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SOME α-OPERATORS VIA IDEALS

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Abstract: The aim of this paper is to introduce some types of α-operators via ideals. We further study some properties of these types. Some propositions, remarks and examples are offered to explain these concepts.

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1. Introduction and Preliminaries

Throughout the present paper, $(X,\tau),(Y,\sigma)$ and (Z,μ) will denote topological spaces with no separation properties assumed. For a subset A of a topological space $(X,\tau), cl(A)$ and $int(A)$ will denote the closure and interior of A in (X,τ), respectively.

The subject of ideals in topological spaces has been studied by Kuratowski [5] and Vaidyanathaswamy [9]. An ideal I on a set X is a nonempty collection of subsets of X which satisfies: (1) $A \in I$ and $B \subseteq A$ implies $B \in I$, (2) $A \in I$ and $B \in I$ implies $A \cup B \in I$. Given a topological space (X,τ) with an ideal I on X and if $\mathcal{P}(\mathcal{X})$ is the set of all subsets of X, a set operator $(\ )^* : \mathcal{P}(\mathcal{X}) \rightarrow \mathcal{P}(\mathcal{X})$, called a local function [9] of A with respect to τ and I, is defined as follows: for

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International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

$A \subseteq X, A^*(I,\tau) = \{x \in X : A \cap U \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau : x \in U\}$. A Kuratowski closure operator $cl^*(\ )$ for a topology $\tau^*(I,\tau)$, called the $*$-topology, finer than τ is defined by $cl^*(A) = A \cup A^*(I,\tau)$ [9]. When there is no chance for confusion, we will simply write A^* for $A^*(I,\tau)$ and τ^* for $\tau^*(I,\tau)$. If I is an ideal on X, then (X,τ,I) is called an ideal topological space.

In an ideal topological space (X,τ,I), if $A \subseteq X, int^*(A)$ will denote the interior of A in (X,τ^*). Closed subsets in (X,τ^*) are called τ^*-closed sets. A subset A of an ideal topological space (X,τ,I) is τ^*-closed if and only if $A^* \subseteq A$ [9].

For any ideal topological space (X,τ,I), the collection $\{V - J : V \in \tau$ and $J \in I\}$ is a basis for τ^*. The elements of τ^* are called τ^*-open sets. A subset A of an ideal topological space (X,τ,I) is said to be τ^*-dense if $cl^*(A) = X$. It is clear that, in a space (X,τ,I) if $I = \{\varnothing\}$, then $\tau = \tau^*$ [3], [4].

The following collections form important ideals [4] on a topological space (X,τ):

(i) $\{\varnothing\}$ or $I_{\{\varnothing\}}$: the trivial ideal.

(ii) $\mathcal{P}(\mathcal{X})$: the improper ideal.

(iii) $\mathcal{F} : (\mathcal{I}_{\{})$ the ideal of all finite sets.

(iv) $\mathcal{C} : (\mathcal{I}_{\rfloor})$ the ideal of all countable subsets of X.

Given (X,τ,I) and (Y,σ,J) are two ideal topological spaces. If $f : (X,\tau,I) \to (Y,\sigma)$ is a function, then $f(I) = \{f(A) : A \in I\}$ is an ideal on Y. Also, If $f : (X,\tau) \to (Y,\sigma,J)$ is an injection function, then $f^{-1}(J) = \{f(B) : B \in J\}$ is an ideal on X [8].

Given a topological space (X,τ). A subset A of a space X is said to be α-open if $A \subseteq int(cl(int(A)))$. The family of all α-open subsets of a space (X,τ) forms a topology on X, called the α-topology on X and denoted by τ_α, finer than τ. If every nowhere dense set in a space (X,τ) is closed, then $\tau_\alpha = \tau$ [6].

Given an ideal topological space (X,τ,I). The ideal I is said to be compatible with τ denoted by $I \sim \tau$, if the following holds for every A subset of X; if for every $x \in A$ there exists an open set U containing x such that $U \cap A \in I$, then $A \in I$ [7].

