ON THE SUBORBITS OF THE ALTERNATING GROUP $A_n$
ACTING ON ORDERED $r$–ELEMENT SUBSETS

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Abstract: Transitivity and primitivity of the action of the alternating group $A_n$ on ordered $r$–element subsets of the set $X = \{1, 2, \ldots, n\}$ of $n$ letters are investigated in this paper. In addition, the rank and subdegrees of the action are calculated. Finally, some properties of the suborbits corresponding to this action are explored.

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Key Words: group, action, suborbits, rank, subdegrees

1. Introduction

Calculation of ranks and subdegrees of the symmetric group $S_n$ acting on $X^{[r]}$, the ordered $r$–element subsets of $X$, appears in [4] and [5]. However, little appears to have been done on the action of $A_n$, a subgroup of $S_n$, on $X^{[r]}$. Section 2 of this paper gives definitions of some terms as well as theorems to be used in subsequent sections. Section 3 investigates the transitivity and primitivity of $A_n$ acting on $X^{[r]}$. On the other hand, Section 4 determines the rank and subdegrees of $A_n$ on $X^{[2]}$, $X^{[3]}$ and $X^{[4]}$ while Section 5 generalizes these invariants. Finally, Section 6 explores some criteria for determining if a suborbit of $A_n$ is self-paired or paired with another. This section also derives a formula for finding the number of self-paired suborbits.

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2. Notation and Preliminary Results

Let $G$ be a group and $X$ a non-empty set. Then $G$ acts on the left of $X$ if there exists a function $G \times X \to X$ such that $(g_1 g_2)x = g_1(g_2)x$ and $ex = x$ where $e$ is the identity in $G$, $x \in X$ and $g_1, g_2 \in G$. The action of $G$ on the right of $X$ can be defined in a similar way. In this case, $X$ is called a $G-$set.

Suppose a group $G$ acts on a set $X$. Define a relation $x \sim y$ on $X$ if and only if there exists $g \in G$ such that $y = gx$. This defines an equivalence relation on $X$. The equivalence class containing $x$ is $\text{Orb}_G x = \{gx | g \in G\}$, and is called the orbit (transitivity class) of $x$. Since any set is a disjoint union of equivalence classes under an equivalence relation, it follows that if $G$ acts on $X$, then $X$ is a union of disjoint orbits.

**Theorem 2.1.** (Cauchy-Frobenius Lemma, [2], p. 223) If $G$ is a finite group and $X$ is a finite $G-$set, then the number of orbits in $X$ under $G$ is $\sum_{g \in G} |\text{Fix}(g)|$, where $\text{Fix}(g) = \{x \in X | gx = x\}$.

The action of a group $G$ on a set $X$ is said to be transitive if for each $x$ and $y$ in $X$, there exists $g \in G$ such that $y = gx$; in other words $\text{Orb}_G x = X$ if $x \in X$. A group which is not transitive is called intransitive.

The stabilizer in $G$ of $x$ is the subset $\text{Stab}_G x = \{g \in G | gx = x\}$ of $G$. It is also denoted by $G_x$ and it is a subgroup of $G$, called the isotropy subgroup of $G$. If $G_x$ is trivial, i.e., $G_x = \{e\}$, then $G$ is said to act faithfully on $X$.  

**Theorem 2.2.** (Orbit-Stabilizer Theorem, [2], p. 218) Let $X$ be a $G-$set and let $x \in X$. Then $|\text{Orb}_G x| = \frac{|G|}{|\text{Stab}_G x|}$, the index of $G_x$ in $G$.

Let $G$ act transitively on a finite set $X$. Then a subset $Y$ of $X$ is called a block (set of imprimitivity) for the action if for each $g \in G$, either $gY = Y$ or $gY \cap Y = \emptyset$; i.e., if $gY$ and $Y$ do not overlap partially. In particular, $\emptyset, X$ and all 1-element subsets of $X$ are blocks, called the trivial blocks. The action is said to be primitive if the only blocks are the trivial blocks; otherwise the action is imprimitive.

