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# ON THE SUBORBITS OF THE ALTERNATING GROUP $A_n$ ACTING ON ORDERED $r-$ELEMENT SUBSETS

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**Abstract:** Transitivity and primitivity of the action of the alternating group $A_n$ on ordered $r-$element subsets of the set $X = \{1, 2, \ldots, n\}$ of $n$ letters are investigated in this paper. In addition, the rank and subdegrees of the action are calculated. Finally, some properties of the suborbits corresponding to this action are explored.

**AMS Subject Classification:** 05E18
**Key Words:** group, action, suborbits, rank, subdegrees

## 1. Introduction

Calculation of ranks and subdegrees of the symmetric group $S_n$ acting on $X^{[r]}$, the ordered $r-$element subsets of $X$, appears in [4] and [5]. However, little appears to have been done on the action of $A_n$, a subgroup of $S_n$, on $X^{[r]}$. Section 2 of this paper gives definitions of some terms as well as theorems to be used in subsequent sections. Section 3 investigates the transitivity and primitivity of $A_n$ acting on $X^{[r]}$. On the other hand, Section 4 determines the rank and subdegrees of $A_n$ on $X^{[2]}$, $X^{[3]}$ and $X^{[4]}$ while Section 5 generalizes these invariants. Finally, Section 6 explores some criteria for determining if a suborbit of $A_n$ is self-paired or paired with another. This section also derives a formula for finding the number of self-paired suborbits.

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## 2. Notation and Preliminary Results

Let $G$ be a group and $X$ a non-empty set. Then $G$ acts on the left of $X$ if there exists a function $G \times X \to X$ such that $(g_1g_2)x = g_1(g_2)x$ and $ex = x$ where $e$ is the identity in $G$, $x \in X$ and $g_1, g_2 \in G$. The action of $G$ on the right of $X$ can be defined in a similar way. In this case, $X$ is called a $G-$*set*.

Suppose a group $G$ acts on a set $X$. Define a relation $x \sim y$ on $X$ if and only if there exists $g \in G$ such that $y = gx$. This defines an equivalence relation on $X$. The equivalence class containing $x$ is $Orb_G x = \{gx | g \in G\}$, and is called the *orbit* (*transitivity class*) of $x$. Since any set is a disjoint union of equivalence classes under an equivalence relation, it follows that if $G$ acts on $X$, then $X$ is a union of disjoint orbits.

**Theorem 2.1.** *(Cauchy-Frobenius Lemma, [2], p. 223) If $G$ is a finite group and $X$ is a finite $G-$set, then the number of orbits in $X$ under $G$ is $\frac{1}{|G|}\sum_{g\in G}|Fix(g)|$, where $Fix(g) = \{x \in X | gx = x\}$.*

The action of a group $G$ on a set $X$ is said to be *transitive* if for each $x$ and $y$ in $X$, there exists $g \in G$ such that $y = gx$; in other words $Orb_G x = X$ if $x \in X$. A group which is not transitive is called *intransitive.*

The *stabilizer* in $G$ of $x$ is the subset $Stab_G x = \{g \in G | gx = x\}$ of $G$. It is also denoted by $G_x$ and it is a subgroup of $G$, called the *isotropy subgroup* of $G$. If $G_x$ is trivial, i.e., $G_x = \{e\}$, then $G$ is said to act *faithfully* on $X$.

**Theorem 2.2.** *(Orbit-Stabilizer Theorem, [2], p. 218) Let $X$ be a $G-$set and let $x \in X$. Then $|Orb_G x| = \frac{|G|}{|G_x|}$, the index of $G_x$ in $G$.*

Let $G$ act transitively on a finite set $X$. Then a subset $Y$ of $X$ is called a *block ( set of imprimitivity )* for the action if for each $g \in G$, either $gY = Y$ or $gY \cap Y = \phi$; i.e., if $gY$ and $Y$ do not overlap partially. In particular, $\phi, X$ and all 1-element subsets of $X$ are blocks, called the *trivial blocks.* The action is said to be *primitive* if the only blocks are the trivial blocks; otherwise the action is *imprimitive.*

Suppose $G$ acts transitively on $X$ and let $G_x$ be the stabilizer of $x \in X$. The orbits $\triangle_0 = \{x\}, \triangle_1, \triangle_2, ..., \triangle_{r-1}$ of $G_x$ on $X$ are known as *suborbits* of $G$. The *rank* of $G$ is then $r$. The sizes $n_i = |\triangle_i|$, $(i = 0, 1, 2, ..., r-1)$ are known as the *subdegrees* of $G$. Both the rank and the subdegrees of $G$ are independent of the choice of $x \in X$.

