International Electronic Journal of Pure and Applied Mathematics

Volume 9 No. 2 2015, 95-104
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v9i2.6

MULTIVALENCE OF THE BESSEL FUNCTION

Olubunmi A. Fadipe-Joseph[1 §], Bolade O. Moses[2], Timothy O. Opoola[3]

[1,3]Department of Mathematics
University of Ilorin
P.M.B. 1515, Ilorin, NIGERIA

[2]Department of Industrial Mathematics
Landmark University
P.M.B. 1001, Omu Aran, NIGERIA

Abstract: The particular solution of the second order differential equation:

$$z^2 w''(z) + zw'(z) + (z^2 - p^2)w(z) = 0$$

is

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} (\frac{z}{2})^{2n+p}$$

and is called the Bessel function of the first kind of order p.

In this work, the p-valence of the generalised Bessel function of the first kind is established. In particular the coefficient inequalities conditions for starlikeness, convexity and close to convexity are given.

AMS Subject Classification: 30C45
Key Words: multivalent functions, Bessel function, starlike, convex and close to convex functions

1. Introduction

In 2007, Patel and Cho [9] established certaln sufficient condition for close to convexity. Also Neng and Ding [7] established some properties of certain multivalent functions involving Ruscheweyh derivative. In particular; subordination relation, inclusion relations, convolution properties and a sharp coefficient estimate were obtained.

Received: June 18, 2014

§Correspondence author

2. Bessel Functions

The function

$$J_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(p+n+1)} (\frac{z}{2})^{2n+p}, \tag{1}$$

which is defined for all $z \in C$ and where p is an unrestricted real or complex number is called the Bessel function of the first kind of order p. It is obtained from the second order differential equation, known as Bessel equation (see [1], [12], [10]):

$$z^2 w''(z) + zw'(z) + (z^2 - p^2)w(z) = 0. \tag{2}$$

In (2010) Baricz defined the function

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma(p+n+\frac{b+1}{2})} (\frac{z}{2})^{2n+p} \tag{3}$$

is the generalized Bessel function of the first kind of order p. By the ratio test, the radius of convergence of the $w_p(z)$ is infinity and thus is convergent for all $b, c, p, z \in C$. The function generalized the Bessel, modified Bessel, Spherical Bessel and Modified Spherical Bessel functions of the first kind of order p. It is the solution of the generalized Bessel equation of the first kind, defined as

$$z^2 w''(z) + bzw'(z) + [cz^2 - p^2 + (1-b)p]w(z) = 0. \tag{4}$$

The normalized the function defined in (3) is used to study the univalence of it in the unit disk using some established results in geometric function theory.

3. Multivalent Function

Let $f(z)$ be a function analytic in the open unit disk $U = \{z : |z| < 1\}$, if the equation $f(z) = w$ has never more than p solution in U and if there exist some w for which this equation has exactly p-solution then $f(z)$ is said to be p-valent in U. Thus the p-valent function $f(z) = z^p$ maps U onto U but each image point except $w = 0$ has p different preimages. More picturesquely $f(z) = z^p$ can be visualized or viewed as a mapping of U in the z- plane univalently onto a spira-like surface with p layers(sheets) hovering above U in the w-plane.

By A_p we denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}, \tag{5}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 2 (2015)

which are analytic and p-valent in $U, p \in N \sim \{1\}$, $z \in U$.

A function $f(z) \in A_p$ is said to be p valent starlike of order $\alpha(0 < \alpha < p)$ in U. If

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad z \in U. \tag{6}$$

The class of all such functions is denoted by $S^*(p,\alpha)$.

And $S_p^*(\alpha)$ denote the subclass of $S^*(p,\alpha)$ consisting of functons $f(z) \in A_p$ for which

$$|\frac{zf'(z)}{f(z)} - p| < p - \alpha, \tag{7}$$

for $0 < \alpha < p$, $p \in N \sim \{1\}$.

