

## COMBINATORIAL RELATIONS OF POWERS

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**Abstract:** Formulas for weighted sums of various powers of complex numbers and identities involving powers and binomial coefficients generalizing the binomial identity are given.

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**Key Words:** power series, binomial theorem, Lagrange polynomial

### 1. Introduction

Evaluation of weighted sums of various powers of numbers is necessary in computational statistics and combinatorial geometry. Identities on the finite sums involving binomial coefficients are numerous and well known (see [1], [2]).

In this paper, we give identities of finite series that generalize the binomial identity  $\sum_{i=0}^n (-1)^i \binom{n}{i} = 0$ .

### 2. Identities

The following propositions give identities involving various powers of complex num-

bers  $x_1, \dots, x_n$  and their differences.

**Proposition 1.** *Let  $x_1, \dots, x_n \in \mathbf{C}$  and  $x_i \neq x_j$  if  $i \neq j$ . Then*

$$\sum_{i=1}^n \frac{x_i^k}{\prod_{j \neq i} (x_i - x_j)} = \begin{cases} 0 & \text{for } 0 \leq k \leq n - 2 \\ 1 & \text{for } k = n - 1 \end{cases}$$

*Proof.* Let  $P_i(x) = \prod_{j \neq i} (x - x_j)$  and  $\sum_{k=0}^{n-1} b_{ik} x^k$  its expansion. It follows  $b_{ik} = (-1)^{n-k-1} \sum_j \prod_{\alpha=1}^{n-k-1} x_{j_\alpha}$  where the sum is taken over  $j_1 < j_2 < \dots < j_{n-k-1}$  and  $j_\alpha \neq i$ . For example,  $b_{i,n-1} = 1$  and  $b_{i,0} = (-1)^{n-1} \prod_{j \neq i} x_j$ . Thus the Lagrange polynomial,  $P(x) = \sum_{i=1}^n \frac{y_i}{P_i(x_i)} P_i(x)$  interpolating  $(x_i, y_i)$ , is expanded as:

$$P(x) = \sum_{k=0}^{n-1} \left( \sum_{i=1}^n \frac{b_{ik} y_i}{P_i(x_i)} \right) x^k$$

If  $y_i = x_i^m$ , then  $P(x) = x^m$  (by the uniqueness) for  $0 \leq m \leq n - 1$ . Hence it follows

$$\sum_{i=1}^n \frac{b_{ik} x_i^m}{\prod_{j \neq i} (x_i - x_j)} = \delta_{km} \tag{1}$$

Substitution of  $b_{i,n-1} = 1$  and  $k = n - 1$  into (1) gives

$$\sum_{i=1}^n \frac{x_i^m}{\prod_{j \neq i} (x_i - x_j)} = \delta_{n-1,m} \quad \text{for } 0 \leq m \leq n - 1. \quad \text{q.e.d.}$$

**Proposition 2.** *If  $x_1, \dots, x_n$  are distinct nonzero complex numbers, then*

$$\sum_{i=1}^n \frac{x_i^k}{\prod_{j \neq i} (x_i - x_j)} = \begin{cases} \frac{(-1)^{n-1}}{\prod_{j=1}^n x_j} & \text{for } k = -1 \\ \sum_{i=1}^n x_i & \text{for } k = n \end{cases}$$

*Proof.* Substitution of  $k = 0$  and  $b_{i0} = (-1)^{n-1} \prod_{j \neq i} x_j$  into (1) gives

$$\sum_{i=1}^n \frac{(-1)^{n-1} \prod_{j \neq i} x_j}{\prod_{j \neq i} (x_i - x_j)} = 1$$

and division of both sides by  $\prod_{j=1}^n x_j$  gives

$$\sum_{i=1}^n \frac{1}{x_i \prod_{j \neq i} (x_i - x_j)} = \frac{(-1)^{n-1}}{\prod_{i=1}^n x_i}$$

Similarly, substitution of  $k = n - 2$  and  $b_{i,n-2} = -\sum_{j \neq i} x_j$  into (1) yields

$$\sum_{i=1}^n \frac{x_i^{n-1} \left(-\sum_{j \neq i} x_j\right)}{\prod_{j \neq i} (x_i - x_j)} = 0$$

and replacing  $\sum_{j \neq i} x_j$  by  $\left(\sum_{j=1}^n x_j\right) - x_i$  gives

$$\sum_{i=1}^n \frac{x_i^n}{\prod_{j \neq i} (x_i - x_j)} = \sum_{i=1}^n x_i \quad \text{q.e.d.}$$

**Proposition 3.**

$$\sum_{i=0}^n (-1)^i \binom{n}{i} i^k = \begin{cases} 0 & \text{for } 1 \leq k \leq n - 1 \\ (-1)^n n! & \text{for } k = n \\ (-1)^n \frac{n(n+1)!}{2} & \text{for } k = n + 1 \end{cases}$$

*Proof.* Since  $\prod_{j \neq i} (i - j) = (-1)^{n-i} (i - 1)!(n - i)!$  it follows from (1) that  $\sum_{i=0}^n \frac{(-1)^i i^k}{i!(n-i)!} = 0$  for  $1 \leq k \leq n - 1$  and

$$\sum_{i=0}^n \frac{(-1)^i i^k}{i!(n-i)!} = \begin{cases} (-1)^n & \text{for } k = n \\ (-1)^n \frac{n(n+1)}{2} & \text{for } k = n + 1 \end{cases} \quad \text{q.e.d.}$$

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**References**

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