Let $G$ act transitively on a set $X$ and let $\triangle$ be an orbit of $G_x$ on $X$. Define $\triangle^* = \{gx | g \in G, x \in g\triangle\}$. Then $\triangle^*$ is also an orbit of $G_x$ called the $G_x-$orbit

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($G-$suborbit) *paired* with $\triangle$. Clearly, $\triangle^{**} = \triangle$ and $|\triangle| = |\triangle^*|$. If $\triangle = \triangle^*$, then $\triangle$ is said to be *self-paired*. The trivial suborbit of $G$ is self-paired, and there are other self-paired suborbits of $G$ if and only if $G$ has even order, [6].

**Theorem 2.3.** *(see [1], p. 425) Let $G$ be transitive on $X$ and let $g \in G$. Then the number of self-paired suborbits of $G$ is given by* $\frac{1}{|G|}\sum_{g\in G}|Fix(g^2)|$.

From this point on, $G$ shall be reserved to denote the alternating group $A_n$. Let $G$ act on $X = \{1, 2, \cdots, n\}$. Then, the action of $G$ on $X$ induces an action of $G$ on $X^{[r]}$ that is defined by

$$g[x_1, x_2, \ldots, x_r] = [g(x_1), g(x_2), \cdots, g(x_r)] \ \forall g \in G, \ [x_1, x_2, \ldots, x_r] \in X^{[r]}.$$

## 3. Transitivity and Primitivity of $G$ Acting on $X^{[r]}$

**Theorem 3.1.** *The ction $G$ on $X^{[r]}$ is intransitive if $n = r + 1$.*

*Proof.* If $n = r + 1$, then $|X^{[r]}| = {}^np_{n-1} = n!$ and $|G| = \frac{n!}{2} < |X^{[r]}|$. Hence, if $G$ acts on $X^{[r]}$ and $[x_1, x_2, \ldots, x_r] \in X^{[r]}$, then $|Orb_G[x_1, x_2, \ldots, x_r]| < |X^{[r]}|$, and the conclusion follows. □

**Lemma 3.2.** *Let $G$ act on $X^{[r]}$ with $n \geq r + 2$. If $[x_1, x_2, \ldots, x_r] \in X^{[r]}$, then $|Stab_G[x_1, x_2, \ldots, x_r]| = \frac{(n-r)!}{2}$. In this case $Stab_G[x_1, x_2, \ldots, x_r]$ is trivial if $n = r + 2$, so that $G$ acts faithfully on $X^{[r]}$, and is non-trivial otherwise.*

*Proof.* Suppose $G$ acts on $X^{[r]}$ and let $[x_1, x_2, \ldots, x_r] \in X^{[r]}$. Then $g \in G$ fixes $[x_1, x_2, \ldots, x_r]$ if and only if each element of $\{x_1, x_2, \ldots, x_r\}$ comes from a $1-$cycle in $g$. Hence, the order of $Stab_G[x_1, x_2, \ldots, x_r]$ is equal to the order of the group of all even permutations of the set $\{x_{r+1}, x_{r+2}, \ldots, x_n\}$. But, this group is isomorphic to $A_{n-r}$. Therefore, $|Stab_G[x_1, x_2, \ldots, x_r]| = \frac{(n-r)!}{2}$. Now, if $n = r + 2$, then on rewriting $n-r = 2$, so that $|Stab_G[x_1, x_2, \ldots, x_r]| = \frac{2!}{2} = 1$. Hence, $G$ acts faithfully on $X^{[r]}$. A similar argument shows that if $n > r + 2$, then $|Stab_G[x_1, x_2, \ldots, x_r]| > 1$. □

