International Electronic Journal of Pure and Applied Mathematics

**Volume 9 No. 3 2015, 121-136**
ISSN: 1314-0744
**url:** http://www.e.ijpam.eu
**doi:** http://dx.doi.org/10.12732/iejpam.v9i3.2

# HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC-ADDITIVE TYPE FUNCTIONAL EQUATION IN FUZZY SPACES

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**Abstract:** In this paper, we investigate a fuzzy version of stability for the functional equation

$$f((a-1)x+y)+(a-1)f(x-y)+f(ax)-(a^2+a-1)f(x) \\ -(a-1)^2f(-x)-f(y)-(a-1)f(-y)=0$$

in the sense of A. K. Mirmostafaee and M. S. Moslehian.

**AMS Subject Classification:** 46S40, 39B52
**Key Words:** fuzzy normed space, fuzzy almost quadratic-additive mapping, quadratic-additive type functional equation

## 1. Introduction

In 1940, S. M. Ulam[23] posed a classical question in the theory of functional equations: “when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?”, which is called *a stability problem of the functional equation.* In the next year, D. H. Hyers [6] gave a partial solution of Ulam’s problem for the case of approximate additive mappings. Subsequently, his result was generalized by T. Aoki [1] for additive mappings, and by Th. M. Rassias [22] for linear mappings, to considering the stability problem with unbounded Cauchy differences. During the last decades, the stability

Received: June 18, 2014

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problems of functional equations have been extensively investigated by a number of mathematicians, see [4], [5], [12], [14], [16], [17].

In 1984, A. K. Katsaras [13] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, several types of fuzzy norm have been introduced in different points of view. In particular, T. Bag and S.K. Samanta [2], following Cheng and Mordeson [3], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15]. In 2008, A. K. Mirmostafaee and M. S. Moslehian obtained stability results for *the Cauchy functional equation* [21] and *the quadratic functional equation[20] in a fuzzy sense.* Now, for a fixed integer $a > 1$, we consider the functional equation:

$$\begin{aligned} f((a-1)x+y)+&(a-1)f(x-y)+f(ax)-(a^2+a-1)f(x) \\ &-(a-1)^2 f(-x)-f(y)-(a-1)f(-y)=0 \end{aligned} \tag{1.1}$$

which is called *the quadratic-additive type functional equation.* A solution of (1.1) is called *a quadratic-additive mapping.*

In this paper, we get a general stability result of the functional equation (1.1) in the fuzzy normed linear space in the manner of A. K. Mirmostafaee and M. S. Moslehian [20](see [7]-[11], [18],[19]).

## 2. Stability of the Functional Equation (1.1)

We use the definition of a fuzzy normed space given in [2] to exhibit a reasonable fuzzy version of stability for the Cauchy additive and quadratic type functional equation in the fuzzy normed linear space.

**Definition 2.1.** ([2]) Let $X$ be a real linear space. A function $N : X\times\mathbb{R} \to [0,1]$ is said to be a *fuzzy norm* on $X$ if the following conditions are true for all $x, y \in X$ and all $s, t \in \mathbb{R}$:

($N_1$) $N(x,t) = 0$ for $t \le 0$;

($N_2$) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;

($N_3$) $N(cx,t) = N(x, t/|c|)$ if $c \neq 0$;

($N_4$) $N(x+y, s+t) \ge \min\{N(x,s), N(y,t)\}$;

($N_5$) $N(x,\cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.

The pair $(X,N)$ is called *a fuzzy normed linear space.* Let $(X,N)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called *the limit of the sequence* $\{x_n\}$ and we denote it by $N - \lim_{n\to\infty} x_n = x$.

A sequence $\{x_n\}$ in $X$ is called *Cauchy* if for each $\varepsilon > 0$ and each $t > 0$ there exists $n_0$ such that for all $n \geq n_0$ and all $p > 0$ we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$. It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed space is called *a fuzzy Banach space.*

Let $(X, N)$ be a fuzzy normed space and $(Y, N')$ a fuzzy Banach space. For a given mapping $f : X \to Y$, we use the abbreviations

$$\begin{aligned} Af(x,y) :=& f(x+y) - f(x) - f(y), \\ Qf(x,y) :=& f(x+y) + f(x-y) - 2f(x) - 2f(y), \\ D_af(x,y) :=& f((a-1)x+y) + (a-1)f(x-y) + f(ax) - (a^2+a-1)f(x) \\ & - (a-1)^2 f(-x) - f(y) - (a-1)f(-y) \end{aligned}$$

for all $x, y \in X$, where $a$ is a fixed natural number greater than 1.