2. α-Open Sets with Respect to an Ideal

The concept of a set operator $(\ )^{\alpha^*} : \mathcal{P}(\mathcal{X}) \to \mathcal{P}(\mathcal{X})$ is called an α-local function of I with respect to τ and defined as follows: for $A \subseteq X, A^{\alpha^*}(I,\tau) = \{x \in X : A \cap U \notin I$ for every $U \in \tau_\alpha(x)\}$ where $\tau_\alpha(x) = \{U \in \tau_\alpha : x \in U\}$. When there is no ambiguity, we will simply write A^{α^*} for $A^{\alpha^*}(I,\tau)$. A τ^{α^*}-closure operator, denoted by $cl^{\alpha^*}(\ )$, for a topology $\tau^{\alpha^*}(I)$ which is called τ^{α^*}-topology on X finer than τ is defined as follows: $cl^{\alpha^*}(A)(I,\tau) = A \cup A^{\alpha^*}(I,\tau)$ for every $A \subseteq X$. When there is no ambiguity,

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

we will simply write $cl^{\alpha^*}(A)$ for $cl^{\alpha^*}(A)(I,\tau)$. A basis $\mathcal{B}(\mathcal{I},\tau)$ for τ^{α^*} was described as follows: $\mathcal{B}(\mathcal{I},\tau) = \{\mathcal{V} - \mathcal{J} : \mathcal{V} \in \tau_\alpha \text{ and } J \in I\}$. We will denote by $int^{\alpha^*}(A)$ and $cl^{\alpha^*}(A)$ the interior and closure of $A \subseteq X$, with respect to τ^{α^*}. The elements of τ^{α^*} are called τ^{α^*}-open sets. Closed subsets in (X,τ^{α^*}) are called τ^{α^*}-closed sets. A subset A of an ideal topological space (X,τ,I) is τ^{α^*}-closed (resp., τ^{α^*}-dense) if and only if $A^{\alpha^*} \subseteq A$ (resp., $cl^{\alpha^*}(A) = X$), [1].

Remark 2.1. [1, 2, 6, 9] If (X,τ,I) is an ideal topological space. Then the following implications hold:

(i) $\tau \subseteq \tau_\alpha \subseteq \tau^{\alpha^*}$.

(ii) $\tau \subseteq \tau^* \subseteq \tau^{\alpha^*}$.

(iii) $cl^{\alpha^*}(A) \subseteq cl^*(A) \subseteq cl(A)$.

(iv) $cl^{\alpha^*}(A) \subseteq cl_\alpha(A) \subseteq cl(A)$.

(v) $A^{\alpha^*} \subseteq A^* \subseteq cl(A)$.

Proposition 2.2. *If (X,τ,I) is an ideal topological space and A, B are two subsets of X. Then,*

(i) $X - A^{\alpha^*} = \cup\{U \in \tau_\alpha : A \cap U \in I\}$.

(ii) A^{α^*} *is an α-closed set.*

(iii) $A^{\alpha^*} = cl_\alpha(A^{\alpha^*}) \subseteq cl_\alpha(A)$.

(iv) $(A^{\alpha^*})^{\alpha^*} \subseteq A^{\alpha^*}$.

(v) if $A \subseteq B$, *then* $A^{\alpha^*} \subseteq B^{\alpha^*}$.

(vi) $(A \cup B)^{\alpha^*} = A^{\alpha^*} \cup B^{\alpha^*}$.

(vii) $(A \cap B)^{\alpha^*} \subseteq A^{\alpha^*} \cap B^{\alpha^*}$.

(viii) $A^{\alpha^*} - B^{\alpha^*} = (A - B)^{\alpha^*} - B^{\alpha^*} \subseteq (A - B)^{\alpha^*}$.

(ix)$A^{\alpha^*} - (A^{\alpha^*})^{\alpha^*} \subseteq (A - A^{\alpha^*})^{\alpha^*}$.

(x) if $E \in I$, *then* $(A \cup E)^{\alpha^*} = A^{\alpha^*} = (A - E)^{\alpha^*}$.

(xi) if $E \in I$, *then* $(X - E)^{\alpha^*} = X^{\alpha^*}$.

(xii) if $G \in \tau_\alpha$, *then* $G \cap A^{\alpha^*} = G \cap (G \cap A)^{\alpha^*} \subseteq (G \cap A)^{\alpha^*}$.

Remark 2.3. Let A be a subset of an ideal topological space (X,τ,I).

(i) If $I = \{\varnothing\}$, then $A^{\alpha^*} = cl_\alpha(A) = cl^{\alpha^*}(A)$.

(ii) If $I = \mathcal{P}(\mathcal{X})$, then $A^* = A^{\alpha^*} = \varnothing$ and $cl^*(A) = cl^{\alpha^*}(A) = A$.

