness} & \longrightarrow & I\text{-compactness} \end{array}$$

None of these implications is reversible, in general, as shown in the following examples.

Example 2.11. Consider the ideal topological space $(\mathbb{N}, \tau, I)$, where τ is the discrete topology on $\mathbb{N}$ and I is the power set of $\mathbb{N}$. It is clear that $(\mathbb{N}, \tau)$ is not compact space, but $(\mathbb{N}, \tau, I)$ is I-compact (resp. SI-compact and IS-compact) space.

Example 2.12. Consider the ideal topological space $(\mathbb{N}, \tau, I)$, where $\tau = \{\mathbb{N}, \varnothing, O^+, E^+\}$ and $I = I_f = \{U \subseteq \mathbb{N} : U \text{ is a finite set}\}$. Then, $(\mathbb{N}, \tau, I)$ is SI-compact (I-compact and compact) space which is not IS-compact.

Example 2.13. Consider the ideal topological space $(\mathbb{N}, \tau, I)$, where $\tau = \{\mathbb{N}, \varnothing, \{1\}\}$ and $I = \{\varnothing\}$. Then, $(\mathbb{N}, \tau, I)$ is compact (resp. I-compact) space which is not SI-compact.

Remark 2.14. If I and J are two ideals on a topological space (X, τ), with $I \subseteq J$, then:

(i) The condition (X, τ, I) is IS-compact is not sufficient to make (X, τ, J) is JS-compact. For example;

Consider the topological space $(\mathbb{N}, \tau)$ where τ is the indiscrete topology on $\mathbb{N}$, with two ideals $I = \{\varnothing\}$ and $J = J_f$ is the ideal of all finite subsets of $\mathbb{N}$. It is clear that $(\mathbb{N}, \tau, I)$ is IS-compact but $(\mathbb{N}, \tau, J)$ is not JS-compact, since $\{\{n\} : n \in \mathbb{N}\}$ is JS-open cover for $\mathbb{N}$ which has no finite subfamily $\{\{n\} : n = 1, 2, \ldots, r\}$ such that $\mathbb{N} - \cup\{\{n\} : n = 1, 2, \ldots, r\} \in J$.

(ii) The condition (X, τ, J) is JS-compact is not sufficient to make (X, τ, I) is IS-compact. For example;

Consider the topological space $(\mathbb{N}, \tau)$ where $\tau = \{\mathbb{N}, \varnothing, \{1\}\}$ with two ideals $I = \{\varnothing\}$ and J is the power set of $\mathbb{N}$. It is clear that $(\mathbb{N}, \tau, J)$ is JS-compact but $(\mathbb{N}, \tau, I)$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

is not IS-compact, since $\{\{1, n\} : n \in \mathbb{N}\}$ is IS-open cover for $\mathbb{N}$ which has no finite subfamily $\{\{1, n\} : n = 1, 2, \ldots, r\}$ such that $\mathbb{N} - \cup\{\{1, n\} : n = 1, 2, \ldots, r\} \in I$.

Proposition 2.15. *If A and B are two IS-compact subsets of an ideal topological space (X, τ, I), then $A \cup B$ is IS-compact.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be an IS-open cover for $A \cup B$, then it is an IS-open cover for both A and B. So, there is a finite subsets Λ_1 and Λ_2 of Λ such that $A - \cup\{U_\lambda : \lambda \in \Lambda_1\} \in I$ and $B - \cup\{U_\lambda : \lambda \in \Lambda_2\} \in I$. Implies, $(A - \cup\{U_\lambda : \lambda \in \Lambda_1\}) \cup (B - \cup\{U_\lambda : \lambda \in \Lambda_2\}) \in I$. Hence, $(A \cup B) - ((\cup\{U_\lambda : \lambda \in \Lambda_1\}) \cup (\cup\{U_\lambda : \lambda \in \Lambda_2\})) \subseteq (A - \cup\{U_\lambda : \lambda \in \Lambda_1\}) \cup (B - \cup\{U_\lambda : \lambda \in \Lambda_2\}) \in I$. Therefore, $(A \cup B)$ is IS-compact. □