Suppose $G$ acts transitively on $X$ and let $G_x$ be the stabilizer of $x \in X$. The orbits $\Delta_0 = \{x\}, \Delta_1, \Delta_2, \ldots, \Delta_{r-1}$ of $G_x$ on $X$ are known as suborbits of $G$. The rank of $G$ is then $r$. The sizes $n_i = |\Delta_i|$, $(i = 0, 1, 2, \ldots, r - 1)$ are known as the subdegrees of $G$. Both the rank and the subdegrees of $G$ are independent of the choice of $x \in X$.

Let $G$ act transitively on a set $X$ and let $\Delta$ be an orbit of $G_x$ on $X$. Define $\Delta^* = \{gx | g \in G, x \in g\Delta\}$. Then $\Delta^*$ is also an orbit of $G_x$ called the $G_x-$orbit.
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(G–subbit) paired with $\triangle$. Clearly, $\triangle^{**} = \triangle$ and $|\triangle| = |\triangle^*|$. If $\triangle = \triangle^*$, then $\triangle$ is said to be self-paired. The trivial suborbit of $G$ is self-paired, and there are other self-paired suborbits of $G$ if and only if $G$ has even order, [6].

**Theorem 2.3.** (see [1], p. 425) Let $G$ be transitive on $X$ and let $g \in G$. Then the number of self-paired suborbits of $G$ is given by $\frac{1}{|G|} \sum_{g \in G} |\Fix(g^2)|$.

From this point on, $G$ shall be reserved to denote the alternating group $A_n$. Let $G$ act on $X = \{1, 2, \ldots, n\}$. Then, the action of $G$ on $X$ induces an action of $G$ on $X^r$ that is defined by

$$g[x_1, x_2, \ldots, x_r] = [g(x_1), g(x_2), \ldots, g(x_r)] \ \forall g \in G, \ [x_1, x_2, \ldots, x_r] \in X^r.$$  

**3.Transitivity and Primitivity of $G$ Acting on $X^r$**

**Theorem 3.1.** The action $G$ on $X^r$ is intransitive if $n = r + 1$.

**Proof.** If $n = r + 1$, then $|X^r| = n!$ and $|G| = \frac{n!}{2} < |X^r|$. Hence, if $G$ acts on $X^r$ and $[x_1, x_2, \ldots, x_r] \in X^r$, then $|\Orb_G[x_1, x_2, \ldots, x_r]| < |X^r|$, and the conclusion follows.

**Lemma 3.2.** Let $G$ act on $X^r$ with $n \geq r + 2$. If $[x_1, x_2, \ldots, x_r] \in X^r$, then $|\Stab_G[x_1, x_2, \ldots, x_r]| = \frac{(n-r)!}{2}$. In this case $\Stab_G[x_1, x_2, \ldots, x_r]$ is trivial if $n = r + 2$, so that $G$ acts faithfully on $X^r$, and is non-trivial otherwise.

**Proof.** Suppose $G$ acts on $X^r$ and let $[x_1, x_2, \ldots, x_r] \in X^r$. Then $g \in G$ fixes $[x_1, x_2, \ldots, x_r]$ if and only if each element of $\{x_1, x_2, \ldots, x_r\}$ comes from a 1-cycle in $g$. Hence, the order of $\Stab_G[x_1, x_2, \ldots, x_r]$ is equal to the order of the group of all even permutations of the set $\{x_1, x_2, \ldots, x_n\}$. But, this group is isomorphic to $A_{n-r}$. Therefore, $|\Stab_G[x_1, x_2, \ldots, x_r]| = \frac{(n-r)!}{2}$. Now, if $n = r + 2$, then on rewriting $n-r = 2$, so that $|\Stab_G[x_1, x_2, \ldots, x_r]| = \frac{n!}{2} = 1$. Hence, $G$ acts faithfully on $X^r$. A similar argument shows that if $n > r + 2$, then $|\Stab_G[x_1, x_2, \ldots, x_r]| > 1$.