A function $f(z) \in A_p$ is said to be p-valent convex of order $\alpha(0 \leq \alpha < p)$ in U

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, \quad z \in U. \tag{8}$$

The class of all such functions is denoted by $K(p,\alpha)$. It follows from (6) and (8) that $f(z) \in K(p,\alpha) \iff \frac{zf'(z)}{p} \in S^*(p,\alpha)$ [1, 3].

Also, by $K_p(\alpha)$ we denote the subclass of $K(p,\alpha)$ consisting of functions $f(z) \in A_p$ for which

$$|\, 1 + \frac{zf''(z)}{f'(z)} - p \,| < p - \alpha, \quad z \in U. \tag{9}$$

The conditionss (6) and (8) are also equivalent to

$$|\frac{zf'(z)}{f(z)} - 1| < 1 - \alpha \tag{10}$$

and

$$|\, \frac{zf''(z)}{f'(z)} \,| < 1 - \alpha, \quad z \in U, \tag{11}$$

for $0 \leq \alpha < 1$ respectively.

It is well known that a function $f(z)$ analytic in the unit disk U is said to be close to convex if there exist a convex function $g(z)$ such that

$$Re\{\frac{f'(z)}{g'(z)}\} > 0, \tag{12}$$

for all $z \in U$, see [3].

Similarly, for a p-valent function analytic in U the condition holds and is stated in the lemma below

Lemma 1. (see [2]) *A p-valent function of the form* (5) *is said to be p-valent close to convex of order $\alpha(0 \leq \alpha < p)$ if and only if*

$$Re\left\{\frac{f'(z)}{z^{p-1}}\right\} > \alpha, \tag{13}$$

for all $z \in U$.

4. p-Valent Bessel Function

Now, we consider the function $g(z)$ defined by the transformation

$$g(z) = 2^n\Gamma(p + \frac{b+1}{2})z^{-n}w_{p,b,c}(z), \tag{14}$$

where Γ is the well known Euler gamma function and $w_{p,b,c}(z)$ defined in (3) we have

$$g(z) = \sum_{n=0}^{\infty} \frac{(-c)\Gamma(p + \frac{b+1}{2})}{n!\Gamma(p + n + \frac{b+1}{2})}(\frac{z}{2})^{n+p}.$$

Taking into consideration

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)...(a+n-1)$$

the well known Pochhammer (or Appell) symbol defined in term of the Euler gamma function for $a \neq 0, -1, -2, ...$ and $(a)_0 = 1$.

Let $(\kappa)_n = \frac{\Gamma(p+n+\frac{b+1}{2})}{\Gamma(p+\frac{b+1}{2})}$, then we have that

$$g(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!(\kappa)_n}(\frac{z}{2})^{n+p}.$$

Now, set

$$g(z) = \sum_{n=0}^{\infty} b_{n+p}(\frac{z}{2})^{n+p}, \tag{15}$$

for all $z \in C$, where

$$b_{n+p} = \frac{(-c)^n(-1)^n}{n!(\kappa)_n},$$

for all $n \geq 0$. Normalising (15) we have

$$g(z) = (\frac{z}{2})^p + \sum_{k=1}^{\infty} b_{p+k}(\frac{z}{2})^{p+k}, \tag{16}$$

for all $p \in N$.

The function defined in (16) above is called the p-valent Bessel function, the function is analytic and multivalent in the unit disk. The class of the form (16) is denoted by $A_{p,k}$. And it is a subclass of the class of functions defined in (5).

Hence, we prove inequalities conditions for starlikeness, convexity and close to convexity of order alpha for the class $A_{p,k}$.

5. Main Results

Theorem 1. *Let $g(z) \in A_{p,k}$. Then $g(z)$ is convex of order alpha if*

$$p(p-1) + \sum_{k=1}^{\infty} b_{p+k}(p+k)(p+k-\alpha) \leq 1-\alpha.$$

Proof. Suppose $g(z)$ is convex of order alpha, then we need to show that $g(z)$ satisfies the conditions (8) and (11).

