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RESOLVABLE SPACES AND COMPACTIFICATIONS

Monerah Al-Hajri[1], Karim Belaid[2 §]

[1,2]Department of Mathematics
Faculty of Sciences of Dammam, Girls College
University of Dammam
P.O. Box 383, Dammam 31113, SAUDI ARABIA

Abstract: This paper deal with spaces such that their compactification is a resolvable space. A characterization of space such that its one point compactification (resp. Wallman compactification) is a resolvable space is given.

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Introduction

In 1943, Hewitt [1] has introduced the notion of resolvable space as follows: A topological space is said to be resolvable if it has two disjoint dense subsets. Hence a topological space X is resolvable if and only if X is written as a union of two disjoint dense subsets. Hewitt in [1] has also called a topological space X maximally irresolvable if each dense subset of X is open. Nowadays, maximally irresolvable spaces are called submaximal spaces.

Recently, Belaid et al. [2], were interested by spaces such that their compactifications are submaximal. They proved that if X is a topological space and $K(X)$ is a compactification of X, then the following statements are equivalent:

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§Correspondence author

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(1) $K(X)$ is submaximal.

(2) For each dense subset D of X, the following properties hold:

   (i) D is co-finite in $K(X)$;

   (ii) for each $x \in K(X)\backslash D$, $\{x\}$ is closed.

It is clear that a compactification of resolvable spaces is resolvable. Hence the following question is natural:

"Characterize spaces X such that a compactification $K(X)$ of X is a resolvable space?".

The first section is devoted to a brief study of spaces X such that their compactification is a resolvable space. The particular case of the one-point compactification is given.

The purpose of the second section is to give an intrinsic topological characterization of spaces X such that the Wallman compactification ωX of X is a resolvable space.

1. Resolvable space and compactifications

First, recall that a compactification of a topological space X is a couple $(K(X), e)$, where $K(X)$ is a compact space and $e : X \to K(X)$ is a continuous embedding (e is a continuous one-to-one map and induces a homeomorphism from X onto $e(X)$) such that $e(X)$ is a dense subspace of $K(X)$. When a compactification $(K(X), e)$ of X is given, X will be identified with $e(X)$ and assumed to be dense in $K(X)$.

Let us give some basic facts about space such that its compactification is a resolvable space.

Lemma 1.1. *Let X be a topological space, $K(X)$ be a compactification of X and A be a subset of $K(X)$. If X is an open set of $K(X)$, then the following statements are equivalent:*

1. *A is a dense subset of $K(X)$.*
2. *$A \cap X$ is a dense subset of X.*

Proof. (1) $\Longrightarrow$ (2) Let O be an open set of X. Since X is an open set of $K(X)$, O is an open set of $K(X)$. Hence $O \cap A \neq \emptyset$. Thus $O \cap (A \cap X) \neq \emptyset$; so that $A \cap X$ is a dense set of X.

(2) $\Longrightarrow$ (1) Let U be an open set of $K(X)$. Since $U \cap X$ is a non-empty open set of X, $(A \cap X) \cap (U \cap X) \neq \emptyset$. Then $U \cap A \neq \emptyset$. Therefore A is a dense set of $K(X)$. □

An immediate consequence of Lemma 1.1 is the following.

Proposition 1.2. *Let X be a topological space and $K(X)$ be a compactification of X. If X is an open set of $K(X)$, then the following statements are equivalent:*

(i) X is resolvable.

(ii) $K(X)$ is resolvable.

Let us recall the construction the one-point compactification: For any non-compact space X the one-point compactification of X is obtained by adding one extra point ∞ (called a point at infinity) and defining the open sets of $\widetilde{X} = X \cup \{\infty\}$ to be the open sets of X together with the sets of the form $O \cup \{\infty\}$, where O is an open set of X such that $X - O$ is a closed compact set of X. The one point compactification $\widetilde{X}$ of X is also called the Alexandroff compactification of X [5].

The following result characterize space such that its one point compactification is a resolvable space. Its proof follows immediately from Proposition 1.2; thus it is omitted.

Proposition 1.3. *Let X be a non compact topological space. Then the following statements are equivalent:*

(i) The one-point compactification $\widetilde{X}$ of X is resolvable.

(ii) X is resolvable.

2. Resolvable space and Wallman compactification

First, recall that the Wallman compactification of T_1-space was introduced, in 1938, by Wallman [4] as follows:

Let $\mathcal{P}$ be a class of subsets of a topological space X which is closed under finite intersections and finite unions.

A $\mathcal{P}$-filter on X is a collection $\mathcal{F}$ of nonempty elements of $\mathcal{P}$ with the properties:

(i) $\mathcal{F}$ is closed under finite intersections;

(ii) $P_1 \in \mathcal{F}$, $P_1 \subseteq P_2$ implies $P_2 \in \mathcal{F}$.

A $\mathcal{P}$-ultrafilter is a maximal $\mathcal{P}$-filter. When $\mathcal{P}$ is the class of closed sets of X, then the $\mathcal{P}$ -filters are called closed filters.

The points of the Wallman compactification wX of a space X are the closed ultrafilters on X. For each closed set $D \subseteq X$, define D^* to be the set $D^* = \{\mathcal{F} \in wX \mid D \in \mathcal{F}\}$. Thus $\mathcal{C} = \{D^* \mid D$ is a closed set of $X\}$ is a base for the closed sets of a topology on wX.