**Theorem 3.3.** The action of $G$ on $X^r$ is transitive if $n \geq r + 2$.

**Proof.** Let $[x_1, x_2, \ldots, x_r] \in X^r$. Now, from Theorem 2.2 and Lemma 3.2,

$$|\Orb_G[x_1, x_2, \ldots, x_r]| = \frac{|G|}{|\Stab_G[x_1, x_2, \ldots, x_r]|}$$
Therefore, the action is transitive.

\[ n = \frac{n!}{(n-r)!} = \frac{n!}{(n-r)!} = |X^r|. \]

By Theorem 3.3, this action is transitive. Now, suppose

\[ Y = \{ [x_1, x_2, \ldots, x_r], [y_1, y_2, \ldots, y_r] \} \subset X^r. \]

Let \( x_1 = y_1 = x', x_2 = y_2 = x'', \ldots, x_{r-2} = y_{r-2} = x^{r-2} \) while \( x_{r-1}, x_r, y_{r-1}, \) and \( y_r \) are all distinct so that \( |\{ x_1, x_2, \ldots, x_r \} \cap \{ y_1, y_2, \ldots, y_r \}| = r - 2 \). Take \( g \in G \). If each of \( x_i \) and \( y_j \) \( (i, j = 1, 2, \ldots, r) \) comes from a 1-cycle of \( g \), then \( g \) fixes each element of \( Y \) so that \( gY = Y \). On the other hand, if each of \( x', x'', \ldots, x^{r-2} \) is in a 1-cycle of \( g \), while \( (x_{r-1}, y_{r-1}) \) and \( (x_r, y_r) \) are transpositions in \( g \), then \( gY = Y \) since \( g[x_1, x_2, \ldots, x_r] = [y_1, y_2, \ldots, y_r] \) and \( g[y_1, y_2, \ldots, y_r] = [x_1, x_2, \ldots, x_r] \). Finally, any other \( g \) takes one element of \( Y \) to an element of \( X^r \) not in \( Y \) so that \( gY \cap Y = \emptyset \).

Hence, \( Y \) is a non-trivial block for the action and the action is therefore imprimitive.

**Theorem 3.4.** The action of \( G \) on \( X^r \) is imprimitive if \( n \geq r + 2 \).

**Proof.** By Theorem 3.3, this action is transitive. Now, suppose

\[ Y = \{ [x_1, x_2, \ldots, x_r], [y_1, y_2, \ldots, y_r] \} \subset X^r. \]

Let \( x_1 = y_1 = x', x_2 = y_2 = x'', \ldots, x_{r-2} = y_{r-2} = x^{r-2} \) while \( x_{r-1}, x_r, y_{r-1}, \) and \( y_r \) are all distinct so that \( |\{ x_1, x_2, \ldots, x_r \} \cap \{ y_1, y_2, \ldots, y_r \}| = r - 2 \). Take \( g \in G \). If each of \( x_i \) and \( y_j \) \( (i, j = 1, 2, \ldots, r) \) comes from a 1-cycle of \( g \), then \( g \) fixes each element of \( Y \) so that \( gY = Y \). On the other hand, if each of \( x', x'', \ldots, x^{r-2} \) is in a 1-cycle of \( g \), while \( (x_{r-1}, y_{r-1}) \) and \( (x_r, y_r) \) are transpositions in \( g \), then \( gY = Y \) since \( g[x_1, x_2, \ldots, x_r] = [y_1, y_2, \ldots, y_r] \) and \( g[y_1, y_2, \ldots, y_r] = [x_1, x_2, \ldots, x_r] \). Finally, any other \( g \) takes one element of \( Y \) to an element of \( X^r \) not in \( Y \) so that \( gY \cap Y = \emptyset \).

Hence, \( Y \) is a non-trivial block for the action and the action is therefore imprimitive.

**4. Ranks and Subdegrees of \( G \) on \( X^2 \), \( X^3 \) and \( X^4 \)**

**Theorem 4.1.** If \( n \geq 6 \), then the rank of \( G \) on \( X^2 \) is 7.

**Proof.** Let \( G \) act on \( X^2 \). Then \( G_{[1,2]} \) has orbits with exactly 2, 1, or no element from \( N = \{ 1, 2 \} \). There is only \( 2C_2 = 1 \) way of selecting two elements from \( N \) and the two can be arranged in the two positions in \( 2P_2 = 2 \) ways. So, there are \( 2C_2 \times 2P_2 = 2 \) suborbits whose elements contain both elements of \( N \). Also, there are \( 2C_1 = 2 \) ways of selecting an element from \( N \), which can occupy any of the two positions in \( 2P_1 = 2 \) ways. Hence, there are \( 2C_1 \times 2P_1 = 4 \) suborbits of \( G \) whose each element has exactly one element from \( N \). Similarly, there is only \( 2C_0 \times 2P_0 = 1 \) suborbit whose each element contains no element from \( N \). So, \( G \) has 7 suborbits in total.

