International Electronic Journal of Pure and Applied Mathematics

Volume 9 No. 2 2015, 89-90
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v9i2.4

HERON'S FORMULA FOR A SIMPLEX

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Abstract: A generalized Heron's formula of the volume of n-dimensional simplices is derived by applying multi-dimensional scaling.

AMS Subject Classification: 15A, 51A, 52A38
Key Words: Heron, volume, simplex, multi-dimensional scaling

1. Introduction

Heron's formula for the area of a triangle $\sqrt{s(s-a)(s-b)(s-c)}$ and its generalizations to quadrilateral and tetrahedron are well known (See Coxeter [1], [2].) But a natural generalization to the volume of higher dimensional simplices in terms of its edge lengths is not available. We derive a formula for the volume by applying the classical MDS (multi-dimensional scaliong) algorithm.

2. Multi-Dimensional Scaling and Volume Formula

Let Δ be an $(m-1)$-simplex and D the $m \times m$ matrix representing the distances between its vertices. The classical MDS algorithm finds vectors $\mathbf{x}_i \in \mathbf{R}^p$ $(p < m)$ such that $\|\mathbf{x}_i - \mathbf{x}_j\| = D_{ij}$, i.e. Δ can be isometrically embedded onto the simplex generated by $\mathbf{x}_i$'s. Hence the volume of Δ can be computed in terms of $\mathbf{x}_i$'s, which are determined by D. We give notation and a brief description of the MDS algorithm first. For details we refer Seber [3].

Let E be the matrix representing the square distance between vertices, i.e. $E_{ij} =$

Received: October 3, 2014

${D_{ij}}^2$ and H the $m \times m$ matrix with $H_{ij} = \delta_{ij} - 1/m$. H is the centering operator, i.e. the row means of HY (resp. the the column means of YH) is $\mathbf{0}$ for any matrix Y. If V is the matrix whose columns are eigenvectors of $A = -HEH/2$ and Λ the diagonal matrix of eigenvalues of A, i.e. $AV = V\Lambda$, then the column vectors, say $\mathbf{x}_i$, of $V\sqrt{\Lambda}$ satisfy the condition $\|\mathbf{x}_i - \mathbf{x}_j\| = D_{ij}$.

Theorem 1. *Let D be the matrix representing the distances between its vertices. of an $(m-1)$-simplex Δ, X the $m \times m$ matrix the j-th column of which is $\mathbf{x}_j$ (given by MDS) and Y the $m \times (m-1)$ matrix defined by $Y_{ij} = \delta_{ij} - \delta_{im}$. Then the volume of Δ:*

$$vol(\Delta) = \frac{\sqrt{detY^tX^tXY}}{(m-1)!} \tag{1}$$

Proof. We may assume the vertices of Δ are the vectors $\mathbf{v}_i = \mathbf{x}_i - \mathbf{x}_m$ for $i = 1, \ldots, m$. Note XY is the $m \times (m-1)$ matrix the j-th column of which is $\mathbf{v}_j$. Since the volume of the parallelepiped generated by the column vectors of XY is $\sqrt{det(Y^tX^tXY)}$, the volume of Δ equals $\sqrt{det(Y^tX^tXY)}/(m-1)!$ q.e.d.

Example. If D is a distance matrix of a regular m-simplex Δ with edge legths 1, i.e. $D_{ij} = 1$ for $0 \le i \ne j, \le m$, then MDS gives $\mathbf{v}_j = (0, \ldots, 1/\sqrt{2}, \ldots, 0)$ ($1/\sqrt{2}$ in the j-th coordinate) and it follows $vol(\Delta) = \sqrt{m+1}/\sqrt{2}^n m!$

Acknowledgments

Eungchun Cho's work at Seoul National University was supported by The Korea Research Foundation and The Korean Federation of Science and Technology Societies Grant funded by Korea Government (MOEHRD, Basic Research Promotion Fund).

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