**Theorem 3.3.** *The action of $G$ on $X^{[r]}$ is transitive if $n \geq r + 2$.*

*Proof.* Let $[x_1, x_2, \ldots, x_r] \in X^{[r]}$. Now, from Theorem 2.2 and Lemma 3.2,

$$|Orb_G[x_1, x_2, \ldots, x_r]| = \frac{|G|}{|Stab_G[x_1, x_2, \ldots, x_r]|}$$

$$\begin{aligned} &= \frac{\frac{n!}{2}}{\frac{(n-r)!}{2}} \\ &= \frac{n!}{(n-r)!} \\ &= |X^{[r]}|. \end{aligned}$$

Therefore, the action is transitive. □

**Theorem 3.4.** The action of $G$ on $X^{[r]}$ is imprimitive if $n \geq r+2$.

*Proof.* By Theorem 3.3, this action is transitive. Now, suppose

$$Y = \{[x_1, x_2, \ldots, x_r], [y_1, y_2, \ldots, y_r]\} \subset X^{[r]}.$$

Let $x_1 = y_1 = x^{'}, x_2 = y_2 = x^{''}, \ldots,\ x_{r-2} = y_{r-2} = x^{r-2}$ while $x_{r-1}, x_r, y_{r-1}$, and $y_r$ are all distinct so that $|\{x_1, x_2, \ldots, x_r\} \cap \{y_1, y_2, \ldots, y_r\}| = r-2$. Take $g \in G$. If each of $x_i$ and $y_j$ $(i, j = 1, 2, \ldots, r)$ comes from a $1-$cycle of $g$, then $g$ fixes each element of $Y$ so that $gY = Y$. On the other hand, if each of $x^{'}, x^{''}, \ldots, x^{r-2}$ is in a $1-$cycle of $g$, while $(x_{r-1}\ y_{r-1})$ and $(x_r\ y_r)$ are transpositions in $g$, then $gY = Y$ since $g[x_1, x_2, \ldots, x_r] = [y_1, y_2, \ldots, y_r]$ and $g[y_1, y_2, \ldots, y_r] = [x_1, x_2, \ldots, x_r]$. Finally, any other $g$ takes one element of $Y$ to an element of $X^{[r]}$ not in $Y$ so that $gY \cap Y = \emptyset$. Hence, $Y$ is a non-trivial block for the action and the action is therefore imprimitive. □

## 4. Ranks and Subdegrees of $G$ on $X^{[2]}$, $X^{[3]}$ and $X^{[4]}$

**Theorem 4.1.** *If $n \geq 6$, then the rank of $G$ on $X^{[2]}$ is 7.*

*Proof.* Let $G$ act on $X^{[2]}$. Then $G_{[1,2]}$ has orbits with exactly 2, 1, or no element from $N = \{1, 2\}$. There is only ${}^2C_2 = 1$ way of selecting two elements from $N$ and the two can be arranged in the two positions in ${}^2P_2 = 2$ ways. So, there are ${}^2C_2 \times {}^2P_2 = 2$ suborbits whose elements contain both elements of $N$. Also, there are ${}^2C_1 = 2$ ways of selecting an element from $N$, which can occupy any of the two positions in ${}^2P_1 = 2$ ways. Hence, there are ${}^2C_1 \times {}^2P_1 = 4$ suborbits of $G$ whose each element has exactly one element from $N$. Similarly, there is only ${}^2C_0 \times {}^2P_0 = 1$ suborbit whose each element contains no element from $N$. So, $G$ has 7 suborbits in total. □

The seven suborbits of $G$ on $X^{[2]}$ are

a) Suborbits whose elements contain both 1 and 2:

| No. of elements from $\{1,2\}$ | Suborbit Length | Corresponding No. of Suborbits |
|---|---|---|
| 2 | 1 | ${}^2C_2 \times {}^2P_2 = 2$ |
| 1 | $(n-2)$ | ${}^2C_1 \times {}^2P_1 = 4$ |
| 0 | $(n-2)(n-3)$ | ${}^2C_0 \times {}^2P_0 = 1$ |
| Total | | 7 |