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Let U be an open set of X, we define $U^* = \{\mathcal{F} \in wX \mid F \subseteq U$ for some F in $\mathcal{F}\}$, it is easily seen that the class $\{U^* \mid U$ is an open set of $X\}$ is a base for open sets of the topology of wX. The following properties of wX are frequently useful:

Proposition 2.1. *Let X be a T_1-space and wX the Wallman compactification of X. Then the following statements hold:*

1. *wX is a T_1-space.*
2. *For $x \in X$ and $\Phi(x) = \{F \mid F$ is a closed set of X and $x \in F\}$. Then Φ is an embedding of X into wX ($\Phi(x)$ will be identified to x).*
3. *If U is an open set of X, then $wX - U^* = (X - U)^*$.*
4. *If U_1 and U_2 are two open sets of X, then $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$ and $(U_1 \cup U_2)^* = U_1^* \cup U_2^*$.*

Recall that Kovar in [5] has characterized space with finite Wallman compactification remainder as following:

Proposition 2.2. *Let X be a T_1-space. Then the following statement are equivalent:*

(i) *$Card(wX - X) = n$.*

(ii) *There exists a collection of n pairwise disjoint non-compact closed sets of X and every family of non-compact pairwise disjoint closed sets of X contain at most n elements.*

The following proposition follows immediately from Proposition 2.3 and Proposition 2.1(1).

Proposition 2.3. *Let X be a T_1-space and $n \in \mathbb{N}$ such that every family of non-compact pairwise disjoint closed sets of X contain at most n elements. Then X is resolvable if and only if wX is resolvable.*

The following lemma has been given in [2] as Remark 4.5 and Remark 4.9.

Lemma 2.4. *Let X be a T_1-space. Then the following properties hold:*

1. *If F is a closed non-compact subset of X, then there exists $\mathcal{F} \in wX - X$ such that $F \in \mathcal{F}$.*
2. *$\mathcal{F} \in wX - X$. Then for each $F \in \mathcal{F}$, F is a non-compact closed set of X.*

The following result is an immediate consequence of Lemma 2.4.

Corollary 2.5. *Let X be a T_1-space, wX be the Wallman compactification of X and U be an open set of X. Then the following statements are equivalent:*

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(*i*) $U \subsetneq U^*$.

(*ii*) *There exists a non compact closed set F of X such that $F \subseteq U$.*

Now, we are in position to give a characterization of spaces such that their Wallman compactification is resolvable.

Theorem 2.6. *Let X be a T_1-space. Then the following statements are equivalent:*

1. *The Wallman compactification wX of X is resolvable.*
2. *There exist two disjoint subsets D_1 and D_2 of X such that:*
   (*i*) $X = D_1 \cup D_2$.
   (*ii*) *For $i \in \{1, 2\}$ and for each non empty open set $O \subseteq D_i$, there exists a non compact closed set F of X such that $F \subseteq O$.*

Proof. (1) $\Longrightarrow$ (2) Since wX is a resolvable space, there exist two disjoint dense sets A_1 and A_2 of wX such that wX is the union of A_1 and A_2. Set $D_1 = X \cap A_1$ and $D_2 = X \cap A_2$.

Let $i \in \{1, 2\}$ and O be a non empty open set of X such that $O \subseteq D_i$. Set $j \in \{1, 2\}$ such that $j \neq i$. Since A_j is a dense subset of wX, $O^* \cap A_j \neq \emptyset$. Now, $O^* \cap (A_j \cap X) = O \cap D_j = \emptyset$ implies that $O^* \cap (A_j \cap wX - X) \neq \emptyset$. It follows that there exists $\mathcal{F} \in O^* \cap (A_j \cap wX - X)$, and thus $O \subsetneq O^*$. According to Corollary C4 there exists a non compact closed set F of X such that $F \in \mathcal{F}$ and $F \subseteq O$.

(2) $\Longrightarrow$ (1) Let D_1, D_2 be two disjoint subsets of X satisfying the condition (ii) and such that $X = D_1 \cup D_2$. Let $i \neq j$ in $\{1, 2\}$ and we define $A_i = D_i \cup \{\mathcal{F} \in wX - X \mid$ there exists $F \in \mathcal{F}$ and an open set O of X such that $F \subseteq O \subseteq D_j\}$. It is immediate that $A_i \cap A_j = \emptyset$.

Now, let U be open set of X. We consider two cases:

Case 1: $U \cap D_i \neq \emptyset$. Then $U \cap A_i \neq \emptyset$. So $U^* \cap A_i \neq \emptyset$.

Case 2: $U \cap D_i = \emptyset$. Then $U \subseteq D_j$. By condition $(ii\ )$, there exists a non compact closed F of X such that $F \subseteq U$. Let $\mathcal{F} \in wX - X$ such that $F \in \mathcal{F}$. Hence $\mathcal{F} \in U^* \cap A_i$. Thus $U^* \cap A_i \neq \emptyset$.

Therefore A_i is a dense set of wX; so that wX is a resolvable space. □

Example 2.7. Let $\mathbb{Q}$ be the set of all rational numbers equipped with the natural topology $T_{\mathbb{Q}}$. Let $X = \mathbb{Q} \times \{0\} \cup \mathbb{Q} \times \{1\}$ equipped with the topology $T = \{U \times \{0\} \mid U \in T_{\mathbb{Q}}\} \cup \{\mathbb{V} \times \{1\} \mid V \in T_{\mathbb{Q}}\}$. It is immediate that the topological space X satisfy the condition (2) of the Theorem 2.6. Then wX is a resolvable space.

The previous result incites us to ask the following question.

Question 2.8. Let X be a space. We denote by $\beta_\omega X$ (resp. βX) the T_0-compactifcation of X introduced by Herrlich in [6] (resp. the Stone Cech compactification). When is $\beta_\omega X$ (resp. βX) a resolvable space?

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