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FUZZY L-FILTERS AND L-IDEALS

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Abstract: The object of this paper is to introduce and study the notions of fuzzy L-filters and fuzzy L-ideals of a semi lattice with truth values in the unit interval of real numbers. For a given collection of semi filters (semi ideals) of a semi lattice, necessary and sufficient conditions are given under which the family represents a collection of level subsets of a fuzzy L-filters (fuzzy L-ideals). It is proved that the class of all fuzzy L-filters (fuzzy L-ideals) of a given semi lattice are complete lattice under fuzzy inclusion.

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1. Introduction

Fuzzy lattices and their algebraic structures have been widely studied from N. Ajmal [1]. They appear when the membership grades can be represented by numbers in the unit interval [0,1]. Even though a lot of work have been done for fuzzy algebraic structures by many authors, it has a prominent role in Mathematics with wide ranging applications in many disciplines such as Theoretical Physics, Computer Sci-

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ences, Control Engineering, Information Science, Coding Theory, Topological space and the like. Further they can be more appropriate to model natural problems. It turns out that recently this approach to fuzzyness attracts more and more interest, in particular when a structure has a residuated or similar lattice as a domain. Thus in the last decade, lattice-valued Mathematics have undergone a significant development. Due to the importance of semi-filters and semi-ideals in the classical representation theory, the main purpose of this paper is to study the generalization of these concepts for fuzzy L-subsets.

In this paper, we introduce fuzzy L-filters and fuzzy L-ideals, where L is a meet semi lattice and further it turns out that the level subsets of these are precisely crisp semi-filter and semi-ideal of L. It can be observed that, the definition of fuzzy L-filters is quite dual to the definition of fuzzy L-ideals. The duality between these two concepts is shown in the Proposition 2.8 by means of complement operator C on U, where U is unit interval [0,1], see [14]. Also we investigate for a given collection of semi-filters (semi-ideals) of a meet semi lattice L, necessary and sufficient conditions are given under which the family represents a collection of level subsets of a fuzzy L-filters (fuzzy L-ideals). Further it is proved that the class of all fuzzy L-filters (fuzzy L-ideals) of a given meet semi lattice L are complete lattices under fuzzy inclusion.

Throughout this paper, $L = (L, \wedge)$ stands for a non trivial meet semi lattice, with least element 0_L. We consider fuzzy L-subsets of L in the sense of N. Ajmal, see [1]. Accordingly, fuzzy L-subset of L is a mapping of L into U, where U is the unit interval [0,1] of real numbers.

2. Preliminaries

In this section, we shall recall some basic definitions and results that are useful throughout this paper.

Definition 1.1. A poset is a non-empty set X equipped with an ordering relation $\leq$. A poset is usually denoted as an ordered pair $(X, \leq)$ or simply by the underlying set X.

Definition 1.2. A semi-filter on a poset X is any subposet U, satisfying the following for $x \in U, y \in X$ and $x \leq y$ implies $y \in U$.

Definition 1.3. A semi-ideal on a poset X is any subposet D, satisfying the following for $x \in D, y \in X$ and $y \leq x$ implies $y \in D$.

Definition 1.4. A lattice is a poset L in which for each pair of elements x, y there is a greatest lower bound (glb) and a least upper bound (lub), denoted by $x \wedge y$ i.e., $\min\{x, y\}$ and $x \vee y$ i.e., $\max\{x, y\}$ respectively.

Definition 1.5. A non-empty poset L is said to be a complete lattice if glb i.e 1 and lub i.e 0 exists for each subset of L.

Definition 1.6. Let $L = (L, \wedge, \vee)$ be a lattice.Let C be an operator from L into L. Consider the following identities:

$C_0 : C(0) = 0,$

$C_1 : x \leq C(x),$

$C_2 : C(x) = C(C(x)),$

$C_3 : if x \leq y then C(x) \leq C(y),$

$C_4 : C(x \vee y) = C(x) \vee C(y),$

$C_5 : C(x \wedge y) = C(x) \wedge C(y),$

(i) If C is satisfies C_1, C_2 and C_3 conditions, then it is called a closure operator and the set of all closure operators on L is denoted by $C(L)$.