Remark 2.4. If (X, τ, I) is an ideal topological space and $\{A_i : i \in \Lambda\}$ be a family of subsets of X. Then,

(i) $\cup\{A_i^{\alpha^*} : i \in \Lambda\} \subseteq (\cup\{A_i : i \in \Lambda\})^{\alpha^*}$.

(ii) $(\cap\{A_i : i \in \Lambda\})^{\alpha^*} \subseteq \cap\{A_i^{\alpha^*} : i \in \Lambda\}$.

The implication in Remark 2.3 is not reversible, as the following example shows.

Example 2.5. Consider the ideal topological space $(\mathbb{N}, \tau, I)$, where $\tau = \{\mathbb{N}, \varnothing\}$ and $I = I_f = \{U \subseteq \mathbb{N} : U \text{ is a finite set}\}$. Then, $\tau_\alpha = \tau$ and

(i) $\{x\}^{\alpha^*} = \varnothing$ for each $x \in \mathbb{N}$, so $\cup\{\{x\}^{\alpha^*} : x \in \mathbb{N}\} = \varnothing$ and $(\cup\{x\} : x \in \mathbb{N}\})^{\alpha^*} = \mathbb{N}$.

(ii) $(E^+)^{\alpha^*} = \mathbb{N}$, where E^+ is the set of all even numbers, so $(O^+)^{\alpha^*} = \mathbb{N}$, where O^+ is the set of all odd numbers. Also, $(E^+)^{\alpha^*} \cap (O^+)^{\alpha^*} = \mathbb{N}$ but $(E^+ \cap O^+)^{\alpha^*} = \varnothing$.

Remark 2.6. If (X, τ, I) is an ideal topological space, then τ^* is the smallest topology on X make that E is closed set in (X, τ^*) for each $E \in I$. In other words, if E is closed set in (X, τ) for each $E \in I$, then $\tau^* = \tau$. Also, if E is τ^*-closed set for each $E \in I$, then $\tau^{\alpha^*} = \tau_\alpha$.

Proposition 2.7. *If (X, τ, I) is an ideal topological space, then*

(i) E is τ^{α^*}-closed set for each $E \in I$.

(ii) $(\tau^{\alpha^*})^* = \tau^{\alpha^*}$. That means, τ^{α^*} is invariant under the operator $(\ )^*$.

(iii) $(\tau^{\alpha^*})^{\alpha^*} = (\tau^{\alpha^*})_\alpha$.

Proposition 2.8. *If (X, τ, I) is an ideal topological space and A, B are two subsets of X, then*

(i) $cl^{\alpha^*}(A) \subseteq cl^{\alpha^*}(B)$, where $A \subseteq B$.

(ii) $cl^{\alpha^*}(A \cup B) = cl^{\alpha^*}(A) \cup cl^{\alpha^*}(B)$.

(iii) $cl^{\alpha^*}(A \cap B) \subseteq cl^{\alpha^*}(A) \cap cl^{\alpha^*}(B)$.

(iv) $cl^{\alpha^*}(cl^{\alpha^*}(A)) = cl^{\alpha^*}(A)$.

(v) $cl^{\alpha^*}(\varnothing) = \varnothing$ and $cl^{\alpha^*}(X) = X$.

Proposition 2.9. *Let (X, τ, I) be an ideal topological space. If σ is a topology on X generated by the closure operator cl^{α^*}, then $\sigma = \tau^{\alpha^*}$.*

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

Proof. It is clear that $U \in \sigma$ iff $X - U$ is σ-closed iff $(X-U)^{\alpha^*} \subseteq (X-U)$ iff $U \subseteq X - (X-U)^{\alpha^*}$. Therefore, $x \in U$ implies that there exists an α-open set V containing x such that $V \cap (X-U) \in I$. Let $E = V \cap (X-U)$, then $X - E = U \cup (X-V)$ which implies that $V - E = V \cap (X-E) = V \cap (U \cup (X-V)) = U \cap V$. Since $x \in U$ and $x \in V$, then $x \in U \cap V = V - E$. Therefore, $U \subseteq V - E$. So, $U \cap V = V - E$ implies that $V - E \subseteq U$. Hence, $U = V - E$. Therefore, $\mathcal{B}(\mathcal{I}, \tau) = \{\mathcal{V} - \mathcal{E} : \mathcal{V} \in \tau_\alpha$ and $E \in I\}$ is a basis for the topology σ on X. This means that $\sigma \subseteq \tau^{\alpha^*}$.