Now

$$g'(z) = \frac{p}{2}(\frac{z}{2})^{p-1} + \sum_{k=1}^{\infty} b_{p+k}\frac{p+k}{2}(\frac{z}{2})^{p+k-1},$$

$$zg''(z) = \frac{p(p-1)}{2}(\frac{z}{2})^{p-1} + \sum_{k=1}^{\infty} b_{p+k}\frac{(p+k)(p+k-1)}{2}(\frac{z}{2})^{p+k-1},$$

$$|\frac{zg''(z)}{g'(z)}| = |\frac{\frac{p(p-1)}{2}(\frac{z}{2})^{p-1} + \sum_{k=1}^{\infty} b_{p+k}\frac{(p+k)(p+k-1)}{2}(\frac{z}{2})}{\frac{p}{2}(\frac{z}{2})^{p-1} + \sum_{k=1}^{\infty} b_{p+k}\frac{(p+k)}{2}(\frac{z}{2})^{p+k-1}}|,$$

$$\|\frac{\frac{1}{2}(p(p-1)(\frac{z}{2})^{p-1} + \sum_{k=1}^{\infty} b_{p+k}(p+k)(p+k-1)(\frac{z}{2})^{p+k-1})}{\frac{1}{2}(p(\frac{z}{2})^{p} + \sum_{k=1}^{\infty} b_{p+k}(p+k)(\frac{z}{2})^{p+k-1})}|$$

$$< \frac{p(p-1) + \sum_{k=1}^{\infty} |b_{p+k}|(p+k)(p+k-1)}{p - \sum_{k=1}^{\infty} |b_{p+k}|(p+k)}.$$

The above is bounded by $1-\alpha$, if

$$p(p-1)+\sum_{k=1}^{\infty}|b_{p+k}|(p+k)(p+k-1)\leq(1-\alpha)(p-\sum_{k=1}^{\infty}|b_{p+k}|(p+k)),$$

which is equivalent to

$$p(p-1)+\sum_{k=1}^{\infty}|b_{p+k}|(p+k)(p+k-\alpha)<1-\alpha.$$

Theorem 2. *Let $A_{p,k}(p,\alpha)$ be the class of p-valent starlike function of order alpha, and $A^{*}_{p,k}(\alpha)$ the subclass $A_{p,k}(p,\alpha)$ consisting of function $g(z)\in A_{p,k}$ such that*

$$|\frac{zg'(z)}{g(z)}-p|<p-\alpha.$$

Then

$$\sum_{k=1}^{\infty}|b_{p+k}|(p+k-\alpha)\leq p-\alpha,$$

holds for $0\leq\alpha<p, p\in N, z\in U$.

Proof. We have

$$|\frac{zg'(z)}{g(z)}-p|=|\frac{p(\frac{z}{2})^{p}+\sum_{K=1}^{\infty}b_{p+k}(p+k)(\frac{z}{2})^{p+k}-p(\frac{z}{2})^{p}-\sum_{k=1}^{\infty}pb_{p+k}(\frac{z}{2})^{p+k}}{(\frac{z}{2})^{p}+\sum_{k=1}^{\infty}b_{p+k}(\frac{z}{2})^{p+k}}|$$
$$<\frac{\sum_{k=1}^{\infty}|b_{p+k}|k}{1-\sum_{k=1}^{\infty}|b_{p+k}|}.$$

This last expression is bounded above by $p-\alpha$ if:

$$\sum_{k=1}^{\infty}|b_{p+k}|k<(p-\alpha)(1-\sum_{k=1}^{\infty}|b_{p+k}|).$$

Therefore

$$\sum_{k=1}^{\infty}|b_{p+k}|(p+k-\alpha)\leq p-\alpha.$$

Remark A. The result in Theorem 2 above agrees with Theorem 1 in [4].

Theorem 3. *Let $k(p,\alpha)$ be the class of p-valent Bessel convex function of order alpha, and $k_p(\alpha)$ the subclass of $k(p,\alpha)$ consisting of $g(z)\in A_(p,k)$ such that*

$$|1+\frac{zg''(z)}{g'(z)}-p|<p-\alpha.$$

Then

$$\sum_{k=1}^{\infty} |b_{p+k}|(p+k)(p+k-\alpha) < p-\alpha$$

holds for all $z \in U, 0 \leq \alpha < p, p \in N$.