(ii) If C is satisfies C_1, C_2 and C_4 conditions, then it is called an additive closure operator, (It can be easily observe that C_3 follows from either C_4 or C_5) and the set of all additive closure operator on L is denoted by $C_A(L)$.

(iii) If C is satisfies C_1, C_2 and C_5 conditions, then it is called a multiplicative closure operator, and the set of all multiplicative closure operator on L is denoted by $C_M(L)$.

It is clear that $C_A(L) \subseteq C(L)$ and $C_M(L) \subseteq C(L)$.

Definition 1.7. If C is a operator on L, we say that $x \in L$ is an invariant under C or C-invariant if $C(x) = x$ and we denote by C_L the set of all invariant elements of L under C, i.e.

$$C_L = \{x \in L / C(x) = x\}.$$

Lemma 1.8. *If C satisfies C_2 iff $C(L) = C_L$.*

3. Fuzzy L-Filters and L-Ideals

In this section, we introduce and studied the concepts of fuzzy semi L-filters and L-ideals, where L is meet semi lattice with least element 0_L and which is renamed as fuzzy L-filters and L-ideals.

We begin with the following.

Definition 2.1. A fuzzy L-subset $A : L \to U$ is called Fuzzy L-filter of L, if for every $x, y \in L$, $A(x \wedge y) \leq \min\{A(x), A(y)\}$.

Let us consider the following examples.

Example 2.2. Let $L = (0_L, a, b, c)$. Then $(L, \wedge)$is a semi lattice, with least element 0_L, under the meet operation $a \wedge b = \min\{a, b\}$. This can represented by Hasse diagram

Now define a fuzzy L-subset $A : L \to U(= [0, 1])$ by

$$A(x) = \begin{cases} 0.6 & \text{if } x = 0_L \\ 0.8 & \text{if } x = a, b \\ 0.9 & \text{if } x = c \end{cases}$$

Then A is a fuzzy L-filter of L.

Example 2.3. Let $D_{12} = \{1, 2, 3, 4, 6, 12\}$ be divisors of 12.

Let $L = D_{12} - \{12\} = \{1_L, 2_L, 3_L, 4_L, 6_L\}$. Then $(L, \wedge)$ is a semi lattice with least element 1_L, under the meet operation $a \wedge b = g.c.d\ of\ a, b$. This can represented by Hasse diagram Now define a fuzzy L-subset $A : L \to U$ by

$$A(x) = \begin{cases} 0.1 & \text{if } x = 1_L \\ 0.3 & \text{if } x = 2_L, 3_L \\ 0.9 & \text{if } x = 4_L, 6_L \end{cases}$$

Then A is a fuzzy L-filter of L.

Proposition 2.4. *Let L be a meet semi lattice with least element 0_L. If A is fuzzy L-filter of L, then $A(x) \geq A(0_L)$, $\forall\ x \in L$.*

Definition 2.5. (Level Subset or Cut Set) If $A : L \to U$ is an fuzzy L-subset on L then, for $t \in U$, the set

$$A_t = \{x \in L / A(x) \geq t\}$$

is called t-level subset or cutset of A.

Example 2.6: Let $L = (0_L, a, b, c)$ be semi lattice and let $A : L \to U$ is fuzzy L-set, define

$$A(x) = \begin{cases} 0.6 & \text{if } x = 0_L \\ 0.8 & \text{if } x = a, b, c \end{cases}$$

Then A is a fuzzy L-filter of L.

Let $t = 0.6$, then $A_t = \{0_L, a, b, c\} = L$.

We now introduce the fuzzy L-ideal of a semi lattice L.

Definition 2.7. A fuzzy L-subset $A : L \to U$ is called Fuzzy L-ideal of L, if for every $x, y \in L$, $A(x \wedge y) \geq \max\{A(x), A(y)\}$.

It can be observe that, the definition of fuzzy L-filters is quite dual to the definition of fuzzy L-ideals. The duality between these two concepts is showed in the following proposition, by means of complement operators, see [14].