Now, suppose that $V - E$ is a member in the base $\mathcal{B}(\mathcal{I}, \tau) = \{\mathcal{V} - \mathcal{E} : \mathcal{V} \in \tau_\alpha$ and $E \in I\}$. Then, $X - V$ is an α-closed set. By using Proposition 2.2 (x) and Remark 2.1 (iv) we get, $X - (V-E) = (X-V) \cup E$ and $(X-(V-E))^{\alpha^*} = ((X-V) \cup E)^{\alpha^*} = (X-V)^{\alpha^*} \subseteq cl_\alpha(X-V) = X - V \subseteq X - (V-E)$. Implies, $X - (V-E)$ is σ-closed set. So $(V-E) \in \sigma = \{U \subseteq X : cl^{\alpha^*}(X-U) = X - U\}$. Hence, $\sigma = \tau^{\alpha^*}$. □

Proposition 2.10. *If (X, τ) is a topological space, A is a subset of X and I, J are two ideals on X such that $I \subseteq J$. Then,*

(i) $A^{\alpha^*}(J) \subseteq A^{\alpha^*}(I)$.

(ii) $\tau^{\alpha^*}(I) \subseteq \tau^{\alpha^*}(J)$.

Remark 2.11. [4] If X is a non-empty set and I, J are two ideals on X, then $I \cap J$ and $I \vee J = \{E \cup H : E \in I$ and $H \in J\}$ are also ideals on X.

Proposition 2.12. *If (X, τ) is a topological space, A is a subset of X and I, J are two ideals on X. Then, $A^{\alpha^*}(I \cap J) = A^{\alpha^*}(I) \cup A^{\alpha^*}(J)$.*

Proposition 2.13. *If (X, τ) is a topological space, A is a subset of X and I, J are two ideals on X. Then, $A^{\alpha^*}(I, \tau^{\alpha^*}(J)) \cap A^{\alpha^*}(J, \tau^{\alpha^*}(I)) \subseteq A^{\alpha^*}(I \vee J, \tau)$.*

Proof. Suppose that $x \notin A^{\alpha^*}(I \vee J, \tau)$, then there is an α-open set U containing x such that $U \cap A \in I \vee J$. So, there are $E \in I$ and $H \in J$ such that $U \cap A = E \cup H$. Also, $U \cap A = E_1 \cup H$ where $E_1 = E - H \in I$ and $E_1 \cap H = \varnothing$. It is clear that, $(U \cap A) - H = (U - H) \cap A = E_1 \in I$. And $(U \cap A) - E_1 = (U - E_1) \cap A = H \in J$. Now; if $x \in A$, then $x \in (U - E_1)$ or $x \in (U - H)$ but not in both and if $x \notin A$, then $x \notin U \cap A = E_1 \cup H$. So, $x \in (U - E_1)$ and $x \in (U - H)$. Since, $U - E_1 \in \tau^{\alpha^*}(I) \subseteq (\tau^{\alpha^*}(I))_\alpha$ and $U - H \in \tau^{\alpha^*}(J) \subseteq (\tau^{\alpha^*}(J))_\alpha$, then $x \notin A^{\alpha^*}(I, \tau^{\alpha^*}(J))$ or $x \notin A^{\alpha^*}(J, \tau^{\alpha^*}(I))$. Therefore, $A^{\alpha^*}(I, \tau^{\alpha^*}(J)) \cap A^{\alpha^*}(J, \tau^{\alpha^*}(I)) \subseteq A^{\alpha^*}(I \vee J, \tau)$. □

Proposition 2.14. *If $f : (X, \tau^*(I)) \to (Y, \sigma)$ is a continuous function, then $f : (X, \tau^{\alpha^*}(I)) \to (Y, \sigma)$ is also continuous.*

Corollary 2.15. *If $f : (X, \tau) \to (Y, \sigma)$ is a continuous function, then $f : (X, \tau^{\alpha^*}(I)) \to (Y, \sigma)$ is also continuous.*

Proposition 2.16. *Let (X, τ, I) be an ideal topological space. If (X, τ) is disconnected, then (X, τ^{α^*}) is also.*

3. Ψ^α-Classes

In this section we will introduce the set operator $\Psi^\alpha(I, \tau)$ and study some properties of it.