Proof. We have

$$|1+\frac{zg''(z)}{g'(z)}-p| =|\frac{g'(z)+zg''(z)-pg'(z)}{g'(z)}| < \frac{\sum_{k=1}^{\infty}|b_{p+k}|(p+k)+(p+k)(p+k-1)-(p+k)p}{p-\sum_{k=1}^{\infty}|b_{p+k}|}.$$

The expression above is bounded by $p-\alpha$, if

$$\sum_{k=1}^{\infty} |b_{p+k}|(p+k)+(p+k)(p+k-1)-(p+k)p < [(p-\alpha)(p-\sum_{k=1}^{\infty}|b_{p+k}|(p+k))].$$

The last is equivalent to

$$\sum_{k=1}^{\infty} |b_{p+k}|(p+k)(p+k-\alpha) < p(p-\alpha).$$

Remark B. Theorem 3 is equivalent with Theorem 1 when $p = 1$.

Theorem 4. *Let*

$$g(z) = (\frac{z}{2})^p + \sum_{k=1}^{\infty} +b_{p+k}(\frac{z}{2})^{p+k}$$

be in the class $A_{p,k}$, *then* $g(z)$ *is* p*-valent close to convex of order* $\alpha(0 \leq \alpha < p)$, *if and only if*

$$Re\{\frac{g'(z)}{(\frac{z}{2})^{p-1}}\} > \alpha,$$

for all $z \in U$.

Proof. Suppose

$$g(z) = (\frac{z}{2})^p + \sum_{k=!}^{\infty} b_{p+k}(\frac{z}{2})^{p+k}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 2 (2015)

is in the class $A_{p,k}$ and

$$f(z) = \frac{z^p}{p2^{p-1}},$$

then $f(z)$ is convex because

$$Re\{(1 + \frac{zf''(z)}{f'(z)})\} = Re\{1 + \frac{(p-1)(\frac{z}{2})^{p-1}}{(\frac{z}{2})^{p-1}}\} = p > 0.$$

Now $f'(z) = (\frac{z}{2})^{p-1}$.
Therefore

$$Re\{\frac{g'(z)}{f'(z)}\} = Re\{\frac{g'(z)}{(\frac{z}{2})^{p-1}}\} > 0,$$

by the condition of the theorem.

Conversely: Suppose

$$Re\{\frac{g'(z)}{(\frac{z}{2})^{p-1}}\} > \alpha$$

then

$$\frac{1}{2^{p-1}} Re(\frac{g'(z)}{z^{p-1}}) > 0,$$

such that $\frac{1}{2^{p-1}} > 0$ this implies that $Re(\frac{g'(z)}{z^{p-1}}) > \alpha$.
Hence, by Lemma 1 $g(z)$ i p-valent.

Lemma 2. *Let the non (constant) function $w(z)$ be analytic in U with $w(0) = 0$ if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then $z_0 w'(z_0) = kw(z_0)$ where k is a real number and $k \geq 1$.*

In 2010, Fadipe-Joseph and Opoola (see [5]) defined the class functions $O_p \in A_p$ satisfing the condition

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} < \frac{2(p+\alpha)+1}{1+(p+\alpha)},$$

for $\alpha(0 \leq \alpha < p)$ using lemma 2. Furthermore, the result in [4] is extended in the following theorem.

Theorem 5. *Let $O_{p,k}$ be the class of functions $g(z) \in A_{p,k}$, then $O_{p,k}$ satisfies*

$$Re\{1 + \frac{zg''(z)}{g'(z)}\} < \frac{2^\alpha + 2(p+\alpha)}{2\alpha + (p+\alpha)}.$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 2 (2015)

Proof. Suppose

$$g(z) = (\frac{z}{2})^p + \sum_{k=1}^{\infty} b_{p+k}(\frac{z}{2})^{p+k}$$

and

$$f'(z) = \frac{g'(z)}{(\frac{z}{2})^{p-1}} = \frac{p}{2} + \frac{p+\alpha}{2^{\alpha+1}} w(z).$$