We can observe that $U = [0, 1]$ is a complete lattice with greatest element 1 and least element 0.

Proposition 2.8. *Let $(L, \leq)$ be a semi lattice. If there exists a complement operator C on U that is, a map $C : U \to U$ fulfilling:*
(i) $C(0) = 1$ and $C(1) = 0$,
(ii) If $\alpha \leq \beta \Rightarrow C(\alpha) \leq C(\beta), \forall\ \alpha, \beta \in U$,
(iii) $C(C(\alpha)) = \alpha; \forall\ \alpha \in U$,
(iv) $C(\min\{\alpha, \beta\}) = \max\{C(\alpha), C(\beta)\}, \forall\ \alpha, \beta \in U$,
(v) $C(\max\{\alpha, \beta\}) = \min\{C(\alpha), C(\beta)\}, \forall\ \alpha, \beta \in U$.
Then the fuzzy L-subset of L, $A : L \to U$ is an fuzzy L-filter iff its C-complement A^C is an fuzzy L-ideal, where A^C denotes the fuzzy L-set of L, defined by $A^C(x) = C(A(x))$.

Proof. Suppose $A : L \to U$ is an fuzzy L-filter of L, then

$$A(x \wedge y) \leq \min\{A(x), A(y)\},\ \forall\ x, y \in L.$$

$$\Rightarrow C(A(x \wedge y) \geq C(\min\{A(x), A(y)\}), by\ property(ii)$$

$$\Rightarrow A^C(x \wedge y) \geq \max\{A^C(x), A^C(y)\}, by\ property(iv)$$

i.e., $A^C(x \wedge y) \geq \max\{A^C(x), A^C(y)\},\ \forall\ x, y \in L$, which implies that A^C is a fuzzy L-ideal of L.

Similarly the converse can be prove, with this we obtain the result. □

4. The Level Subsets of Fuzzy Semi L-Filters

In this section, we introduce the concept of fuzzy semi L-filters and fuzzy semi L-ideals on a semi lattice $L = (L, \wedge)$, in terms of level subsets.

We begin with the following definition.

Definition 3.1. Let $L = (L, \wedge)$ be a semi lattice. If $A : L \to U$ is a fuzzy semi L-set on L, then, for $t \in ImA$, the set

$$A_t = \{x \in L / A(x) \geq t\}$$

is called t-cut of A or t-level subset of A. It can be observe that $A_t \subseteq L$.

Immediately we have following characterization.

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Proposition 3.2. *Let L be a semi lattice, a function $A : L \to U$ is an fuzzy semi L-filter on L if and only if any t-cut A_t of A is a crisp semi-filter on L, for every $t \in ImA$.*

Proof. Let A be a fuzzy semi L-filter on L, then

$$A(x \wedge y) \leq \min\{A(x), A(y)\}, \ \forall \ x, y \in L.$$

Now for any $p \in A_t$ and $q \in L$ such that $p \leq q$. Then $A(p \wedge q) \leq \min\{A(p), A(q)\}$, which implies $t \leq A(p) \leq \min\{A(p), A(q)\}$. This gives $t \leq A(q)$, hence $q \in A_t$.

Hence A_t is crisp semi filter on L, for any $t \in ImA$.

Conversely, let the t-level subset A_t of A be crisp semi filter of L, for any $t \in ImA$.

By definition, any $x \in A_t$ and $y \in L$ such that $x \leq y$, which implies $y \in A_t$. For this x *and* y, it seems that $x, y \in L$.

Let $A(x) = a, A(y) = b \ and \ a < b$. This gives $a = \min\{a, b\}$, i.e., $A(x) = \min\{A(x), A(y)\}$.

But $x \wedge y = x$, hence $A(x \wedge y) \leq \min\{A(x), A(y)\}$.

This gives A is fuzzy semi L-filter on L. □

The proof of this theorem is a immediate consequence of the definitions of level subsets and fuzzy semi L-filters.