The seven suborbits of \( G \) on \( X^2 \) are

a) Suborbits whose elements contain both 1 and 2:
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<table>
<thead>
<tr>
<th>No. of elements from ${1, 2}$</th>
<th>Suborbit Length</th>
<th>Corresponding No. of Suborbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>$2C_2 \times 2P_2 = 2$</td>
</tr>
<tr>
<td>1</td>
<td>$(n - 2)$</td>
<td>$2C_1 \times 2P_1 = 4$</td>
</tr>
<tr>
<td>0</td>
<td>$(n - 2)(n - 3)$</td>
<td>$2C_0 \times 2P_0 = 1$</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>7</td>
</tr>
</tbody>
</table>

Table 1: Rank and Subdegrees of $G$ on $X^{[2]}$ for $n \geq 6$

$\triangle_0 = Orb_{G_{[1, 2]}}[1, 2] = \{[1, 2]\}$

$\triangle_1 = Orb_{G_{[1, 2]}}[2, 1] = \{[2, 1]\}$

b) Suborbits whose each element contains exactly one of 1 and 2:

$\triangle_2 = Orb_{G_{[1, 2]}}[1, 3] = \{[1, 3], [1, 4], [1, 5], \ldots, [1, n]\}$

$\triangle_3 = Orb_{G_{[1, 2]}}[3, 1] = \{[3, 1], [4, 1], [5, 1], \ldots, [n, 1]\}$

$\triangle_4 = Orb_{G_{[1, 2]}}[2, 3] = \{[2, 3], [2, 4], [2, 5], \ldots, [2, n]\}$

$\triangle_5 = Orb_{G_{[1, 2]}}[3, 2] = \{[3, 2], [4, 2], [5, 2], \ldots, [n, 2]\}$

c) Suborbit whose each element contains neither 1 nor 2:

$\triangle_6 = Orb_{G_{[1, 2]}}[3, 4] = \{[3, 4], [3, 5], \ldots, [3, n], [4, 3], \ldots, [4, n], \ldots, [n, n - 1]\}$.

The subdegrees and corresponding number of suborbits of $G$ on $X^{[2]}$ for $n \geq 6$ are summarized in Table 1 below.

**Theorem 4.2.** If $n \geq 8$, then the rank of $G$ on $X^{[3]}$ is 34.

**Proof.** Suppose $G$ acts on $X^{[3]}$. Then $G_{[1, 2, 3]}$ has orbits with exactly 3, 2, 1, or no element from $N = \{1, 2, 3\}$. An argument similar to the one in the proof of Theorem 4.1 shows that there are $3C_3 \times 3P_3 = 6$ suborbits of $G$ with exactly three elements from $N$ and $3C_2 \times 3P_2 = 18$ suborbits with exactly 2 elements from $N$. Also, there are $3C_1 \times 3P_1 = 9$ suborbits with exactly 1 element from $N$ and $3C_0 \times 3P_0 = 1$ suborbit with no element from $N$. Therefore, the rank of $G$ on $X^{[3]}$ is 34.

The subdegrees and corresponding number of suborbits of $G$ on $X^{[3]}$ for $n \geq 8$ are summarized in Table 2 below.

**Theorem 4.3.** If $n \geq 10$, then the rank of $G$ on $X^{[4]}$ is 209.

**Proof.** It is analogous to the proofs of Theorems 4.1 and 4.2 above.

The subdegrees and corresponding number of suborbits of $G$ on $X^{[4]}$ for $n \geq 10$ are summarized in Table 3 below.
Suppose $G$ acts on $X^{[r]}$ with $n \geq 2(r + 1)$. Suppose $\triangle_i$ is a suborbit of $G$ whose each element has exactly $r - i$ elements from $\{1, 2, \ldots, r\}$ for $0 \leq i \leq r$. Then, adding an extra element to $X$ increases $|\triangle_i|$ by

$$i(n - r)(n - r - 1)(n - r - 2) \ldots (n - r - i + 3)(n - r - i + 2)$$

units, but does not affect the rank of $G$.