Remark 2.16. The intersection of two IS-compact subsets of an ideal topological space (X, τ, I) need not be IS-compact. For example;

Consider the ideal topological space (X, τ, I), where $X = \mathbb{N} \cup \{-1, 0\}$ with the indiscrete topology and $I = I_f = \{U \subseteq \mathbb{N} : U$ is a finite set$\}$. Let $A = \mathbb{N} \cup \{0\}$ and $B = \mathbb{N} \cup \{-1\}$. Then A is IS-compact subsets of X, since every cover $\{U_\lambda : \lambda \in \Lambda\}$ for A by IS-open subsets of X has at least one element $U_{\lambda 0}$. So $A - U_{\lambda 0}$ is a finite subset of $\mathbb{N}$. Hence, A is IS-compact set. Similarly, B is IS-compact.

But $A \cap B = \mathbb{N}$ is not IS-compact, since the I-semi-open cover $\{\{n\} : n \in \mathbb{N}\}$ has no finite subcover $\{\{n\} : n = 1, 2, \ldots, r\}$ such that $\mathbb{N} - \cup\{\{n\} : n = 1, 2, \ldots, r\} \in I$.

Remark 2.17. IS-compactness is not hereditary property. See the example of Remark 2.16, X is IS-compact space but $\mathbb{N}$ is not.

Proposition 2.18. *I-semi-closed subset of an IS-compact space is IS-compact.*

Remark 2.19. [12]

(i) If $f : (X, \tau, I) \to (Y, \sigma)$ is a function, then $f(I) = \{f(J) : J \in I\}$ is an ideal on Y.

(ii)If $f : (X, \tau) \to (Y, \sigma, J)$ is an injection function, then $f^{-1}(J) = \{f(B) : B \in J\}$ is an ideal on X.

Definition 2.20. A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be

(i) I-semi-open function if $f(A)$ is J-semi-open set in Y for each I-semi-open set A in X.

(ii) I-semi-irresolute function if $f^{-1}(B)$ is I-semi-open set in X for each J-semi-open set B in Y.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

Proposition 2.21. *If $f:(X,\tau,I)\to(Y,\sigma,f(I))$ is an I-semi-irresolute surjection function and (X,τ,I) is IS-compact, then $(Y,\sigma,f(I))$ is IS-compact.*

Corollary 2.22. *If $f:(X,\tau,f^{-1}(J))\to(Y,\sigma,J)$ is $f^{-1}(J)$-semi-open bijection function and (Y,σ,J) is JS-compact space, then $(X,\tau,f^{-1}(J))$ is $f^{-1}(J)S$-compact.*

Definition 2.23. A subset A of an ideal topological space (X,τ,I) is said to be countably IS-compact if for every countable cover $\{U_n : n\in\mathbb{N}\}$ of A by I-semi- open sets of X, there exists a finite subset $\mathbb{N}_0$ of $\mathbb{N}$ such that $A-\cup\{U_n : n\in\mathbb{N}_0\}\in I$. An ideal topological space (X,τ,I) is said to be countably IS-compact if X is countably IS-compact as a subset.

Proposition 2.24. *Every IS-compact space is countably IS-compact.*

The implication of Proposition 2.24, is not reversible as the following example shows.

Example 2.25. Consider the ideal topological space $(\mathbb{R},\tau,I)$ where $\mathbb{R}$ is the set of all real numbers, $\tau=\mathcal{P}(\mathcal{I}\nabla\nabla)\cup\{\mathcal{Q}\cup\mathcal{A} : \mathcal{A}\subseteq\mathcal{I}\nabla\nabla\}$ and $I=\{\varnothing\}$, such that Irr is the set of irrational numbers, Q is the set of rational numbers and $\mathcal{P}(\mathcal{I}\nabla\nabla)$ is the set of all subsets of the irrational numbers.