Table 1: Rank and Subdegrees of $G$ on $X^{[2]}$ for $n \geq 6$

$\triangle_0 = Orb_{G_{[1,2]}}[1,2] = \{[1,2]\}$
$\triangle_1 = Orb_{G_{[1,2]}}[2,1] = \{[2,1]\}$

b) Suborbits whose each element contains exactly one of 1 and 2:
$\triangle_2 = Orb_{G_{[1,2]}}[1,3] = \{[1,3],[1,4],[1,5],\ldots,[1,n]\}$
$\triangle_3 = Orb_{G_{[1,2]}}[3,1] = \{[3,1],[4,1],[5,1],\ldots,[n,1]\}$
$\triangle_4 = Orb_{G_{[1,2]}}[2,3] = \{[2,3],[2,4],[2,5],\ldots,[2,n]\}$
$\triangle_5 = Orb_{G_{[1,2]}}[3,2] = \{[3,2],[4,2],[5,2],\ldots,[n,2]\}$

c) Suborbit whose each element contains neither 1 nor 2:
$\triangle_6 = Orb_{G_{[1,2]}}[3,4] = \{[3,4],[3,5],\ldots,[3,n],[4,3],\ldots,[4,n],\ldots,[n,n-1]\}$.

The subdegrees and corresponding number of suborbits of $G$ on $X^{[2]}$ for $n \geq 6$ are summarized in Table 1 below.

**Theorem 4.2.** *If $n \geq 8$, then the rank of $G$ on $X^{[3]}$ is 34.*

*Proof.* Suppose $G$ acts on $X^{[3]}$. Then $G_{[1,2,3]}$ has orbits with exactly 3, 2, 1, or no element from $N = \{1,2,3\}$. An argument similar to the one in the proof of Theorem 4.1 shows that there are ${}^3C_3 \times {}^3P_3 = 6$ suborbits of $G$ with exactly three elements from $N$ and ${}^3C_2 \times {}^3P_2 = 18$ suborbits with exactly 2 elements from $N$. Also, there are ${}^3C_1 \times {}^3P_1 = 9$ suborbits with exactly 1 element from $N$ and ${}^3C_0 \times {}^3P_0 = 1$ suborbit with no element from $N$. Therefore, the rank of $G$ on $X^{[3]}$ is 34. □

The subdegrees and corresponding number of suborbits of $G$ on $X^{[3]}$ for $n \geq 8$ are summarized in Table 2 below.

**Theorem 4.3.** *If $n \geq 10$, then the rank of $G$ on $X^{[4]}$ is 209.*

*Proof.* It is analogous to the proofs of Theorems 4.1 and 4.2 above □

The subdegrees and corresponding number of suborbits of $G$ on $X^{[4]}$ for $n \geq 10$ are summarized in Table 3 below.

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| No. of elements from $\{1,2,3\}$ | Suborbit Length | Corresponding No. of Suborbits |
|---|---|---|
| 3 | 1 | ${}^3C_3 \times {}^3P_3 = 6$ |
| 2 | $(n-3)$ | ${}^3C_2 \times {}^3P_2 = 18$ |
| 1 | $(n-3)(n-4)$ | ${}^3C_1 \times {}^3P_1 = 9$ |
| 0 | $(n-3)(n-4)(n-5)$ | ${}^3C_0 \times {}^3P_0 = 1$ |
| Total | | 34 |

Table 2: Rank and Subdegrees of $G$ on $X^{[3]}$ for $n \geq 8$

| No. of elements from $\{1,2,3,4\}$ | Suborbit Length | Corresponding No. of Suborbits |
|---|---|---|
| 4 | 1 | ${}^4C_4 \times {}^4P_4 = 24$ |
| 3 | $(n-4)$ | ${}^4C_3 \times {}^4P_3 = 96$ |
| 2 | $(n-4)(n-5)$ | ${}^4C_2 \times {}^4P_2 = 72$ |
| 1 | $(n-4)(n-5)(n-6)$ | ${}^4C_1 \times {}^4P_1 = 16$ |
| 0 | $(n-4)(n-5)(n-6)(n-7)$ | ${}^4C_0 \times {}^4P_0 = 1$ |
| Total | | 209 |

Table 3: Rank and Subdegrees of $G$ on $X^{[4]}$ for $n \geq 10$

## 5. Rank and Subdegrees of $G$ Acting on $X^{[r]}$

Suppose $G$ acts on $X^{[r]}$ and let $N = \{1, 2, \ldots, r\}$. If $n \geq 2(r+1)$, then it is clear from Section 4 that $G$ has suborbits whose each element contains exactly $r, (r-1), (r-2), (r-3), \ldots, (r-i), \ldots, 3, 2, 1$, or no element from $N$ where $0 \leq i \leq r$. The subdegrees and corresponding number of suborbits of this action are obtained by generalizing the results in, respectively, the second and third columns of Tables 1, 2 and 3. This is done as in Table 4 below.