Definition 3.1. If (X, τ, I) is an ideal topological space. Then, we define an operator; $\Psi^\alpha(I, \tau) : \mathcal{P}(\mathcal{X}) \to \tau_\alpha$ as follows: for every subset A of $X, \Psi^\alpha(I, \tau)(A) = \{x \in X : U - A \in I$ for every $U \in \tau_\alpha(x)\}$. Equivalently, $\Psi^\alpha(A) = X - (X - A)^{\alpha^*}$ for every subset A of X. We denoted $\Psi^\alpha(I, \tau)$ simply by Ψ^α when no ambiguity if present. It would appear intuitively that the operator Ψ^α is a natural complement to the operator $(\ )^{\alpha^*}$.

Remark 3.2. Let (X, τ, I) be an ideal topological space. If A is a subset of X, then

(i) $I = \varnothing$ implies $\Psi^\alpha(A) = int_\alpha(A)$.

(ii) $I = \mathcal{P}(\mathcal{X})$ implies $\Psi^\alpha(A) = X$.

(iii) U is an α-open subset of X, implies $U \subseteq \Psi^\alpha(U)$.

Proposition 3.3. *Let (X, τ, I) be an ideal topological space. If A is a subset of X, then $\Psi^\alpha(A) = \cup\{U \in \tau_\alpha : U - A \in I\}$.*

Proposition 3.4. *Let (X, τ, I) be an ideal topological space. If U is an α-open set, then $\Psi^\alpha(U) = \cup\{M \in \tau_\alpha : (M - U) \cup (U - M) \in I\}$.*

Proof. For any subsets M and U of $X, (M - U) \cup (U - M) \in I$ implies that $M - U \in I$. Then, $\cup\{M \in \tau_\alpha : (M - U) \cup (U - M) \in I\} \subseteq \Psi^\alpha(U)$. Conversely, if $x \in \Psi^\alpha(U)$, then there exists an α-open set M containing x such that $M - U \in I$. If $U \in \tau_\alpha$ and $L = M \cup U$, then L is an α-open set containing x and $(L-U) \cup (U-L) = M - U \in I$. Therefore, $x \in \cup\{M \in \tau_\alpha : (M - U) \cup (U - M) \in I\}$ and so, $\Psi^\alpha(U) \subseteq \cup\{M \in \tau_\alpha : (M - U) \cup (U - M) \in I\}$. Hence, $\Psi^\alpha(U) = \cup\{M \in \tau_\alpha : (M - U) \cup (U - M) \in I\}$. □

Proposition 3.5. *Let (X, τ, I) be an ideal topological space and A, B are two subsets of X. The following statements are hold,*

(i) $\Psi^{\alpha}(A)$ *is an* α*-open set.*

(ii) If $A \subseteq B$, *then* $\Psi^{\alpha}(A) \subseteq \Psi^{\alpha}(B)$.

(iii) $\Psi^{\alpha}(A \cap B) = \Psi^{\alpha}(A) \cap \Psi^{\alpha}(B)$.

(iv) $\Psi^{\alpha}((X - A)^{\alpha^*}) = X - (\Psi^{\alpha}(A))^{\alpha^*}$.

(v) $(A^{\alpha^*})^{\alpha^*} = X - \Psi^{\alpha}(X - A^{\alpha^*})$.

(vi) If $U \in \tau^{\alpha^*}$, *then* $U \subseteq \Psi^{\alpha}(U)$.

(vii) $\Psi^{\alpha}(A) \subseteq \Psi^{\alpha}(\Psi^{\alpha}(A))$.

(viii) $\Psi^{\alpha}(A) = \Psi^{\alpha}(\Psi^{\alpha}(A))$ *if and only if* $(X - A)^{\alpha^*} = ((X - A)^{\alpha^*})^{\alpha^*}$.

(ix) If $A \in I$, *then* $\Psi^{\alpha}(A) = X - X^{\alpha^*}$.

(x) If $E \in I$, *then* $\Psi^{\alpha}(A - E) = \Psi^{\alpha}(A) = \Psi^{\alpha}(A \cup E)$.

(xi) If $(A - B) \cup (B - A) \in I$, *then* $\Psi^{\alpha}(A) = \Psi^{\alpha}(B)$.