Example 3.3. Let $L = (\{0_L, a, b\}, \wedge)$ be the meet semi lattice with three elements, and this can be represent by these diagram

The fuzzy set on L, $A : L \to U$ by

$$A(x) = \begin{cases} 0.1 & \text{if} \quad x = 0_L \\ 0.6 & \text{if} \quad x = a \\ 0.9 & \text{if} \quad x = b \end{cases}$$

is fuzzy semi L-filter on L.

Further, consider the family of its level-subsets formed by the elements

$$A_{0.1} = \{0_L, a, b\} = L, \ A_{0.5} = \{a, b\}, \ A_{0.9} = \{b\}$$

as a consequence of the previous theorem, we know that all these crisp sets are semi filter. Conversely, since all the level-subsets are crisp semi filters on L, the previous theorem says that A is a fuzzy semi L-filter which is easy to check.

Now we introduce the $F_U(L)$, which contains the set of all fuzzy semi L-filters on L, where U is unit interval [0, 1], i.e.

$$F_U(L) = \{A : L \to U/A \ is \ a \ fuzzy \ L-filter \ on \ L\}$$

.

This set becomes poset by naturally using the order induced by the one from the lattice L. Thus for any $A, B \in F_U(L)$, we say $A \subseteq B$ if and only if for each $x \in L, A(x) \leq B(x)$.

Since $U = [0,1]$ is complete lattice, and hence the minimum and maximum values exists for any subset of $F_U(L)$. This can be stated in the following.

Proposition 3.4. *The poset $(F_U(L), \subseteq)$ is complete lattice.*

In fact, the poset $(F_U(L), \subseteq)$ is complete sublattice of the lattice $(I^L, \leq)$. Further due to the definition of the order in $F_U(L)$, we can observe that, the identity holds in unit interval U, (U is complete lattice satisfying distributive law), then the same identity is satisfied in the lattice $(F_U(L), \subseteq)$.

5. The Level Subsets of Fuzzy Semi L-Ideals

All the previous studies made for fuzzy semi L-filters could be repeated for fuzzy semi L-ideals. By the duality between these two concepts (see the definitions) the obtained results are totally analogues.

We begin with the following.

Proposition 4.1. *Let $L = (L, \wedge)$ be a semi lattice and let A be a fuzzy lattice set on L. Then the following are equivalent:*

(i) A is a fuzzy semi L-ideal on L.

(ii) The level subset A_t of A is semi ideal on L for any $t \in ImA$.

Proof. The proof is obvious. □

Now consider the set $I_U(L)$ formed by the collection of all fuzzy semi L-ideals on L, where U is unit interval [0,1]. i.e.,

$$I_U(L) = \{A/A : L \to U; A \; is \; a \; fuzzy \; semi \; L - ideal\}$$

.

This set, under inclusion order defined for any $A, B \in I_U(L)$ such that $A \subseteq B$ if and only if $A(x) \leq B(x), \forall \; x \in L$ becomes a poset. Further, the minimum and maximum values exists for any subset of $I_U(L)$, as it stated in the following.

Proposition 4.2. *The poset $(I_U(L), \subseteq)$ is complete lattice.*

Further, by the duality between these two concepts the properties satisfied by fuzzy semi L-filter could be repeated for fuzzy semi L-ideals. Since the unit interval

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[0,1] is a complete lattice satisfying distributive law, the same identity is satisfied in the lattice $(I_U(L), \subseteq)$.

Conclusion. In this paper, the concept of fuzzy L-filters and fuzzy L-ideals, where L is a meet semi lattice has been include and it is general version of existing notion of fuzzy lattice filters and ideals. Moreover, we have carried out an in depth study of some particular classes of fuzzy L-filters and fuzzy L-ideals and it was expected that several results from theory of lattices, Boolean algebras and Topological operators can be extended to the concept of fuzzy lattice sets.

In future works we would like to obtain some necessary and sufficient conditions not only for the existence of a fuzzy L-filters (resp. fuzzy L-ideals) for a given family of semi-filters (resp. semi-ideals) but for the uniqueness of such fuzzy L-filters (resp. fuzzy L-ideals).

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