Then

$$f''(z) = \frac{z(p+\alpha)w'(z)}{2^{\alpha+1}}.$$

By Lemma 2 we have $z_0 w'(z_0) = kw(z_0)$, $k \geq 1$, $w(z_0) = pe^{i\theta}$:$\theta \in \Re$, then

$$Re\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\} = Re\{1 + \frac{\frac{z_0(p+\alpha)w'(z)}{2^{\alpha+1}}}{\frac{p}{2} + \frac{p+\alpha}{2^{\alpha+1}} w(z)}\}$$

$$= 1 + Re\{\frac{\frac{(p+\alpha)kw(z_0)}{2^{\alpha+1}}}{\frac{p}{2} + \frac{p+\alpha}{2^{\alpha+1}} w(z)}\} \leq 1 + \frac{\frac{(p+\alpha)p}{2^{\alpha+1}}}{\frac{2^\alpha p + (p+\alpha)p}{2^{\alpha+1}}}$$

If $z \in (0, 1]$, we see

$$Re\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\} = Re\{1 + \frac{\frac{z_0(p+\alpha)w'(z)}{2^{\alpha+1}}}{\frac{p}{2} + \frac{p+\alpha}{2^{\alpha+1}} w(z)}\}$$

$$= 1 + \frac{(p+\alpha)p}{2^\alpha p + (p+\alpha)p} = \frac{2^\alpha p + (p+\alpha)p + (p+\alpha)p}{2^\alpha p + (p+\alpha)p} = \frac{2^\alpha + 2(p+\alpha)}{2^\alpha + (p+\alpha)},$$

which complete the proof.

Corollary 1. *Let* $p = 1$, *then*

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} < \frac{2^\alpha + 2 + 2\alpha}{2^\alpha + 1 + \alpha}.$$

Corrollary 2. *Let* $p = 1$ *and* $\alpha = 0$, *then*

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} < \frac{3}{2}$$

The class $O_{p,k}$ is equivalent to the class of functions $f \in A$ satisfing the condition

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} < \frac{3}{2}$$

for all $z \in U$.

This class was remarked in [9] and was considered earlier in [5] and [8].

6. Conclusion

In this work, the p-valent function of the Generalised Bessel function of the first kind of order p is established. The function was used to establish some results in geometric functions theory.

References

[1] A. Baricz, *Generalized Bessel Function of the First Kind*, Springer Verlag, Berlin-Heidelberg, 2010.

[2] V. B. Daniel, I. Dorca, General integral operator of p-valent functions, *Euro. J. Math. Sci.* , **1**, No. 1 (2012), 105-118.

[3] P. Duren, *Univalent Functions*, Springer Verlag, 1983.

[4] R. M. El-Ashwal, M. K. Aouf, A. O. Moustafa, Starlike and convexity properties for p-valent hyper geometric functions, *Acta. Math.* , **LXX** (2010), 55-64.

[5] O. A. Fadipe-Joseph, T. O. Opoola, On p-valent function, *ICASTOR J. Math. Sci.* , **4**, No. 1 (2010), 83-86.

[6] A. W. Goodman, *Univalent Functions*, Mariner Publishing Co. , Tampa Florida, 1983.

[7] N. Xu, D. Yang, Properties of certain multivalent functions involving Ruscheweyh derivatives, *Aust. J. Math. Anal. App.* , **10**, No. 1 (2013), 1-11.

[8] M. Obradovic, On certain differential operators and some classes of univalent function, *Serija Mathematik*, **6** (1992), 107-112.

[9] J. Patel, N. E Cho On certain sufficient conditions for close-to-convexity, *Applied Mathematics and Computation*, **187** (2007), 369-378.

[10] R. Szasz, P. A. Kupan, About the univalence of the Bessel functions, *Studia UNIV, "BABES-BOLYAT", Mathematica*, **44**, No. 1 (2009), 127-132.

[11] R. Singh, S. Singh, Some sufficient condition for univalent and starlikeness, *colloq. Math.* , **47** (1982), 309-314.

[12] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.