Then $(\mathbb{R},\tau,I)$ is countably IS-compact, since every countable cover of $\mathbb{R}$ by I-semi-open sets is a finite cover. But $(\mathbb{R},\tau,I)$ is not IS-compact, since $\{Q\cup\{x\} : x\in Irr\}$ is an I-semi-open cover for $\mathbb{R}$ which has no finite subcover $\mathcal{M}$ such that $\mathbb{R}-\mathcal{M}\in\mathcal{I}$.

Remark 2.26. The concepts I-compactness and countably IS-compactness are independent as the following examples show.

(i) See Example 2.25, $(\mathbb{R},\tau,I)$ is countably IS-compact space which is not I-compact.

(ii) See Example 2.14, the ideal topological space $(\mathbb{N},\tau,I)$, where $\tau=\{\mathbb{N},\varnothing,\{1\}\}$ and $I=\{\varnothing\}$, is an I-compact space which is not countably IS-compact.

3. Connectedness with Respect to an I-Semi-Open Set

In this section, we will introduce new types of connectedness in ideal topological spaces by using the concept of I-semi- open set. Some properties of these types are studied.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

Definition 3.1. [5] An ideal topological space (X, τ, I) is said to be $*$-hyperconnected if A is τ^*-dense ($cl^*(A) = X$) for every nonempty open subset A of X.

Definition 3.2. An ideal topological space (X, τ, I) is said to be $I*$-hyperconnected if $X - cl^*(A) \in I$ for every nonempty open subset A of X.

Definition 3.3. An ideal topological space (X, τ, I) is said to be $IS*$-hyperconnected if $X - cl^*(A) \in I$ for every nonempty I-semi-open subset A of X.

Remark 3.4. (i) Every $*$-hyperconnected is an $I*$-hyperconnected, not conversely.

(ii) Every $IS*$-hyperconnected is an $I*$-hyperconnected, not conversely.

(iii) The concepts $IS*$-hyperconnectedness and $*$-hyperconnectedness are independent.

Example 3.5. Consider the ideal topological space (X, τ, I), where X is any set has more than one element with $\tau = \{X, \varnothing, \{x\}\}$ and $I = \mathcal{P}(\mathcal{X})$. Then X is $I*$-hyperconnected and $IS*$-hyperconnected which is not $*$-hyperconnected.

Example 3.6. Consider the ideal topological space (X, τ, I), where $X = \{1, 2, 3\}$ with the indiscrete topology and $I = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. Then X is $*$-hyperconnected and $I*$-hyperconnected which is not $IS*$-hyperconnected.

Proposition 3.7. *For any ideal topological space (X, τ, I)*

(i) If $I = \{\varnothing\}$, then the concepts $*$-hyperconnectedness and $I*$-hyperconnectedness are the same.

(ii) If $I = \{\varnothing\}$ and every open subset of X is closed, then the concepts $*$-hyperconnectedness, $I*$-hyperconnectedness and $IS*$-hyperconnectedness are equivalent.

Proposition 3.8. *For any ideal topological space (X, τ, I), the following statements are equivalent;*

(i) X is $I*$-hyperconnected.

(ii) $X - cl^*(A) \in I$ for every nonempty α-open subset A of X.

(iii) $X - cl^*(A) \in I$ for every nonempty semi-open subset A of X.

(iv) $int^*(B) \in I$ for every proper closed subset B of X.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

(v) $int^*(B) \in I$ *for every proper* α*-closed subset* B *of* X.

(vi) $int^*(B) \in I$ *for every proper semi-closed subset* B *of* X.

Proposition 3.9. *For any ideal topological space* (X, τ, I)*, the following statements are equivalent;*

(i) X *is* $IS*$*-hyperconnected.*

(ii) $int^*(B) \in I$ *for every proper IS-closed subset* B *of* X.