**Lemma 2.2.** *If $f : X \to Y$ is a mapping such that $D_af(x,y) = 0$ for all $x, y \in X \backslash \{0\}$ with $f(0) = 0$, then*

$$D_af(x,y) = 0$$

*for all $x, y \in X$.*

*Proof.* By the hypothesis, we get

$$\begin{aligned} D_af(x,0) =& \frac{a-1}{a(a-2)} D_af(x,(1-a)x) - \frac{1}{a(a-2)} D_af(-x,(a-1)x) \\ & - \frac{2a-3}{2(a-2)} D_af(x,x) + \frac{1}{2(a-2)} D_af(-x,-x) = 0 \end{aligned}$$

for all $x \in X \backslash \{0\}$ if $a > 2$ and

$$D_2f(x,0) = \frac{1}{2} D_2f(x,x) = 0$$

for all $x \in X \backslash \{0\}$ if $a = 2$. It is easy to show that $D_af(0,y) = 0$ for all $y \in X$ as we desired. □

Observe that the equations $Af = 0$, $Qf = 0$, and $D_af = 0$ represent the Cauchy functional equation, the quadratic functional equation, and the quadratic-additive type functional equation, respectively. A solution of $Af = 0$ is call an additive mapping and a solution of $Qf = 0$ is call a quadratic mapping. The following lemma shows that if $f$ is an even or an odd mapping satisfying the equality $D_af(x,y) = 0$ for all $x, y \in X$, then $f$ is a quadratic mapping or an additive mapping, respectively.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)

**Lemma 2.3.** *A mapping $f : X \to Y$ satisfyies the inequalty $D_af(x,y) = 0$ for all $x, y \in X$ if and only if there exist a quadratic mapping $g : X \to Y$ and an additive mapping $h : X \to Y$ such that*

$$f(x) = g(x) + h(x)$$

*for all $x \in X$.*

*Proof.* (Necessity) We decompose $f$ into the even part and the odd part by putting

$$g(x) = \frac{f(x) + f(-x)}{2}, \qquad h(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. Notice that $f(0) = \frac{-D_af(0,0)}{(2a+1)(a-1)} = 0$. From the equalities

$$\begin{aligned} Qg(x,y) =& \frac{D_ag(y,ax) - D_ag(x+y,x)}{a(a-1)} - \frac{D_ag(-x,y-x)}{a} + \frac{(2a-1)D_ag(x,x)}{2a(a-1)} \\ &+ \frac{D_ag(x+y,x+y)}{2a(a-1)} - \frac{D_ag(y,y)}{2a(a-1)} \\ =& 0, \end{aligned}$$

$$\begin{aligned} Ah(x,y) =& \frac{D_ah(y,(a-1)x) - (a-2)D_ah(x,0) - D_ah(x+y,0)0}{a(a-1)} - \frac{D_ah(-x,y)}{a} \\ &+ \frac{(a-1)D_ah(x,x) + D_ah(x+y,x+y) - D_ah(y,y)}{2a(a-1)} \\ =& 0 \end{aligned}$$

for all $x, y \in X$, we conclude that $g$ is a quadratic mapping and $h$ is an additive mapping.

(Sufficiency) If $g$ is a quadratic mapping, then

$$D_2g(x,y) = Qg(x,y) + Qg(x,x) = 0$$

for all $x, y \in X$. Notice that $g(nx) = n^2g(x)$ and $g(x) = g(-x)$ for all $x \in X$ and all $n \in N$. Assume that $D_kg(x,y) = 0$ for all $x, y \in X$ and an integer $k \geq 2$. Then

$$D_{k+1}g(x,y) = D_kg(x,x+y) + kQg(x,y) + g((k+1)x) - g(kx) - (4k+1)g(x) = 0$$

for all $x, y \in X$. By induction, we have $D_ag(x,y) = 0$ for all $x \in X$ and all integers $a \geq 2$. If $h$ is an additive mapping, then $h(nx) = nh(x)$ and $h(x) = h(-x)$ for all $x \in X$ and all $n \in N$. So $h$ satisfies the equality

$$D_ah(x,y) = Ah((a-1)x,y) + (a-1)Ah(x,-y) + h(ax) + h(a-1)x) - (2a-1)h(x) = 0$$

for all $x, y \in X$ and all integers $a.\geq$

Now we get the general stability result of the functional equation (1.1) in a fuzzy space. For given $q > 0$, the mapping $f$ is called *a fuzzy q-almost quadratic-additive mapping,* if

$$N'(D_a f(x,y), t+s) \geq \min\{N(x, s^q), N(y, t^q)\} \tag{2.1}$$

for all $x, y \in X\backslash\{0\}$ and all $s, t \in (0, \infty)$.

**Theorem 2.4.** *Let $q$ be a positive real number with $q \neq \frac{1}{2}, 1$. And let $f$ be a fuzzy q-almost quadratic-additive mapping from a fuzzy normed space $(X, N)$ into a fuzzy Banach space $(Y, N')$ with $f(0) = 0$. Then there is a unique quadratic-additive mapping $F : X \to Y$ such that*

$$N'\big(F(x) - f(x), t\big) \geq \begin{cases} \sup_{t'<t} \{N\left(x, (a-a^p)^q t'^q\right)\} & \text{if } q > 1, \\ \sup_{t'<t} \left\{N\left(x, \frac{(a^2-a^p)^q (a^p-a)^q}{(a^2-a)^q} t'^q\right)\right\} & \text{if } 1/2 < q < 1, \\ \sup_{t'<t} \left\{N\left(x, (a^p - a^2)^q t'^q\right)\right\} & \text{if } 0 < q < 1/2 \end{cases} \tag{2.2}$$