**Lemma 5.1.** *Let $G$ act on $X^{[r]}$ with $n \geq 2(r+1)$. Suppose $\triangle_i$ is a suborbit of $G$ whose each element has exactly $r-i$ elements from $\{1, 2, \ldots, r\}$ for $0 \leq i \leq r$. Then, adding an extra element to $X$ increases $|\triangle_i|$ by*

$$i(n-r)(n-r-1)(n-r-2)\ldots(n-r-i+3)(n-r-i+2)$$

*units, but does not affect the rank of $G$.*

*Proof.* From the second column of Table 4,

$$|\triangle_i| = (n-r)(n-r-1)(n-r-2)\ldots(n-r-i+2)(n-r-i+1).$$

If an extra element is added to $X$, the new value of $|\triangle_i|$ is obtained by replacing $n$ with $n+1$, which equals $(n-r+1)(n-r)(n-r-1)\cdots(n-r-i+3)(n-r-i+2)$.

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| No. of Elements from $N$ | Suborbit Length | No. of Suborbits |
|---|---|---|
| $r$ | $1$ | ${}^rC_r \times {}^rP_r$ |
| $(r-1)$ | $(n-r)$ | ${}^rC_{r-1} \times {}^rP_{r-1}$ |
| $(r-2)$ | $(n-r)(n-r-1)$ | ${}^rC_{r-2} \times {}^rP_{r-2}$ |
| $(r-3)$ | $(n-r)(n-r-1)(n-r-2)$ | ${}^rC_{r-3} \times {}^rP_{r-3}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $(r-i)$ | $(n-r)(n-r-1)\cdots(n-r-i+1)$ | ${}^rC_{r-i} \times {}^rP_{r-i}$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $3$ | $(n-r)(n-r-1)\cdots(n-2r+4)$ | ${}^rC_3 \times {}^rP_3$ |
| $2$ | $(n-r)(n-r-1)\cdots(n-2r+3)$ | ${}^rC_2 \times {}^rP_2$ |
| $1$ | $(n-r)(n-r-1)\cdots(n-2r+2)$ | ${}^rC_1 \times {}^rP_1$ |
| $0$ | $(n-r)(n-r-1)\cdots(n-2r+1)$ | ${}^rC_0 \times {}^rP_0$ |
| Total | | $\sum_{i=0}^{r} ({}^rC_{r-i} \times {}^rP_{r-i})$ |

Table 4: Subdegrees and Corresponding Number of Suborbits of $G$ on $X^{[r]}$.

So, the number of units by which the suborbit length changes is

$$\begin{aligned}(n-r+1)(n-r)(n-r-1)(n-r-2)\cdots(n-r-i+3)(n-r-i+2)\\ -(n-r)(n-r-1)(n-r-2)\cdots(n-r-i+2)(n-r-i+1)\\ = i(n-r)(n-r-1)\cdots(n-r-i+3)(n-r-i+2).\end{aligned}$$

Now, the number of suborbits $\triangle_i$ $(i = 0, 1, \ldots, r)$, which is the corresponding entry in the third column of Table 4, is given purely in terms of the integers $r$ and $i$. It is clear that these integers are unaffected by increasing $|X|$. This in turn implies that increasing $|X|$ does not change the number of suborbits $\triangle_i$ $(i = 0, 1, \ldots, r)$. As a result, the rank of $G$, which is just the sum of entries in this column, is not affected by adding an extra element to $X$. $\square$

**Theorem 5.2.** *The rank of $G$ acting on $X^{[r]}$ is $(r!)^2 \overset{[}{i}= 0]r\sum \frac{1}{(i!)^2(r-i)!}$ for all $n \geq 2(r+1)$.*