Definition 3.10. Let (X, τ, I) be an ideal topological space and A is a subset of X. An I-semi-closure of A is the smallest I-semi-closed set containing A, denoted by $Iscl(A)$.

Definition 3.11. A nonempty subsets A and B of an ideal topological space (X, τ, I) are IS-separated if $Iscl(A) \cap B = \varnothing$ and $A \cap scl(B) = \varnothing$.

Definition 3.12. A subset A of an ideal topological space (X, τ, I) is IS-separated connected if it cannot be written as a union of two IS-separated sets in X. An ideal topological space (X, τ, I) is said to be IS-separated connected if X is IS-separated connected as a subset.

Remark 3.13. Let (X, τ, I) be an ideal topological space. If A and B are nonempty disjoint subsets of X such that A is semi-open and B is I-semi-open, then A and B are IS-separated.

Proposition 3.14. *If* (X, τ, I) *is an IS-separated connected, then* (X, τ) *is connected.*

The implication of Proposition 3.14 is not reversible, in general. As the following example shows;

Example 3.15. The ideal topological space $(\mathbb{N}, \tau, I)$ where $\tau = \{\mathbb{N}, \varnothing, \{1\}\}$ and I is the power set of $\mathbb{N}$ is not IS-separated connected, take $A = \{1, 2\}$ and $B = \mathbb{N} - \{1, 2\}$, whereas $(\mathbb{N}, \tau)$ is connected.

Proposition 3.16. *Let* Y *be an open subset of an ideal topological space* (X, τ, I) *and* A *is a subset of* X*. If* A *is an* I*-semi-open subset of* X*, then* $A \cap Y$ *is* I_Y*-semi-open subset of* Y.

The implication of Proposition 3.16 is not reversible, in general. As the following example shows;

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

Example 3.17. Consider the ideal topological space (X, τ, I) where $X = \{1, 2, 3, 4\}, \tau = \{X, \varnothing, \{1, 3\}\}$ and $I = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. Take $Y = \{1, 3\}$ and $A = \{1, 4\}$.

Proposition 3.18. *Let $A \subseteq Y$ and Y be an open subset of an ideal topological space (X, τ, I). A is an I_Y-semi- open subset of Y if and only if it is I-semi- open in X.*

Proposition 3.19. *Let $A \subseteq Y$ and Y be an open subset of an ideal topological space (X, τ, I). $I_Y scl_Y(A) = Iscl(A) \cap Y$.*

Proof. First, to prove $I_Y scl_Y(A) \subseteq Iscl(A) \cap Y$. Let $x \notin Iscl(A) \cap Y$ implies, $x \in X - Iscl(A)$ and by Proposition 3.16, $(X - Iscl(A)) \cap Y$ is an I_Y-semi-open set in Y containing x. So, $A \subseteq Y - ((X - Iscl(A)) \cap Y) = Iscl(A) \cap Y$.

Now, if $Iscl(A) \cap Y$ is the smallest I_Y-semi-closed set in Y containing A then the proof is over.

If not, then there exists an I_Y-semi-closed set, say B, in Y containing A. So $B \cap (Iscl(A) \cap Y)$ is an I_Y-semi-closed set in Y containing A and $x \notin B \cap (Iscl(A) \cap Y) = B$. Hence, $x \notin I_Y scl_Y(A)$.

Second, to prove $Iscl(A) \cap Y \subseteq I_Y scl_Y(A)$. Let $x \notin I_Y scl_Y(A)$ implies, $x \in Y - I_Y scl_Y(A)$ and by Proposition 3.18, $Y - I_Y scl_Y(A)$ is an I-semi-open set in X containing x. So, $A \subseteq X - (Y - I_Y scl_Y(A)) = I_Y scl_Y(A) \cup (X - Y)$.

Now, if $I_Y scl_Y(A) \cup (X - Y)$ is the smallest I-semi-closed set in X containing A then the proof is over.