Proposition 3.6. *If (X, τ, I) is an ideal topological space, then $\tau^{\alpha^*}(I) = \{A \subseteq X : A \subseteq \Psi^{\alpha}(A)\}$.*

Definition 3.7. [8] Let (X, τ, I) be an ideal topological space and A, B are two subsets of X. Then, $A = B$ [mod I] if $(A - B) \cup (B - A) \in I$ and observes that "=[mod I]" is an equivalence relation. Let us denote by $A \Delta B$ the "symmetric" $(A - B)$ τ^{α^*}-closed $(B - A)$.

Remark 3.8. Let (X, τ, I) be an ideal topological space and A, B are two subsets of X. If $A = B$ [mod I], then $\Psi^{\alpha}(A) = \Psi^{\alpha}(B)$.

Definition 3.9. A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be an α^*-homeomorphism with respect to τ, I, σ and J if and only if $f : (X, \tau^{\alpha^*}) \to (Y, \sigma^{\alpha^*})$ is a homeomorphism or simply an α^*-homeomorphism when no ambiguity is present.

A topological property P will be called an α^*-topological property with respect to τ, I, σ and J if it is preserved by any α^*-homeomorphism.

Proposition 3.10. *If $f : (X, \tau, I) \to (Y, \sigma, J)$ is a bijection with $f(I) = J$, then the following statements are equivalent:*

(i) f *is* α^**-homeomorphism.*

(ii) $f(A^{\alpha^*}) = [f(A)]^{\alpha^*}$, *for every subset* A *of* X.

(iii) $f(\Psi^{\alpha}(A)) = \Psi^{\alpha}(f(A))$, *for every subset* A *of* X.

Proof. (i) $\rightarrow$ (ii): Let A be a subset of X and $y \notin f(A^{\alpha^*})$, then $f^{-1}(y) \notin A^{\alpha^*}$ so there exists an α-open subset U of X such that $f^{-1}(y) \in U$ and $U \cap A \in I$. Then, $y \in f(U) \in \sigma^{\alpha^*}$ and $f(U) \cap f(A) = f(U \cap A) \in J$. This implies that $y \notin [f(A)]^{\alpha^*}$. Hence, $[f(A)]^{\alpha^*} \subseteq f(A^{\alpha^*})$. Conversely, let $y \notin [f(A)]^{\alpha^*}$, then there exists an α-open subset V of Y such that $y \in V$ and $V \cap f(A) \in J$. Implies, $f^{-1}(y) \in f^{-1}(V) \in \tau^{\alpha^*}$ and $f^{-1}(V) \cap A \in I$ so $f^{-1}(y) \notin A^{\alpha^*}$. Therefore, $y \notin f(A^{\alpha^*})$. Hence, $f(A^{\alpha^*}) \subseteq [f(A)]^{\alpha^*}$.

(ii) $\rightarrow$ (iii): Let A be a subset of X. Then, $f(\Psi^{\alpha}(A)) = f(X - (X - A)^{\alpha^*}) = Y - f((X - A)^{\alpha^*}) = Y - [Y - f(A)]^{\alpha^*} = \Psi^{\alpha}(f(A))$.

(iii) $\rightarrow$ (i): Let $U \in \tau^{\alpha^*}$, then by Proposition 3.6, $U \subseteq \Psi^{\alpha}(U)$ which implies that $f(U) \subseteq f(\Psi^{\alpha}(U)) = \Psi^{\alpha}(f(U))$ and $f(U) \in \sigma^{\alpha^*}$. Hence, $f : (X, \tau^{\alpha^*}) \rightarrow (Y, \sigma^{\alpha^*})$ is an open function. Similarly, we can show that $f^{-1} : (Y, \sigma^{\alpha^*}) \rightarrow (X, \tau^{\alpha^*})$ is an open function. Therefore, f is α^*-homeomorphism. □

Definition 3.11. Let (X, τ, I) be an ideal topological space. The ideal I is said to be α-compatible with τ denoted by $I \sim^{\alpha} \tau$, if the following holds for every A subset of X; if for every $x \in A$ there exists an α-open set U containing x such that $U \cap A \in I$, then $A \in I$. One can deduce that if $I \sim^{\alpha} \tau$, then $I \sim \tau$.

The significance of α-compatibility will be given after establishing some results.

Proposition 3.12. *If $I \sim^{\alpha} \tau$, then the following statements are equivalent for a subset A of an ideal topological space (X, τ, I).*

(i) $A \in I$.

(ii) $A^{\alpha^*} = \varnothing$.