*for each $x \in X$ and $t > 0$, where $p = 1/q$.*

*Proof.* We will prove the theorem in three cases, $q > 1$, $\frac{1}{2} < q < 1$, and $0 < q < \frac{1}{2}$.
**Case 1.** Let $q > 1$ and let $J_n f : X \to Y$ be a mapping defined by

$$J_n f(x) = \frac{1}{2}\left(a^{-2n}\left(f(a^n x) + f(-a^n x)\right) + a^{-n}\left(f(a^n x) - f(-a^n x)\right)\right)$$

for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \frac{a^{j+1}-1}{4 \cdot a^{2j+2}} D_a f(-a^j x, -a^j x) - \frac{a^{j+1}+1}{4 \cdot a^{2j+2}} D_a f(a^j x, a^j x) \tag{2.3}$$

for all $x \in X$ and $j \geq 0$. Together with (N3), (N4) and (2.1), this equation implies that if $n + m > m \geq 0$ then

$$\begin{aligned} N'\Bigg(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{1}{a}\left(\frac{a^p}{a}\right)^j t^p\Bigg) \\ &\geq N'\left(\sum_{j=m}^{n+m-1} \left(J_j f(x) - J_{j+1} f(x)\right), \sum_{j=m}^{n+m-1} \frac{1}{a}\left(\frac{a^p}{a}\right)^j t^p\right) \\ &\geq \min \bigcup_{j=m}^{n+m-1} \left\{N'\left(J_j f(x) - J_{j+1} f(x), \frac{1}{a}\left(\frac{a^p}{a}\right)^j t^p\right)\right\} \\ &\geq \min \bigcup_{j=m}^{n+m-1} \left\{\min\left\{N'\left(-\frac{(a^{j+1}+1) D_a f(a^j x, a^j x)}{4 \cdot a^{2j+2}}, \frac{(a^{j+1}+1) a^{jp} t^p}{2 \cdot a^{2j+2}}\right),\right.\right. \end{aligned}$$

$$N'\left(\frac{(a^{j+1}-1)D_af(-a^jx,-a^jx)}{4\cdot a^{2j+2}},\frac{(a^{j+1}-1)a^{jp}t^p}{2\cdot a^{2j+2}}\right)\}\}$$
$$\geq \min\bigcup_{j=m}^{n+m-1}\{N(a^jx,a^jt)\}=N(x,t) \tag{2.4}$$

for all $x\in X$ and $t>0$. Let $\varepsilon>0$ be given. Since $\lim_{t\to\infty}N(x,t)=1$, there is $t_0>0$ such that

$$N(x,t_0)\geq 1-\varepsilon.$$

We observe that for some $\tilde{t}>t_0$, the series $\sum_{j=0}^{\infty}\frac{1}{a}\left(\frac{a^p}{a}\right)^j\tilde{t}^p$ converges for $p=\frac{1}{q}<1$. It guarantees that, for an arbitrary given $c>0$, there exists $n_0\geq 0$ such that

$$\sum_{j=m}^{n+m-1}\frac{1}{a}\left(\frac{a^p}{a}\right)^j\tilde{t}^p<c$$

for each $m\geq n_0$ and $n>0$. By (N5) and (2.4), we have

$$\begin{aligned}&N'(J_mf(x)-J_{n+m}f(x),c)\\&\quad\geq N'\left(J_mf(x)-J_{n+m}f(x),\sum_{j=m}^{n+m-1}\frac{1}{a}\left(\frac{a^p}{a}\right)^j\tilde{t}^p\right)\\&\quad\geq N(x,\tilde{t})\geq N(x,t_0)\geq 1-\varepsilon.\end{aligned}$$

for all $x\in X$. Hence $\{J_nf(x)\}$ is a Cauchy sequence in the fuzzy Banach space $(Y,N')$, and so we can define a mapping $F:X\to Y$ by

$$F(x):=N'-\lim_{n\to\infty}J_nf(x).$$

Moreover, if we put $m=0$ in (2.4), we have

$$N'(f(x)-J_nf(x),t)\geq N\left(x,\frac{t^q}{\left(\sum_{j=0}^{n-1}\frac{1}{a}\left(\frac{a^p}{a}\right)^j\right)^q}\right) \tag{2.5}$$

for all $x\in X$.

Next we will show that $F$ is the desired quadratic additive mapping. Using (N4), we have

$$\begin{aligned}&N'(D_aF(x,y),t)\\&\quad\geq\min\left\{N'\left((F-J_nf)((a-1)x+y),\frac{t}{14}\right),N'\left((a-1)(F-J_nf)(x-y),\frac{t}{14}\right),\right.\\&\qquad N'\left((F-J_nf)(ax),\frac{t}{14}\right),N'\left((a^2+a-1)(J_nf-F)(x),\frac{t}{14}\right),\end{aligned}$$

$$N'\left((a-1)^2(J_nf-F)(-x),\frac{t}{14}\right),N'\left((J_nf-F)(y),\frac{t}{14}\right),$$
$$N'\left((a-1)(J_nf-F)(-y),\frac{t}{14}\right),N'\left(DJ_nf(x,y),\frac{t}{2}\right)\Bigg\}\tag{2.6}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. The first seven terms on the right hand side of (2.6) tend to 1 as $n \to \infty$ by the definition of $F$ and (N2), and the last term holds