If not, then there exists an I-semi-closed set, say B, in X containing A. So $B \cap (I_Y scl_Y(A) \cup (X - Y))$ is an I-semi-closed set in X containing A and $x \notin B \cap (I_Y scl_Y(A) \cup (X - Y)) = B$. Hence, $x \notin Iscl(A)$. □

Proposition 3.20. *Let A and B are two subsets of Y and Y is an open subset of an ideal topological space (X, τ, I). The following are equivalent;*

(i) A and B are $I_Y S$- separated in Y.

(ii) A and B are IS-separated in X.

Proof. Follow from Proposition 3.19. □

Proposition 3.21. *An ideal topological space (X, τ, I) is IS-separated connected space if and only if it is cannot be written as a disjoint union of a nonempty semi-open and I-semi-open subsets of X.*

Proposition 3.22. *Let (X, τ, I) be an ideal topological space and A is an open* *subset of X. If A is IS-separated connected subset of X and H, G are IS-separated subsets of X with $A \subseteq H \cup G$, then either $A \subseteq H$ or $A \subseteq G$.*

Proof. Suppose $A \cap H$ and $A \cap G$ are nonempty. Since $A \subseteq H \cap G$, then $A = (A \cap H) \cup (A \cap G)$. So $Iscl(A \cap H) \cap (A \cap G) \subseteq Iscl(H) \cap G = \varnothing$ and $(A \cap H) \cap scl(A \cap G) \subseteq H \cap scl(G) = \varnothing$. Then, $(Iscl(A \cap H) \cap A) \cap (A \cap G) = \varnothing$ and $(A \cap H) \cap (scl(A \cap G) \cap A) = \varnothing$. By Proposition 3.19, $(I_A scl_A(A \cap H)) \cap (A \cap G) = \varnothing$ and $(A \cap H) \cap (scl_A(A \cap G)) = \varnothing$. Then, A is not IS-separated connected. This is a contradiction. Thus, either $A \cap H$ or $A \cap G$ are empty. Implies, either $A \subseteq H$ or $A \subseteq G$. □

Proposition 3.23. *Let A and B are two IS-separated subsets of an ideal topological space (X, τ, I). If C and D are two nonempty subsets of X such that $C \subseteq A$ and $D \subseteq B$, then C and D are also IS-separated.*

Proposition 3.24. *The continuous image of an IS-separated connected space is connected.*

Proof. Follow from Proposition 3.14. □

Proposition 3.25. *If A is an IS-separated connected subset of an IS-separated connected ideal space (X, τ, I) such that $X - A$ is the union of two IS-separated sets B and C, then $A \cup B$ and $A \cup C$ are IS-separated connected.*

Proof. Suppose $A \cup B$ is not IS-separated connected. Then there exist two nonempty IS-separated sets G and H such that $A \cup B = G \cup H$. Since A is IS-separated connected, $A \subset G \cup H$, by Proposition 3.22, either $A \subset G$ or $A \subset H$. Suppose $A \subset G$ implies that $A \cup B \subset G \cup B$. Since $A \cup B = G \cup H$ then $G \cup H \subset G \cup B$. Hence $H \subset B$. Since B and C are IS-separated, by Proposition 3.23, H and C are also IS-separated. Thus H is IS-separated from G as well as C. Now, $Iscl(H) \cap (G \cup C) = (Iscl(H) \cap G) \cup (Iscl(H) \cap C) = \varnothing$ and $H \cap scl(G \cup C) = (H \cap scl(G)) \cup (H \cap scl(C)) = \varnothing$. Therefore, H is IS-separated from $G \cup C$. Since $X - A = B \cup C, X = A \cup (B \cup C) = (A \cup B) \cup C = (G \cup H) \cup C = H \cup (G \cup C)$. Thus, X is not IS-separated connected. This is a contradiction. Similar, if $A \subset H$ we will get a contradiction. Hence, $A \cup B$ is IS-separated connected. Similarly, we can prove that $A \cup C$ is IS-separated connected. □