**Proof.** From the second column of Table 4,

$$|\triangle_i| = (n - r)(n - r - 1)(n - r - 2) \ldots (n - r - i + 2)(n - r - i + 1).$$

If an extra element is added to $X$, the new value of $|\triangle_i|$ is obtained by replacing $n$ with $n + 1$, which equals $(n - r + 1)(n - r)(n - r - 1) \ldots (n - r - i + 3)(n - r - i + 2)$.

---

Table 2: Rank and Subdegrees of $G$ on $X^{[3]}$ for $n \geq 8$

<table>
<thead>
<tr>
<th>No. of elements from ${1, 2, 3}$</th>
<th>Suborbit Length</th>
<th>Corresponding No. of Suborbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1</td>
<td>$3C_3 \times 3P_3 = 6$</td>
</tr>
<tr>
<td>2</td>
<td>$(n - 3)$</td>
<td>$3C_2 \times 3P_2 = 18$</td>
</tr>
<tr>
<td>1</td>
<td>$(n - 3)(n - 4)$</td>
<td>$3C_1 \times 3P_1 = 9$</td>
</tr>
<tr>
<td>0</td>
<td>$(n - 3)(n - 4)(n - 5)$</td>
<td>$3C_0 \times 3P_0 = 1$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>34</strong></td>
</tr>
</tbody>
</table>

Table 3: Rank and Subdegrees of $G$ on $X^{[4]}$ for $n \geq 10$

<table>
<thead>
<tr>
<th>No. of elements from ${1, 2, 3, 4}$</th>
<th>Suborbit Length</th>
<th>Corresponding No. of Suborbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>1</td>
<td>$4C_4 \times 4P_4 = 24$</td>
</tr>
<tr>
<td>3</td>
<td>$(n - 4)$</td>
<td>$4C_3 \times 4P_3 = 96$</td>
</tr>
<tr>
<td>2</td>
<td>$(n - 4)(n - 5)$</td>
<td>$4C_2 \times 4P_2 = 72$</td>
</tr>
<tr>
<td>1</td>
<td>$(n - 4)(n - 5)(n - 6)$</td>
<td>$4C_1 \times 4P_1 = 16$</td>
</tr>
<tr>
<td>0</td>
<td>$(n - 4)(n - 5)(n - 6)(n - 7)$</td>
<td>$4C_0 \times 4P_0 = 1$</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>209</strong></td>
</tr>
</tbody>
</table>

5. Rank and Subdegrees of $G$ Acting on $X^{[r]}$

Suppose $G$ acts on $X^{[r]}$ and let $N = \{1, 2, \ldots, r\}$. If $n \geq 2(r + 1)$, then it is clear from Section 4 that $G$ has suborbits whose each element contains exactly $r, (r - 1), (r - 2), (r - 3), \ldots, (r - i), \ldots, 3, 2, 1$, or no element from $N$ where $0 \leq i \leq r$. The subdegrees and corresponding number of suborbits of this action are obtained by generalizing the results in, respectively, the second and third columns of Tables 1, 2 and 3. This is done as in Table 4 below.

**Lemma 5.1.** Let $G$ act on $X^{[r]}$ with $n \geq 2(r + 1)$. Suppose $\triangle_i$ is a suborbit of $G$ whose each element has exactly $r - i$ elements from $\{1, 2, \ldots, r\}$ for $0 \leq i \leq r$. Then, adding an extra element to $X$ increases $|\triangle_i|$ by

$$i(n - r)(n - r - 1)(n - r - 2) \ldots (n - r - i + 3)(n - r - i + 2)$$

units, but does not affect the rank of $G$. 