$$N'\left(D_aJ_nf(x,y),\frac{t}{2}\right)\geq \min\left\{N'\left(\frac{D_af(a^nx,a^ny)}{2\cdot a^{2n}},\frac{t}{8}\right),N'\left(\frac{D_af(-a^nx,-a^ny)}{2\cdot a^{2n}},\frac{t}{8}\right),\right.$$
$$\left.N'\left(\frac{D_af(a^nx,a^ny)}{2\cdot a^{n}},\frac{t}{8}\right),N'\left(\frac{D_af(-a^nx,-a^ny)}{2\cdot a^{n}},\frac{t}{8}\right)\right\}$$

for all $x, y \in X$. By (N3) and (2.1), we obtain

$$\begin{aligned}N'\left(\frac{D_af(\pm a^nx,\pm a^ny)}{2\cdot a^{2n}},\frac{t}{8}\right)=&N'\left(D_af(\pm a^nx,\pm a^ny),\frac{a^{2n}t}{4}\right)\\ \geq&\min\left\{N\left(a^nx,\left(\frac{a^{2n}t}{8}\right)^q\right),N\left(a^ny,\left(\frac{a^{2n}t}{8}\right)^q\right)\right\}\\ \geq&\min\left\{N\left(x,\frac{a^{(2q-1)n}}{2^{3q}}t^q\right),N\left(y,\frac{a^{(2q-1)n}}{2^{3q}}t^q\right)\right\}\end{aligned}$$

and

$$N'\left(\frac{D_af(\pm a^nx,\pm a^ny))}{2\cdot a^{n}},\frac{t}{8}\right)\geq\min\left\{N\left(x,\frac{a^{(q-1)n}}{2^{3q}}t^q\right),N\left(y,\frac{a^{(q-1)n}}{2^{3q}}t^q\right)\right\}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Since $q > 1$, together with (N5), we can deduce that the last term of (2.6) also tends to 1 as $n \to \infty$. It follows from (2.6) that

$$N'(D_af(x,y),t)=1$$

for each $x, y \in X$ and $t > 0$. By (N2) and Lemma 2.2, this means that $D_af(x, y) = 0$ for all $x, y \in X$. Next we approximate the difference between $f$ and $F$ in a fuzzy sense. For an arbitrary fixed $x \in X$ and $t > 0$, choose $0 < \varepsilon < 1$ and $0 < t' < t$. Since $F$ is the limit of $\{J_nf(x)\}$, there is $l \in \mathbb{N}$ such that

$$N'\left(F(x)-J_nf(x),t-t'\right)\geq 1-\varepsilon.$$

By (2.5), we have

$$\begin{aligned}N'(F(x)-f(x),t)\geq&\min\left\{N'\left(F(x)-J_nf(x),t-t'\right),N'\left(J_nf(x)-f(x),t'\right)\right\}\\ \geq&\min\left\{1-\varepsilon,N\left(x,\frac{t'^q}{\left(\sum_{j=0}^{n-1}\frac{1}{a}\left(\frac{a^p}{a}\right)^j\right)^q}\right)\right\}\end{aligned}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)



$$\geq \min\left\{1-\varepsilon, N\left(x, (a-a^p)^q t'^q\right)\right\}.$$

Because $0 < \varepsilon < 1$ is arbitrary, we get the inequality (2.2) in this case. Finally, to prove the uniqueness of $F$, let $F' : X \to Y$ be another quadratic-additive mapping satisfying (2.2). Then by (2.3), we get

$$\begin{cases} F(x) - J_nF(x) = \sum_{j=0}^{n-1}\left(J_jF(x) - J_{j+1}F(x)\right) = 0 \\ F'(x) - J_nF'(x) = \sum_{j=0}^{n-1}\left(J_jF'(x) - J_{j+1}F'(x)\right) = 0 \end{cases} \tag{2.7}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Together with (N4) and (2.2), this implies that

$$\begin{aligned} &N'(F(x) - F'(x), t) \\ &\quad = N'(J_nF(x) - J_nF'(x), t) \\ &\quad \geq \min\left\{N'\left(J_nF(x) - J_nf(x),\ \frac{t}{2}\right),\ N'\left(J_nf(x) - J_nF'(x),\ \frac{t}{2}\right)\right\} \\ &\quad \geq \min\left\{N'\left(\frac{(F-f)(n^lx)}{2\cdot n^{2l}},\ \frac{t}{8}\right),\ N'\left(\frac{(f-F')(a^nx)}{2\cdot a^{2n}},\ \frac{t}{8}\right),\right. \\ &\qquad\qquad N'\left(\frac{(F-f)(-a^nx)}{2\cdot a^{2n}},\ \frac{t}{8}\right),\ N'\left(\frac{(f-F')(-a^nx)}{2\cdot a^{2n}},\ \frac{t}{8}\right), \\ &\qquad\qquad N'\left(\frac{(F-f)(a^nx)}{2\cdot a^{n}},\ \frac{t}{8}\right),\ N'\left(\frac{(f-F')(a^nx)}{2\cdot a^{n}},\ \frac{t}{8}\right), \\ &\qquad\qquad \left.N'\left(\frac{(F-f)(-a^nx)}{2\cdot a^{n}},\ \frac{t}{8}\right),\ N'\left(\frac{(f-F')(-a^nx)}{2\cdot a^{n}},\ \frac{t}{8}\right)\right\} \\ &\quad \geq \sup_{t'<t} N\left(x, a^{(q-1)n}2^{-2q}(a-a^p)^q t'^q\right) \end{aligned}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Observe that, for $q = \frac{1}{p} > 1$, the last term of the above inequality tends to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and so we get

$$F(x) = F'(x)$$

for all $x \in X$ by (N2).