Proposition 3.26. *Let (X, τ, I) be an ideal space. If the union of two IS-separated sets is an semi-closed set, then one set is semi-closed and the other is I-semi-closed.*

Proof. Let A and B two IS-separated sets such that $A \cup B$ is semi-closed. Then $Iscl(A) \cap B = A \cap scl(B) = \varnothing$. Since $A \cup B$ is semi-closed, $A \cup B = scl(A \cup B)$. Now, $A \subset A \cup B$ implies that $Iscl(A) \subset Iscl(A \cup B) \subset scl(A \cup B) = A \cup B$ and so $Iscl(A) = Iscl(A) \cap (A \cup B) = A$. Hence, A is I-semi-closed. Also, $scl(B) = scl(B) \cap scl(A \cup B) = scl(B) \cap (A \cup B) = B$. Hence, B is semi-closed. $\square$

Conclusion and Future Work

In this paper begins with a brief survey of the notion of ideal in topological spaces introduced by K. Kuratowski [8], V. Vaidyanathaswamy [14] and D. Jankovic, T. R. Hamlett [7]. We recall different types of collections of sets in topological spaces (X, τ) and in an ideal topological spaces (X, τ, I), like α-open sets [11], semi-open sets [9] and I-semi-open sets [6]. The authors studied some properties of an I-semi-open set in ideal topological spaces, like compactness and connectedness. In particular, the authors investigate the relationship between the different types of compactness (resp. connectedness) that introduced in this paper. Moreover, several examples are given to indicate the connection between these types. It also gives some conditions to make these notions equivalence and prove some properties of these concepts. Finally, by using the notion I-semi-open set, the authors introduced some types of continuity functions in ideal topological spaces. The affecting of these functions on some types of compactness investigated.

The concept I-semi-open set in an ideal topological space with related properties help to study new types of continuity and separation axioms.

References

[1] S.G. Crossley and S.K. Hildebrand, Semi-closed sets and semi-continuity in topological spaces, Texas J. Sci., (2+3)(22)(1971), 123-126.

[2] S.G. Crossley and S.K. Hildebrand, Semi-closure, Texas J. Sci., (2+3)(22)(1971), 99-119.

[3] J. Dontchev, M. Ganster and D. Rose, Ideal resolvability. Topology and its Appl. 93(1991), 1-16.

[4] Ch. Dorsett, Semi compactness, semi separation axioms and product spaces, Bull. Malaysian Math. Soc. (2)4(1981), 21-28.

[5] E. Ekici, T. Noiri, $*$-Hyperconnected ideal topological spaces, (Submitted).

[6] Friday Ifeanyi Michael, On semi-open sets with respect to an ideal, European Journal of Pure and Applied Mathematics, 6(1)2013, 53- 58.

[7] D. Jankovic and T. R. Hamlett, New topologies from old via ideals. Amer. Math. Monthly, 97(1990), 295- 310.

[8] K. Kuratowski, Topology, Vol I. New York: Academic Press, (1933).

[9] N. Levine, Semi-open sets and semi- continuity in topological spaces,American Mathematical Monthly, 70(1963), 36-41.

[10] A. A. Nasef, Some classes of compactness in terms of ideal, Soochow Journal of Mathematics, 27(3)(2001), 343-352.

[11] O. Njastad, On Some Classes of Nearly Open Sets, Pacific J. Math. 15(1965), 961- 970.

[12] R. L. Newcomb, Topologies which are compact modulo an ideal, Ph. D. dissertation, Univ. of Cal. At Santa Barbara, (1967).

[13] D. V. Rancin, Compactness modulo an ideal, Soviet Math. Dokl., 13(1)(1972), 131- 147.

[14] V. Vaidyanathaswamy, The localization theory in set topology, Proc. Indian Acad. Sci. 20(1945), 51- 61.