*Proof.* The proof is by mathematical induction. If $n = 2(r+1)$, the sum of the entries in the third column of Table 4 gives the desired result, i.e.,

$$R(G) = \sum_{i=0}^{r} ({}^rC_{r-i} \times {}^rP_{r-i})$$

$$= \quad (r!)^2 \overset{[}{i} = 0]r \sum \frac{1}{(i!)^2\,(r-i)!}.$$

Thus, the result holds for $n = 2(r+1)$. Now, suppose the result holds for $n = 2(r+1)+k,\ k \in \mathbb{Z}^+$. To show that it holds for $n = 2(r+1)+(k+1)$, add an extra element to the set $\{1, 2, \ldots, 2r, 2r+1, 2(r+1), \ldots, 2(r+1)+k\}$. From Lemma 5.1, the extra element just changes the lengths of the orbits of $G_{[1,2,\ldots,r]}$ with exactly $r, (r-1), (r-2), (r-3), (r-4), \ldots, 3, 2, 1$, and no element from $N = \{1, 2, \ldots, r\}$, respectively, by

$$\begin{gathered}
0, 1, 2(n-r), 3(n-r)(n-r-1), 4(n-r)(n-r-1)(n-r-2),\\
\vdots\\
(r-3)(n-r)(n-r-1)(n-r-2)\ldots(n-2r+6)(n-2r+5),\\
(r-2)(n-r)(n-r-1)(n-r-2)\ldots(n-2r+5)(n-2r+4),\\
(r-1)(n-r)(n-r-1)(n-r-2)\ldots(n-2r+4)(n-2r+3),
\end{gathered}$$

and

$$r(n-r)(n-r-1)(n-r-2)\ldots(n-2r+3)(n-2r+2)$$

units. However, by the same lemma, the number of suborbits of $G$ remain unchanged. So, if $n = 2(r+1)+(k+1)$, then the rank of $G$ on $X^{[r]}$ is the same as that when $n = 2(r+1)+k$. Thus, it holds for $n = 2(r+1)+(k+1)$ whenever it holds for $n = 2(r+1)+k$. Therefore, by the principle of mathematical induction, it holds for all $n \geq 2(r+1)$. □

## 6. Properties of Suborbits of $G$ Acting on $X^{[r]}$

**Theorem 6.1.** *Let $G$ act on $X^{[r]}$ and let $\triangle_i$ and $\triangle_j$ be orbits of $G_{[1,2,\ldots,r]}$. Suppose $[x_1, x_2, \ldots, x_r] \in \triangle_i$ where $x_k \in \{1, 2, \ldots, n\}\ \forall k = 1, 2, \ldots, r$. Then $\triangle_i$ is paired with $\triangle_j$ if and only if there is an element $[y_1, y_2, \ldots, y_r] \in \triangle_j$ with $y_t \in \{1, 2, \ldots, n\}\ \forall t = 1, 2, \ldots, r$ such that the permutations $\sigma_i = \begin{pmatrix} 1 & 2 & \ldots & r \\ y_1 & y_2 & \ldots & y_r \end{pmatrix}$ and $\sigma_j = \begin{pmatrix} 1 & 2 & \ldots & r \\ x_1 & x_2 & \ldots & x_r \end{pmatrix}$ are inverses of each other.*

*Proof.* Suppose $\triangle_i$ is paired with $\triangle_j$ and $[x_1, x_2, \ldots, x_r] \in \triangle_i$. Then there exists $[y_1, y_2, \ldots, y_r] \in \triangle_j$ and $g_i, g_j \in G$ such that

$$g_i[x_1, x_2, \ldots, x_r] = [1, 2, \ldots, r];\ g_i[1, 2, \ldots, r] = [y_1, y_2, \ldots, y_r]$$

and

$$g_j[y_1, y_2, \ldots, y_r] = [1, 2, \ldots, r];\ g_j[1, 2, \ldots, r] = [x_1, x_2, \ldots, x_r].$$

By definition,

$$g_i(x_1) = 1, g_i(x_2) = 2, \ldots, g_i(x_r) = r;\ g_i(1) = y_1, g_i(2) = y_2, \ldots, g_i(r) = y_r$$

and

$$g_j(y_1) = 1, g_j(y_2) = 2, \ldots, g_j(y_r) = r;\ g_j(1) = x_1, g_j(2) = x_2, \ldots, g_j(r) = x_r.$$