**Case 2.** Let $\frac{1}{2} < q < 1$ and let $J_nf : X \to Y$ be a mapping defined by

$$J_nf(x) = \frac{1}{2}\left(a^{-2n}\left(f(a^nx) + f(-a^nx)\right) + a^n\left(f\left(\frac{x}{a^n}\right) - f\left(-\frac{x}{a^n}\right)\right)\right)$$

for all $x \in X$. Then we have $J_0f(x) = f(x)$ and

$$\begin{aligned} J_jf(x) - J_{j+1}f(x) = &-\frac{1}{4\cdot a^{2j+2}}D_af(a^jx, a^jx) - \frac{1}{4\cdot a^{2j+2}}D_af(-a^jx, -a^jx) \\ &+ \frac{a^j}{4}D_af\left(\frac{x}{a^{j+1}}, \frac{x}{a^{j+1}}\right) - \frac{a^j}{4}D_af\left(\frac{-x}{a^{j+1}}, \frac{-x}{a^{j+1}}\right) \end{aligned}$$

for all $x \in X$ and $j \geq 0$. If $n + m > m \geq 0$, then we have

$$\begin{aligned}
N'\Bigg(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left(\frac{1}{a^2}\left(\frac{a^p}{a^2}\right)^j + \frac{1}{a^p}\left(\frac{a}{a^p}\right)^j\right) t^p\Bigg) \\
\geq \min \bigcup_{j=m}^{n+m-1} \Bigg\{ \min \Bigg\{ N'\left(-\frac{D_a f(a^j x, a^j x)}{4 \cdot a^{2j+2}}, \frac{a^{jp} t^p}{2 \cdot a^{2j+2}}\right), \\
N'\left(-\frac{D_a f(-a^j x, -a^j x)}{4 \cdot a^{2j+2}}, \frac{a^{jp} t^p}{2 \cdot a^{2j+2}}\right), \\
N'\left(\frac{a^j}{4} D_a f\left(\frac{x}{a^{j+1}}, \frac{x}{a^{j+1}}\right), \frac{a^j t^p}{2 \cdot a^{(j+1)p}}\right), \\
N'\left(-\frac{a^j}{4} D_a f\left(-\frac{x}{a^{j+1}}, -\frac{x}{a^{j+1}}\right), \frac{a^j t^p}{2 \cdot a^{(j+1)p}}\right) \Bigg\} \Bigg\} \\
\geq \min \bigcup_{j=m}^{n+m-1} \left\{ N(a^j x, a^j t), N\left(\frac{x}{a^{j+1}}, \frac{t}{a^{j+1}}\right) \right\} \\
= N(x, t)
\end{aligned}$$

for all $x \in X$ and $t > 0$. In the similar argument following (2.4) of the previous case, we can define the limit $F(x) := N' - \lim_{n\to\infty} J_n f(x)$ of the Cauchy sequence $\{J_n f(x)\}$ in the Banach fuzzy space $Y$. Moreover, putting $m = 0$ in the above inequality, we have

$$N'(f(x) - J_n f(x), t) \geq N\left(x, \frac{t^q}{\left(\sum_{j=0}^{n-1}\left(\frac{1}{a^2}\left(\frac{a^p}{a^2}\right)^j + \frac{1}{a^p}\left(\frac{a}{a^p}\right)^j\right)\right)^q}\right) \tag{2.8}$$

for each $x \in X$ and $t > 0$. To prove that $F$ is a quadratic additive mapping, we have enough to show that the last term of (2.6) in Case 1 tends to 1 as $n \to \infty$. By (N3) and (2.1), we get

$$\begin{aligned}
N'\left(D_a J_n f(x, y), \frac{t}{2}\right) \\
\geq \min \Bigg\{ N'\left(\frac{D_a f(a^n x, a^n y)}{2 \cdot a^{2n}}, \frac{t}{8}\right), N'\left(\frac{D_a f(-a^n x, -a^n y}{2 \cdot a^{2n}}, \frac{t}{8}\right), \\
N'\left(\frac{a^n}{2} D_a f\left(\frac{x}{a^n}, \frac{y}{a^n}\right), \frac{t}{8}\right), N'\left(\frac{a^n}{2} D_a f\left(\frac{-x}{a^n}, \frac{-y}{a^n}\right), \frac{t}{8}\right) \Bigg\} \\
\geq \min \Big\{ N(x, a^{(2q-1)n} 2^{-3q} t^q), N(y, a^{(2q-1)n} 2^{-3q} t^q), \\
N(x, a^{(1-q)n} 2^{-3q} t^q), N(y, a^{(1-q)n} 2^{-3q} t^q) \Big\}
\end{aligned}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 3 (2015)