(iii) $A \cap A^{\alpha^*} = \varnothing$.

Proposition 3.13. *If $I \sim^{\alpha} \tau$, then $A - A^{\alpha^*} \in I$ for every subset A of an ideal topological space (X, τ, I).*

Lemma 3.14. *Let (X, τ, I) be an ideal topological space such that $I \sim^{\alpha} \tau$ and A is a subset of X, then A is a τ^{α^*}-closed if and only if $A = B \cup E$ such that B is an α-closed and $E \in I$.*

Proof. Necessity, if $A = B \cup E$ such that B is an α-closed set and $E \in I$, then from Proposition 2.2 (x) we get that $A^{\alpha^*} = B^{\alpha^*} \cup E^{\alpha^*} = B^{\alpha^*} \subseteq cl^{\alpha^*}(B) \subseteq cl_{\alpha}(B) = B \subseteq A$. Implies that A is an τ^{α^*}-closed.

Sufficient, if A is a τ^{α^*}-closed set, then $A^{\alpha^*} \subseteq A$. Implies, $A = A \cup A^{\alpha^*} = (A - A^{\alpha^*}) \cup A^{\alpha^*}$. Then, from Proposition 2.2 (ii), A^{α^*} is an α-closed set and from Proposition 3.13, $A - A^{\alpha^*} \in I$. □

Proposition 3.15. *Let (X, τ, I) be an ideal topological space. If $I \sim^{\alpha} \tau$, then $\mathcal{B}(\mathcal{I}, \tau) = \{\mathcal{V} - \mathcal{J} : \mathcal{V} \in \tau_{\alpha}$ and $J \in I\} = \tau^{\alpha^*}$.*

Proof. Let G be a τ^{α^*}-open set, then $X - G$ is a τ^{α^*}-closed. From Lemma 3.14, there are α-closed set B and $E \in I$ such that $X - G = B \cup E$. Hence, $G = X - (B \cup E)$ and $G = (U - E)$, where $U = (X - B) \in \tau_{\alpha}$. Therefore, $\mathcal{B}(\mathcal{I}, \tau) = \{\mathcal{V} - \mathcal{J} : \mathcal{V} \in \tau_{\alpha}$ and $J \in I\} = \tau^{\alpha^*}$. □

Proposition 3.16. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $I \sim^{\alpha} \tau$, then $(A^{\alpha^*})^{\alpha^*} = A^{\alpha^*}$.*

Proof. From Proposition 3.13, $A - A^{\alpha^*} \in I$ and by Proposition 3.12 (ii), $(A - A^{\alpha^*})^{\alpha^*} = \varnothing$. So by Proposition 2.2 (ix), $A^{\alpha^*} - (A^{\alpha^*})^{\alpha^*} \subseteq (A - A^{\alpha^*})^{\alpha^*}$. Implies, $A^{\alpha^*} \subseteq (A^{\alpha^*})^{\alpha^*}$. And from Proposition 2.2 (iv), $(A^{\alpha^*})^{\alpha^*} \subseteq A^{\alpha^*}$. Therefore, $(A^{\alpha^*})^{\alpha^*} = A^{\alpha^*}$. □

Corollary 3.17. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $I \sim^{\alpha} \tau$, then $\Psi^{\alpha}(X - A^{\alpha^*}) = X - A^{\alpha^*}$.*

Proposition 3.18. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. Then, $I \sim^{\alpha} \tau$ if and only if $\Psi^{\alpha}(A) - A \in I$.*

Proof. Necessity, if $I \sim^{\alpha} \tau$, then for each $x \in \Psi^{\alpha}(A) - A$ there exists an α-open set U containing x such that $U - A \in I$. So, by hereditary property $U \cap (\Psi^{\alpha}(A) - A) \in I$. Hence, $\Psi^{\alpha}(A) - A \in I$.