**Proof.** From the second column of Table 4,

$$|\triangle_i| = (n - r)(n - r - 1)(n - r - 2) \ldots (n - r - i + 2)(n - r - i + 1).$$

If an extra element is added to $X$, the new value of $|\triangle_i|$ is obtained by replacing $n$ with $n + 1$, which equals $(n - r + 1)(n - r)(n - r - 1) \ldots (n - r - i + 3)(n - r - i + 2)$. 
So, the number of units by which the suborbit length changes is
\[
(n - r + 1)(n - r)(n - r - 1)(n - r - 2) \cdots (n - r - i + 3)(n - r - i + 2) - (n - r)(n - r - 1)(n - r - 2) \cdots (n - r - i + 2)(n - r - i + 1) = i(n - r)(n - r - 1) \cdots (n - r - i + 3)(n - r - i + 2).
\]

Now, the number of suborbits \( \triangle_i \) \((i = 0, 1, \ldots , r)\), which is the corresponding entry in the third column of Table 4, is given purely in terms of the integers \( r \) and \( i \). It is clear that these integers are unaffected by increasing \(|X|\). This in turn implies that increasing \(|X|\) does not change the number of suborbits \( \triangle_i \) \((i = 0, 1, \ldots , r)\). As a result, the rank of \( G \), which is just the sum of entries in this column, is not affected by adding an extra element to \( X \). \( \Box \)

**Theorem 5.2.** The rank of \( G \) acting on \( X^{[r]} \) is \((r!)^2 \sum_{i=0}^{r} \frac{1}{(i)!^2 (r-i)!} \) for all \( n \geq 2(r + 1) \).

**Proof.** The proof is by mathematical induction. If \( n = 2(r + 1) \), the sum of the entries in the third column of Table 4 gives the desired result, i.e.,
\[
R(G) = \sum_{i=0}^{r} (rC_{r-i} \times rP_{r-i})
\]
Thus, the result holds for \( n = 2(r + 1) \). Now, suppose the result holds for \( n = 2(r + 1) + k, k \in \mathbb{Z}^+ \). To show that it holds for \( n = 2(r + 1) + (k + 1) \), add an extra element to the set \{1, 2, \ldots, 2r, 2r + 1, 2(r + 1), \ldots, 2(r + 1) + k\}. From Lemma 5.1, the extra element just changes the lengths of the orbits of \( G_{[1,2,\ldots,r]} \) with exactly \( r, (r - 1), (r - 2), (r - 3), (r - 4), \ldots, 3, 2, 1 \), and no element from \( N = \{1, 2, \ldots, r\} \), respectively, by

\[
0, 1, 2(n-r), 3(n-r)(n-r-1), 4(n-r)(n-r-1)(n-r-2),
\]

\[
\vdots
\]

\[
(r-3)(n-r)(n-r-1)(n-r-2) \ldots (n-2r+6)(n-2r+5),
\]

\[
(r-2)(n-r)(n-r-1)(n-r-2) \ldots (n-2r+5)(n-2r+4),
\]

\[
(r-1)(n-r)(n-r-1)(n-r-2) \ldots (n-2r+4)(n-2r+3),
\]

and

\[
r(n-r)(n-r-1)(n-r-2) \ldots (n-2r+3)(n-2r+2)
\]

units. However, by the same lemma, the number of suborbits of \( G \) remain unchanged. So, if \( n = 2(r + 1) + (k + 1) \), then the rank of \( G \) on \( X^r \) is the same as that when \( n = 2(r + 1) + k \). Thus, it holds for \( n = 2(r + 1) + (k + 1) \) whenever it holds for \( n = 2(r + 1) + k \). Therefore, by the principle of mathematical induction, it holds for all \( n \geq 2(r + 1) \). 

\[
\square
\]

6. Properties of Suborbits of \( G \) Acting on \( X^r \)

**Theorem 6.1.** Let \( G \) act on \( X^r \) and let \( \triangle_i \) and \( \triangle_j \) be orbits of \( G_{[1,2,\ldots,r]} \). Suppose \( [x_1,x_2,\ldots,x_r] \in \triangle_i \) where \( x_k \in \{1, 2, \ldots, n\} \forall k = 1, 2, \ldots, r \). Then \( \triangle_i \) is paired with \( \triangle_j \) if and only if there is an element \( [y_1,y_2,\ldots,y_r] \in \triangle_j \) with \( y_t \in \{1, 2, \ldots, n\} \forall t = 1, 2, \ldots, r \) such that the permutations

\[
\sigma_i = \begin{pmatrix} 1 & 2 & \cdots & r \\ y_1 & y_2 & \cdots & y_r \end{pmatrix} \text{ and } \sigma_j = \begin{pmatrix} 1 & 2 & \cdots & r \\ x_1 & x_2 & \cdots & x_r \end{pmatrix}
\]

are inverses of each other.