for each $x, y \in X$ and $t > 0$. Observe that all the terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, since $\frac{1}{2} < q < 1$. Hence, together with the similar argument after (2.6), we can say that $D_a f(x, y) = 0$ for all $x, y \in X$. Recall, in Case 1, the inequality (2.2) follows from (2.5). By the same reasoning, we get (2.2) from (2.8) in this case. Now to prove the uniqueness of $F$, let $F'$ be another quadratic additive mapping satisfying (2.2). Then, together with (N4), (2.2), and (2.7), we have

$$\begin{aligned}
&N'(F(x) - F'(x), t) \\
&\quad = N'(J_nF(x) - J_nF'(x), t) \\
&\quad \geq \min\left\{N'\left(J_nF(x) - J_nf(x), \frac{t}{2}\right), N'\left(J_nf(x) - J_nF'(x), \frac{t}{2}\right)\right\} \\
&\quad \geq \min\left\{N'\left(\frac{(F-f)(a^n x)}{2\cdot a^{2n}}, \frac{t}{8}\right), \left(\frac{(f-F')(a^n x)}{2\cdot a^{2n}}, \frac{t}{8}\right),\right. \\
&\qquad N'\left(\frac{(F-f)(-a^n x)}{2\cdot a^{2n}}, \frac{t}{8}\right), N'\left(\frac{(f-F')(-a^n x)}{2\cdot a^{2n}}, \frac{t}{8}\right), \\
&\qquad N'\left(\frac{a^n}{2}\left((F-f)\left(\frac{x}{a^n}\right)\right), \frac{t}{8}\right), N'\left(\frac{a^n}{2}\left((f-F')\left(\frac{x}{a^n}\right)\right), \frac{t}{8}\right), \\
&\qquad \left. N'\left(\frac{a^n}{2}\left((F-f)\left(\frac{-x}{a^n}\right)\right), \frac{t}{8}\right), N'\left(\frac{a^n}{2}\left((f-F')\left(\frac{-x}{a^n}\right)\right), \frac{t}{8}\right)\right\} \\
&\quad \geq \min\left\{\sup_{t'<t} N\left(x, a^{(2q-1)n}2^{-2q}(a^2-a)^{-q}(a^2-a^p)^q(a^p-a)^q t'^q\right),\right. \\
&\qquad \left.\sup_{t'<t} N\left(x, a^{(1-q)n}2^{-2q}(a^2-a)^{-q}(a^2-a^p)^q(a^p-a)^q t'^q\right)\right\}
\end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. Since $\lim_{n\to\infty} a^{(2q-1)n} = \lim_{n\to\infty} a^{(1-q)n} = \infty$ in this case, both terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x) - F'(x),\ t) = 1$ and so $F(x) = F'(x)$ for all $x \in X$ by (N2). **Case 3.** Finally, we take $0 < q < \frac{1}{2}$ and define $J_nf : X \to Y$ by

$$J_nf(x) = \frac{1}{2}\left(a^{2n}\left(f\left(\frac{x}{a^n}\right) - f\left(-\frac{x}{a^n}\right)\right) + a^n\left(f\left(\frac{x}{a^n}\right) - f\left(-\frac{x}{a^n}\right)\right)\right)$$

for all $x \in X$. Then we have $J_0f(x) = f(x)$ and

$$J_jf(x) - J_{j+1}f(x) = \frac{a^{2j}+a^j}{4}D_af\left(\frac{x}{a^{j+1}}, \frac{x}{a^{j+1}}\right) + \frac{a^{2j}-a^j}{4}D_af\left(\frac{-x}{a^{j+1}}, \frac{-x}{a^{j+1}}\right)$$

which implies that if $n + m > m \geq 0$ then

$$N'\left(J_mf(x) - J_{n+m}f(x), \sum_{j=m}^{n+m-1}\left(\frac{a^2}{a^p}\right)^j \frac{t^p}{a^p}\right)$$

$$\begin{aligned}
&\geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min\left\{ N'\left(\frac{a^{2j}+a^j}{4} D_a f\left(\frac{x}{a^{j+1}}, \frac{x}{a^{j+1}}\right), \frac{(a^{2j}+a^j)t^p}{2\cdot a^{(j+1)p}}\right),\right.\right. \\
&\qquad \left.\left. N'\left(\frac{a^{2j}-a^j}{4} D_a f\left(-\frac{x}{a^{j+1}}, -\frac{x}{a^{j+1}}\right), \frac{(a^{2j}-a^j)t^p}{2\cdot a^{(j+1)p}}\right) \right\}\right\} \\
&\geq \min \bigcup_{j=m}^{n+m-1} \left\{ N\left(\frac{x}{a^{j+1}}, \frac{t}{a^{j+1}}\right)\right\} \\
&= N(x,t)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Similar to the previous cases, it leads us to define the mapping $F : X \to Y$ by $F(x) := N' - \lim_{n\to\infty} J_n f(x)$. Putting $m = 0$ in the above inequality, we have

$$N'(f(x) - J_n f(x), t) \geq N\left(x, \frac{t^q}{\left(\frac{1}{a^p}\sum_{j=0}^{n-1}\left(\frac{a^2}{a^p}\right)^j\right)^q}\right) \tag{2.9}$$