Sufficient, suppose that for every $x \in A$ there exists an α-open set U containing x such that $U \cap A \in I$. Implies, $U - (X - A) \in I$ so, $x \in \Psi^{\alpha}(X - A) - (X - A)$. Therefore, $A \subseteq \Psi^{\alpha}(X - A) - (X - A)$. And since $\Psi^{\alpha}(X - A) - (X - A) \in I$, then $A \in I$. Hence, $I \sim^{\alpha} \tau$. □

Proposition 3.19. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $I \sim^{\alpha} \tau$, then $\Psi^{\alpha}(A) = \Psi^{\alpha}(\Psi^{\alpha}(A))$.*

Proposition 3.20. *Let (X, τ, I) be an ideal topological space and $I \sim^{\alpha} \tau$. If U and V are two τ^{α^*}-open sets such that $\Psi^{\alpha}(U) = \Psi^{\alpha}(V)$, then $U = V$ [mod I].*

Proof. By Proposition 3.6, $U, V \in \tau^{\alpha^*}$ implies that $U \subseteq \Psi^{\alpha}(U)$. $U - V \subseteq \Psi^{\alpha}(U) - V = \Psi^{\alpha}(V) - V$. By Proposition 3.18, $\Psi^{\alpha}(V) - V \in I$. So, by hereditary property $U - V \in I$. Similarly, $V - U \in I$. Hence, $(U - V) \cup (V - U) \in I$ and so $U = V$ [mod I]. □

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

Proposition 3.21. *Let (X, τ, I) be an ideal topological space. If $I \sim^{\alpha} \tau$, then $I \sim \tau^{\alpha^*}$.*

Proof. For each $x \in A$, there exists a τ^{α^*}-open set V containing x such that $V \cap A \in I$. By Proposition 3.15, there exists an α-open set U containing x and $E \in I$ such that $V = U - E$. Since, $U \subseteq V \cup E$ and $U \cap A \subseteq (V \cap A) \cup (E \cap A) \in I$. Implies that $U \cap A \in I$ and so $A \in I$. Hence, $I \sim \tau^{\alpha^*}$. □

Proposition 3.22. *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $I \sim^{\alpha} \tau$, then $\Psi^{\alpha}(A) = \cup\{\Psi^{\alpha}(U) : U \in \tau_{\alpha}$ and $\Psi^{\alpha}(U) - A \in I\}$.*

Proof. It is not difficult to show that $\cup\{\Psi^{\alpha}(U) : U \in \tau_{\alpha}$ and $\Psi^{\alpha}(U) - A \in I\} \subseteq \Psi^{\alpha}(A)$. Conversely, if $x \in \Psi^{\alpha}(A)$, then there exists an α-open set U containing x such that $U - A \in I$. By Proposition 3.5 (vi), $U \subseteq \Psi^{\alpha}(U)$ and $\Psi^{\alpha}(U) - A \subseteq (\Psi^{\alpha}(U) - U) \cup (U - A)$. Since $(\Psi^{\alpha}(U) - U) \in I$ by Proposition 3.18, then $(\Psi^{\alpha}(U) - U) \cup (U - A) \in I$. Hence, $x \in \cup\{\Psi^{\alpha}(U) : U \in \tau_{\alpha}$ and $\Psi^{\alpha}(U) - A \in I\}$. □

Definition 3.23. Let X be a nonempty set and I be an ideal on X. Two topologies τ and σ on a set X are called α^*-equivalent, denoted by $\tau = \sigma$ [mod I], if $\tau^{\alpha^*}(I) = \sigma^{\alpha^*}(I)$.

Proposition 3.24. *Let (X, τ, I) and (X, σ, I) are two ideal topological spaces such that $\tau = \sigma$ [mod I]. If $I \sim^{\alpha} \tau$, then $I \sim^{\alpha} \sigma$.*

4. Conclusion and Future Work

In this paper begins with a brief survey of the notion of ideal in topological spaces introduced by K. Kuratowski [5], V. Vaidyanathaswamy [9], D. Jankovic and T. R. Hamlett [4]. We recall different types of collections of sets in topological spaces (X, τ) and in an ideal topological spaces (X, τ, I), like α-open sets [6], $I \sim \tau$ [7]and τ^{α^*}-open sets [1]. The authors study some properties about the notions τ^{α^*}-open sets, set operator $\Psi^{\alpha}(I, \tau)$ and $I \sim^{\alpha} \tau$ in the ideal topological spaces. In particular, the authors investigate the relationship between the different types of notions that introduced in this paper. Moreover, several examples are given to explain these notions. Finally, by using the constructed concepts, the authors introduced some types of continuity functions in ideal topological spaces. The relationships with some properties of these functions are investigated.

The concepts τ^{α^*}-open sets,set operator $\Psi^{\alpha}(I, \tau)$ and $I \sim^{\alpha} \tau$ in the ideal topological spaces with related properties help to study some topological properties like connectedness and compactness.

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