**Proof.** Suppose \( \triangle_i \) is paired with \( \triangle_j \) and \( [x_1,x_2,\ldots,x_r] \in \triangle_i \). Then there exists \( [y_1,y_2,\ldots,y_r] \in \triangle_j \) and \( g_i, g_j \in G \) such that

\[
g_i[x_1,x_2,\ldots,x_r] = [1,2,\ldots,r]; \ g_i[1,2,\ldots,r] = [y_1,y_2,\ldots,y_r]
\]

and

\[
g_j[y_1,y_2,\ldots,y_r] = [1,2,\ldots,r]; \ g_j[1,2,\ldots,r] = [x_1,x_2,\ldots,x_r].
\]
By definition,
\[ g_i(x_1) = 1, g_i(x_2) = 2, \ldots, g_i(x_r) = r; \quad g_j(1) = y_1, g_j(2) = y_2, \ldots, g_j(r) = y_r \]
and
\[ g_j(y_1) = 1, g_j(y_2) = 2, \ldots, g_j(y_r) = r; \quad g_j(1) = x_1, g_j(2) = x_2, \ldots, g_j(r) = x_r. \]
This implies that
\[ (g_i g_j)(1) = 1, (g_i g_j)(2) = 2, \ldots, (g_i g_j)(r) = r \]
and
\[ (g_j g_i)(1) = 1, (g_j g_i)(2) = 2, \ldots, (g_j g_i)(r) = r, \]
so that the permutations \( \sigma_i \) and \( \sigma_j \) are inverses of each other. Conversely, suppose \( \sigma_i \) and \( \sigma_j \) are inverses of each other. If \( g_i, g_j \in G \) such that
\[ g_i = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & n \\ y_1 & y_2 & \ldots & y_r & \ldots & y_n \end{pmatrix} \quad \text{and} \quad g_j = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & n \\ x_1 & x_2 & \ldots & x_r & \ldots & x_n \end{pmatrix}, \]
then \( g_i \) takes \([x_1, x_2, \ldots, x_r] \) to \([1, 2, \ldots, r] \) and \([1, 2, \ldots, r] \) to \([y_1, y_2, \ldots, y_r] \). Similarly, \( g_j \) takes \([y_1, y_2, \ldots, y_r] \) to \([1, 2, \ldots, r] \) and \([1, 2, \ldots, r] \) to \([x_1, x_2, \ldots, x_r] \). Hence \( \Delta_i \) and \( \Delta_j \) are paired.

**Theorem 6.2.** Let \( \Delta \) be an orbit of \( G_{[1, 2, \ldots, r]} \) on \( X^r \) and let
\[ [x_1, x_2, \ldots, x_r], [y_1, y_2, \ldots, y_r] \in \Delta. \]
Then \( \Delta \) is self-paired if and only if the permutations
\[ \sigma_i = \begin{pmatrix} 1 & 2 & \ldots & r \\ y_1 & y_2 & \ldots & y_r \end{pmatrix} \quad \text{and} \quad \sigma_j = \begin{pmatrix} 1 & 2 & \ldots & r \\ x_1 & x_2 & \ldots & x_r \end{pmatrix} \]
are inverses of each other. In this case, if \([x_1, x_2, \ldots, x_r] = [y_1, y_2, \ldots, y_r] \), then \( \sigma_i = \sigma_j = \sigma \) and
\[ \sigma = \begin{pmatrix} 1 & 2 & \ldots & r \\ x_1 & x_2 & \ldots & x_r \end{pmatrix} \]
is self-inverse.