for all $x \in X$ and $t > 0$. Notice that

$$\begin{aligned}
&N'\left(D_a J_n f(x,y), \frac{t}{2}\right) \\
&\quad\geq \min\left\{N'\left(\frac{a^{2n}}{2} D_a f\left(\frac{x}{a^n}, \frac{y}{a^n}\right), \frac{t}{8}\right), N'\left(\frac{a^{2n}}{2} D_a f\left(\frac{-x}{a^n}, \frac{-y}{a^n}\right), \frac{t}{8}\right),\right. \\
&\qquad \left. N'\left(\frac{a^{n}}{2} D_a f\left(\frac{x}{a^n}, \frac{y}{a^n}\right), \frac{t}{8}\right), N'\left(\frac{a^{n}}{2} D_a f\left(\frac{-x}{a^n}, \frac{-y}{a^n}\right), \frac{t}{8}\right)\right\} \\
&\quad\geq \min\left\{N\left(x, a^{(1-2q)n}2^{-3q}t^q\right), N\left(y, a^{(1-2q)n}2^{-3q}t^q\right),\right. \\
&\qquad \left. N\left(x, a^{(1-q)n}2^{-3q}t^q\right), N\left(y, a^{(1-q)n}2^{-3q}t^q\right)\right\}
\end{aligned}$$

for each $x, y \in X$ and $t > 0$. Since $0 < q < \frac{1}{2}$, oth terms on the right hand side tend to 1 as $n \to \infty$, which implies that the last term of (2.6) tends to 1 as $n \to \infty$. Therefore, we can say that $D_a f \equiv 0$. Moreover, using the similar argument after (2.6) in Case 1, we get the inequality (2.2) from (2.9) in this case. To prove the uniqueness of $F$, let $F' : X \to Y$ be another quadratic additive function satisfying (2.2). Then by (2.7), we get

$$\begin{aligned}
&N'(F(x) - F'(x), t) \\
&\quad\geq \min\left\{N'\left(J_n F(x) - J_n f(x), \frac{t}{2}\right), N'\left(J_n f(x) - J_n F'(x), \frac{t}{2}\right)\right\}
\end{aligned}$$

$$\geq \min\Big\{N'\left(\frac{a^{2n}}{2}\left((F-f)\left(\frac{x}{a^n}\right)\right),\frac{t}{8}\right),N'\left(\frac{a^{2n}}{2}\left((f-F')\left(\frac{x}{a^n}\right)\right),\frac{t}{8}\right),$$
$$N'\left(\frac{a^{2n}}{2}\left((F-f)\left(-\frac{x}{a^n}\right)\right),\frac{t}{8}\right),N'\left(\frac{a^{2n}}{2}\left((f-F')\left(-\frac{x}{a^n}\right)\right),\frac{t}{8}\right),$$
$$N'\left(\frac{a^{n}}{2}\left((F-f)\left(\frac{x}{a^n}\right)\right),\frac{t}{8}\right),N'\left(\frac{a^{n}}{2}\left((f-F')\left(\frac{x}{a^n}\right)\right),\frac{t}{8}\right),$$
$$N'\left(\frac{a^{n}}{2}\left((F-f)\left(\frac{-x}{a^n}\right)\right),\frac{t}{8}\right),N'\left(\frac{a^{n}}{2}\left((f-F')\left(\frac{-x}{a^n}\right)\right),\frac{t}{8}\right)\Big\}$$
$$\geq \sup_{t'<t} N\left(x, a^{(1-2q)n}2^{-2q}(a^p-a^2)^q t'^q\right)$$

for all $x \in X$ and all $n \in \mathbb{N}$. Observe that, for $0 < q < \frac{1}{2}$, the last term tends to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x)-F'(x),t) = 1$ and $F(x) = F'(x)$ for all $x \in X$ by (N2). □

**Remark 2.5.** Consider a mapping $f : X \to Y$ satisfying (2.1) for all $x, y \in X\backslash\{0\}$ and a real number $q < 0$. Take any $t > 0$. If we choose a real number $s$ with $0 < 2s < t$, then we have

$$N'(D_af(x,y),t) \geq N'(D_af(x,y),2s) \geq \min\{N(x,s^q),N(y,s^q)\}$$

for all $x, y \in X\backslash\{0\}$. Since $q < 0$, we have $\lim_{s\to0^+} s^q = \infty$. This implies that

$$\lim_{s\to0^+} N(x,s^q) = \lim_{s\to0^+} N(y,s^q) = 1$$

and so

$$N'(D_af(x,y),t) = 1$$

for all $x, y \in X\backslash\{0\}$ and $t > 0$. By (N2), it allows us to get $D_af(x,y) = 0$ for all $x, y \in X$. In other words, $f$ is itself a quadratic additive mapping if $f$ is a fuzzy $q$-almost quadratic-additive mapping for the case $q < 0$.