This implies that

$$(g_i g_j)(1) = 1, (g_i g_j)(2) = 2, \ldots, (g_i g_j)(r) = r$$

and

$$(g_j g_i)(1) = 1, (g_j g_i)(2) = 2, \ldots, (g_j g_i)(r) = r,$$

so that the permutations $\sigma_i$ and $\sigma_j$ are inverses of each other. Conversely, suppose $\sigma_i$ and $\sigma_j$ are inverses of each other. If $g_i, g_j \in G$ such that $g_i = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & n \\ y_1 & y_2 & \ldots & y_r & \ldots & y_n \end{pmatrix}$ and $g_j = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & n \\ x_1 & x_2 & \ldots & x_r & \ldots & x_n \end{pmatrix}$, then $g_i$ takes $[x_1, x_2, \ldots, x_r]$ to $[1, 2, \ldots, r]$ and $[1, 2, \ldots, r]$ to $[y_1, y_2, \ldots, y_r]$. Similarly, $g_j$ takes $[y_1, y_2, \ldots, y_r]$ to $[1, 2, \ldots, r]$ and $[1, 2, \ldots, r]$ to $[x_1, x_2, \ldots, x_r]$. Hence $\triangle_i$ and $\triangle_j$ are paired. □

**Theorem 6.2.** *Let $\triangle$ be an orbit of $G_{[1,2,\ldots,r]}$ on $X^{[r]}$ and let $[x_1, x_2, \ldots, x_r], [y_1, y_2, \ldots, y_r] \in \triangle$. Then $\triangle$ is self-paired if and only if the permutations $\sigma_i = \begin{pmatrix} 1 & 2 & \ldots & r \\ y_1 & y_2 & \ldots & y_r \end{pmatrix}$ and $\sigma_j = \begin{pmatrix} 1 & 2 & \ldots & r \\ x_1 & x_2 & \ldots & x_r \end{pmatrix}$ are inverses of each other. In this case, if $[x_1, x_2, \cdots, x_r] = [y_1, y_2, \ldots, y_r]$, then $\sigma_i = \sigma_j = \sigma$ and $\sigma = \begin{pmatrix} 1 & 2 & \ldots & r \\ x_1 & x_2 & \ldots & x_r \end{pmatrix}$ is self-inverse.*

*Proof.* Suppose $\triangle$ is self-paired. Then, there exists $g_i, g_j \in G$ such that

$$g_i[x_1, x_2, \ldots, x_r] = [1, 2, \ldots, r];\ g_i[1, 2, \ldots, r] = [y_1, y_2, \ldots, y_r]$$

and

$$g_j[y_1, y_2, \ldots, y_r] = [1, 2, \ldots, r];\ g_j[1, 2, \ldots, r] = [x_1, x_2, \ldots, x_r].$$

By definition,

$$g_i(x_1) = 1, g_i(x_2) = 2, \ldots, g_i(x_r) = r;\ g_i(1) = y_1, g_i(2) = y_2, \ldots, g_i(r) = y_r$$

and

$$g_j(y_1) = 1, g_j(y_2) = 2, \ldots, g_j(y_r) = r;\ g_j(1) = x_1, g_j(2) = x_2, \ldots, g_j(r) = x_r.$$