**Proof.** Suppose \( \Delta \) is self-paired. Then, there exists \( g_i, g_j \in G \) such that
\[ g_i[x_1, x_2, \ldots, x_r] = [1, 2, \ldots, r]; \quad g_i[1, 2, \ldots, r] = [y_1, y_2, \ldots, y_r] \]
and
\[ g_j[y_1, y_2, \ldots, y_r] = [1, 2, \ldots, r]; \quad g_j[1, 2, \ldots, r] = [x_1, x_2, \ldots, x_r]. \]
By definition,
\[ g_i(x_1) = 1, g_i(x_2) = 2, \ldots, g_i(x_r) = r; \quad g_i(1) = y_1, g_i(2) = y_2, \ldots, g_i(r) = y_r \]
and
\[ g_j(y_1) = 1, g_j(y_2) = 2, \ldots, g_j(y_r) = r; \quad g_j(1) = x_1, g_j(2) = x_2, \ldots, g_j(r) = x_r. \]
This argument implies that
\[(g, g_j)(1) = 1, (g, g_j)(2) = 2, \ldots, (g, g_j)(r) = r\]
and
\[(g, g_i)(1) = 1, (g, g_i)(2) = 2, \ldots, (g, g_i)(r) = r,\]
so that the permutations \(\sigma_i\) and \(\sigma_j\) are inverses of each other. Conversely, suppose \(\sigma_i\) and \(\sigma_j\) are inverses of each other. If \(g_i, g_j \in G\) such that
\[g_i = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & n \\ y_1 & y_2 & \ldots & y_r & \ldots & y_n \end{pmatrix}\]
and \(g_j = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & n \\ x_1 & x_2 & \ldots & x_r & \ldots & x_n \end{pmatrix}\), then \(g_i\) takes \([x_1, x_2, \ldots, x_r] \in \Delta\) to \([1, 2, \ldots, r]\) and \([1, 2, \ldots, r]\) to \([y_1, y_2, \ldots, y_r] \in \Delta\). Similarly, \(g_j\) takes \([y_1, y_2, \ldots, y_r] \in \Delta\) to \([1, 2, \ldots, r]\) and \([1, 2, \ldots, r]\) to \([x_1, x_2, \ldots, x_r] \in \Delta\). Hence, \(\Delta\) is self-paired. Now, if \([x_1, x_2, \ldots, x_r] = [y_1, y_2, \ldots, y_r]\), then, clearly \(\sigma_i = \sigma_j = \sigma\) and it is, trivially, self-inverse. \(\square \)

**Lemma 6.3.** Let the cycle type of \(g \in G\) be \((\alpha_1, \alpha_2, \ldots, \alpha_n)\). If \(\alpha_1 \geq r\), then the number of elements in \(X^\alpha\) fixed by \(g\) is given by \(|\text{Fix}(g)| = r! \binom{\alpha_1}{r}\).

**Proof.** Let \([x_1, x_2, \ldots, x_r] \in X^\alpha\) and \(g \in G\). Then \(g\) fixes \([x_1, x_2, \ldots, x_r]\) if and only if each of the elements \(x_1, x_2, \ldots, x_r\) comes from a 1–cycle in \(g\). Now, out of the set of \(\alpha_1\) elements of \(X\) that are fixed by \(g\), the total number of ordered \(r\)–element subsets that can be formed is \(\alpha_1 P_r = r! \binom{\alpha_1}{r}\). Each of these elements of \(X^\alpha\) will be fixed by \(g\) so that \(|\text{Fix}(g)| = r! \binom{\alpha_1}{r}\). \(\square \)

**Theorem 6.4.** Let \(G\) act on \(X^\alpha\) and let \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) be the cycle type of an element \(g \in G\). Then the number of self-paired suborbits of \(G\) is given by
\[2r! \sum_{g \in G} \binom{\alpha_1 + 2\alpha_2}{r}.\]

**Proof.** The number of 1–cycles in \(g^2\) is \((\alpha_1 + 2\alpha_2)\), (see [3]). By Lemma 6.3, the number of elements in \(X^\alpha\) fixed by \(g^2\) is \(|\text{Fix}(g^2)| = r! \binom{\alpha_1 + 2\alpha_2}{r}\). By Theorem 2.3, the number of self-paired suborbits of \(G\) on \(X^\alpha\) is given by
\[\frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g^2)| = \frac{1}{2r!} \sum_{g \in G} r! \binom{\alpha_1 + 2\alpha_2}{r} = \frac{2r!}{n!} \sum_{g \in G} \binom{\alpha_1 + 2\alpha_2}{r}.\]
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References