**Corollary 2.6.** *Let $f$ be an even mapping satisfying all of the conditions of Theorem 2.4 with $f(0) = 0$. Then there is a unique quadratic mapping $F : X \to Y$ such that*

$$N'(F(x)-f(x),t) \geq \sup_{t'<t} N\left(x, \left(|a^2-a^p|t'\right)^q\right) \tag{2.10}$$

*for all $x \in X$ and $t > 0$, where $p = 1/q$.*

*Proof.* Let $J_nf$ be defined as in Theorem 2.4. Since $f$ is an even mapping, we obtain

$$J_nf(x) = \begin{cases} \frac{f(a^nx)+f(-a^nx)}{2\cdot a^{2n}} & \text{if } q > \frac{1}{2}, \\ \frac{1}{2}\left(a^{2n}\left(f(\frac{x}{a^n})+f(\frac{-x}{a^n})\right)\right) & \text{if } 0 < q < \frac{1}{2} \end{cases}$$

for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{-D_a f(a^j x, a^j x)}{2 \cdot a^{2j+2}} & \text{if } q > \frac{1}{2}, \\ \frac{a^{2j}}{2} D_a f\left(\frac{x}{a^{j+1}}, \frac{x}{a^{j+1}}\right) & \text{if } 0 < q < \frac{1}{2} \end{cases}$$

for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$. From these, using the similar method in Theorem 2.4, we obtain the quadratic-additive mapping $F$ satisfying (2.10). Notice that $F$ is also even, $F(0) = 0$, and $D_a F(x, y) = 0$ for all $x, y \in X$. This means that $F$ is a quadratic mapping. □

**Corollary 2.7.** *Let $f$ be an odd mapping satisfying all of the conditions of Theorem 2.4. Then there is a unique additive mapping $F : X \to Y$ such that*

$$N'(F(x) - f(x), t) \geq \sup_{t' < t} N\left(x, \left(|a - a^p| t'\right)^q\right) \tag{2.11}$$

*for all $x \in X$ and $t > 0$, where $p = 1/q$.*

*Proof.* Let $J_n f$ be defined as in Theorem 2.4. Since $f$ is an odd mapping, we obtain

$$J_n f(x) = \begin{cases} \frac{f(a^n x) - f(-a^n x)}{2 \cdot a^n} & \text{if } q > 1, \\ \frac{n^l}{2}\left(f(\frac{x}{a^n}) - f(\frac{-x}{a^n})\right) & \text{if } 0 < q < 1 \end{cases}$$

for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{-1}{2 \cdot a^{j+1}} D_a f(a^j x, a^j x) & \text{if } q > 1, \\ \frac{a^j}{2} D_a f\left(\frac{x}{a^{j+1}}, \frac{x}{a^{j+1}}\right) & \text{if } 0 < q < 1 \end{cases}$$

for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$. From these, using the similar method in Theorem 2.4, we obtain the quadratic-additive mapping $F$ satisfying (2.11). Notice that $F$ is also odd and $D_a F(x, y) = 0$ for all $x, y \in X$. This means that $F$ is an additive mapping. □

We can use Theorem 2.4 to get a classical result in the framework of normed spaces. Let $(X, \|\cdot\|)$ be a normed linear space. Then we can define a fuzzy norm $N_X$ on $X$ by following

$$N_X(x, t) = \begin{cases} 0, & t \leq \|x\| \\ 1, & t > \|x\| \end{cases}$$

where $x \in X$ and $t \in \mathbb{R}$, see [20]. Suppose that $f : X \to Y$ is a mapping into a Banach space $(Y, |||\cdot|||)$ such that

$$|||D_a f(x, y)||| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$. Let $N_Y$ be a fuzzy norm on $Y$. Then we get

$$N_Y(D_a f(x,y), s+t) = \begin{cases} 0, & s+t \leq |||D_a f(x,y)||| \\ 1, & s+t > |||D_a f(x,y)||| \end{cases}$$

for all $x, y \in X$ and $s, t \in \mathbb{R}$. Consider the case $N_Y(D_a f(x,y), s+t) = 0$. This implies that

$$\|x\|^p + \|y\|^p \geq |||D_a f(x,y)||| \geq s+t$$

and so either $\|x\|^p \geq s$ or $\|y\|^p \geq t$ in this case. Hence, for $q = \frac{1}{p}$, we have

$$\min\{N_X(x, s^q), N_X(y, t^q)\} = 0$$

for all $x, y \in X$ and $s, t > 0$. Therefore, in every case, the inequality

$$N_Y(D_a f(x,y), s+t) \geq \min\{N_X(x, s^q), N_X(y, t^q)\}$$

holds. It means that $f$ is a fuzzy $q$-almost quadratic additive mapping, and by Theorem 2.2, we get the following stability result.

**Corollary 2.8.** *Let $(X, \|\cdot\|)$ be a normed linear space and let $(Y, |||\cdot|||)$ be a Banach space. If*

$$|||D_a f(x,y)||| \leq \|x\|^p + \|y\|^p$$

*for all $x, y \in X$, where $p > 0$ and $p \neq 1, 2$, then there is a unique quadratic-additive mapping $F : X \to Y$ such that*

$$|||F(x) - f(x)||| \leq \begin{cases} \frac{||x||^p}{a-a^p} & \text{if } 0 < p < 1, \\ \frac{(a^2-a)||x||^p}{(a^2-a^p)(a^p-a)} & \text{if } 1 < p < 2, \\ \frac{||x||^p}{a^p-a^2} & \text{if } p > 2 \end{cases}$$

*for all $x \in X$.*

## Acknowledgments

This work was supported by Gongju National University of Education Grant.

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