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This argument implies that

$$(g_ig_j)(1) = 1, (g_ig_j)(2) = 2, \ldots, (g_ig_j)(r) = r$$

and

$$(g_jg_i)(1) = 1, (g_jg_i)(2) = 2, \ldots, (g_jg_i)(r) = r,$$

so that the permutations $\sigma_i$ and $\sigma_j$ are inverses of each other. Conversely, suppose $\sigma_i$ and $\sigma_j$ are inverses of each other. If $g_i, g_j \in G$ such that $g_i = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & n \\ y_1 & y_2 & \ldots & y_r & \ldots & y_n \end{pmatrix}$ and $g_j = \begin{pmatrix} 1 & 2 & \ldots & r & \ldots & n \\ x_1 & x_2 & \ldots & x_r & \ldots & x_n \end{pmatrix}$, then $g_i$ takes $[x_1, x_2, \ldots, x_r] \in \triangle$ to $[1, 2, \ldots, r]$ and $[1, 2, \ldots, r]$ to $[y_1, y_2, \ldots, y_r] \in \triangle$. Similarly, $g_j$ takes $[y_1, y_2, \ldots, y_r] \in \triangle$ to $[1, 2, \ldots, r]$ and $[1, 2, \ldots, r]$ to $[x_1, x_2, \ldots, x_r] \in \triangle$. Hence, $\triangle$ is self-paired. Now, if $[x_1, x_2, \ldots, x_r] = [y_1, y_2, \ldots, y_r]$, then, clearly $\sigma_i = \sigma_j = \sigma$ and it is, trivially, self-inverse. □

**Lemma 6.3.** *Let the cycle type of $g \in G$ be $(\alpha_1, \alpha_2, \ldots, \alpha_n)$. If $\alpha_1 \geq r$, then the number of elements in $X^{[r]}$ fixed by $g$ is given by $|Fix(g)| = r!\binom{\alpha_1}{r}$.*

*Proof.* Let $[x_1, x_2, \ldots, x_r] \in X^{[r]}$ and $g \in G$. Then $g$ fixes $[x_1, x_2, \ldots, x_r]$ if and ony if each of the elements $x_1, x_2, \ldots, x_r$ comes from a $1-$cycle in $g$. Now, out of the set of $\alpha_1$ elements of $X$ that are fixed by $g$, the total number of ordered $r-$element subsets that can be formed is ${}^{\alpha_1}P_r = r!\binom{\alpha_1}{r}$. Each of these elements of $X^{[r]}$ will be fixed by $g$ so that $|Fix(g)| = r!\binom{\alpha_1}{r}$. □

**Theorem 6.4.** *Let $G$ act on $X^{[r]}$ and let $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the cycle type of an element $g \in G$. Then the number of self-paired suborbits of $G$ is given by $\frac{2r!}{n!}\sum_{g\in G}\binom{\alpha_1 + 2\alpha_2}{r}$.*

*Proof.* The number of $1-$cycles in $g^2$ is $(\alpha_1 + 2\alpha_2)$, (see [3]). By Lemma 6.3, the number of elements in $X^{[r]}$ fixed by $g^2$ is $|Fix(g^2)| = r!\binom{\alpha_1 + 2\alpha_2}{r}$. By Theorem 2.3, the number of self-paired suborbits of $G$ on $X^{[r]}$ is given by

$$\begin{aligned} \frac{1}{|G|}\sum_{g\in G}|Fix(g^2)| &= \frac{1}{\frac{n!}{2}}\sum_{g\in G} r!\binom{\alpha_1 + 2\alpha_2}{r} \\ &= \frac{2r!}{n!}\sum_{g\in G}\binom{\alpha_1 + 2\alpha_2}{r}. \end{aligned}$$

□

**References**


[1] P.J. Cameron, Suborbits in transitive permutation groups, In: *In Proceedings of NATO Advanced Study Institute of Combinatorics*, Bruekelen, Netherlands (1974).

[2] J.B. Fraleigh, *A First Course in Abstract Algebra*, Addison-Wesley, Reading (1994).

[3] I.N. Kamuti, J.O. Obon'go, The derivation of cycle index of $S_n^{[3]}$, *Quaestiones Mathematicae*, **25** (2002), 99-108.

[4] J.K. Rimberia, I.N. Kamuti, B.M. Kivunge, F. Kinyua, Subdegrees of the symmetric group $S_n$ acting on ordered $r-$element subsets, *International Journal of Science, Commerce and Humanities*, **1**, No. 6 (2013), 112-118.

[5] J.K. Rimberia, I.N. Kamuti, B.M. Kivunge, F. Kinyua, Suborbits and suborbital graphs of the symmetric group $S_n$ acting on ordered $r-$element subsets, *Journal of Mathematical Theory and Modelling*, **3**, No. 11 (2012), 115-120.

[6] C.C. Sims, Graphs and finite permutation groups, *Mathematische Zeitschrift*, **95** (1967), 76-86.

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