

SOME ISOMORPHISM THEOREMS FOR MVD-ALGEBRAS

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Abstract: In the papers [5, 6, 7] many Stone-type duality theorems for the category of locally compact Hausdorff spaces and continuous maps and some of its subcategories were proved. The dual objects in all these theorems are the local contact algebras. In [17] the notion of an MVD-algebra was introduced and it was shown that it is equivalent to the notion of a local contact algebra. In this paper we express the duality theorems mentioned above in a new form using MVD-algebras and appropriate morphisms between them instead of local contact algebras and the respective morphisms.

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1. Introduction

The idea of building a *region-based theory of space* belongs to A. N. Whitehead [18] and T. de Leguna [3]. Survey papers describing various aspects and historical remarks on region-based theory of space are [10, 1, 16, 14]. With the help of the notion of a *region-based topology* (which is called *local contact algebra* (briefly, *LC-algebra*) in [8]) Roeper [15] gave one of the possible first-order axiomatizations for region-based theory of space. The notion of the region-based topology is based on two primitive spatial relations: *contact* and the one-place relation of *limitedness*. An attempt to give a different formulation of the same theory using only one

primitive relation, called *interior parthood*, was made by Mormann [12] but, as it was pointed out in [17], the obtained notion of an *enriched Boolean algebra* was more general than it was necessary; the right axiomatization with only one primitive relation was given in [17], where the appropriate notion of an *MVD-algebra* was introduced. In [5, 6] Dimov defined categories **DHLC**, **DSkeLC**, **DSkePerLC**, **DOpLC** and **DOpPerLC** whose objects are all complete local contact algebras and whose morphisms are some appropriate functions between them. These categories are dual to the categories of all locally compact Hausdorff spaces and, respectively, all continuous maps, all continuous skeletal maps, all skeletal perfect maps, all open maps and all open perfect maps. Here we define five categories **MVDSkeLC**, **MVDSkePerLC**, **MVDOpLC**, **MVDOpPerLC** and **MVDHLC** whose objects are all complete MVD-algebras and whose morphisms are some appropriate functions between them, and we prove that these categories are isomorphic, respectively, to the categories **DSkeLC**, **DSkePerLC**, **DOpLC**, **DOpPerLC** and **DHLC**.

We now fix the notations.

If \mathcal{C} denotes a category, we write $X \in |\mathcal{C}|$ if X is an object of \mathcal{C} , and $f \in \mathcal{C}(X, Y)$ if f is a morphism of \mathcal{C} with domain X and codomain Y .

All lattices will be with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0. We do not require the elements 0 and 1 to be distinct.

If (X, τ) is a topological space and M is a subset of X , we denote by $\text{cl}_{(X, \tau)}(M)$ (or simply by $\text{cl}(M)$) the closure of M in (X, τ) and by $\text{int}_{(X, \tau)}(M)$ (or briefly by $\text{int}(M)$) the interior of M in (X, τ) .

The closed maps and the open maps between topological spaces are assumed to be continuous but are not assumed to be onto. Recall that a map is *perfect* if it is closed and compact (i.e. point inverses are compact sets). A continuous map $f : X \rightarrow Y$ is called *quasi-open* ([11]) if for every non-empty open subset U of X , $\text{int}(f(U)) \neq \emptyset$; a function $f : X \rightarrow Y$ is called *skeletal* if $\text{int}(\text{cl}(f(U))) \neq \emptyset$, for every non-empty open subset U of X .

2. Preliminaries

Definition and Proposition 2.1. Let us recall the notion of *lower adjoint* for posets. Let $\varphi : A \rightarrow B$ be an order-preserving map between posets. If

$$\varphi_{\Lambda} : B \rightarrow A$$

is an order-preserving map satisfying the following condition

$$(\Lambda) \text{ for all } a \in A \text{ and all } b \in B, b \leq \varphi(a) \text{ iff } \varphi_{\Lambda}(b) \leq a$$

(i.e., the pair $(\varphi_{\Lambda}, \varphi)$ forms a *Galois connection between posets B and A*) then we will say that φ_{Λ} is a *lower adjoint* of φ . It is easy to see that condition (Λ) is equivalent to the following two conditions:

($\Lambda 1$) $\forall b \in B, \varphi(\varphi_{\Lambda}(b)) \geq b$;

($\Lambda 2$) $\forall a \in A, \varphi_{\Lambda}(\varphi(a)) \leq a$.

Fact 2.2. ([5]) If A and B are Boolean algebras, $\varphi : A \rightarrow B$ is a Boolean homomorphism, A has all meets and φ preserves them, then $\forall a \in A$ and $\forall b \in B$, $\varphi_{\Lambda}(\varphi(a) \wedge b) = a \wedge \varphi_{\Lambda}(b)$.

Definition 2.3. An algebraic system $(B, 0, 1, \vee, \wedge, *, C)$ is called a *contact Boolean algebra* or, briefly, *contact algebra* (abbreviated as CA or C-algebra) ([8]) if the system $(B, 0, 1, \vee, \wedge, *)$ is a Boolean algebra (where the operation “complement” is denoted by “ $*$ ”) and C is a binary relation on B , satisfying the following axioms:

(C1) If $a \neq 0$ then aCa ;

(C2) If aCb then $a \neq 0$ and $b \neq 0$;

(C3) aCb implies bCa ;

(C4) $aC(b \vee c)$ iff aCb or aCc .

We shall simply write (B, C) for a contact algebra. The relation C is called a *contact relation*. When B is a complete Boolean algebra, we will say that (B, C) is a *complete contact Boolean algebra* or, briefly, *complete contact algebra* (abbreviated as CCA or CC-algebra). If $D \subseteq B$ and $E \subseteq B$, we will write “ DCE ” for “ $(\forall d \in D)(\forall e \in E)(dCe)$ ”.

We will say that two C-algebras (B_1, C_1) and (B_2, C_2) are *CA-isomorphic* iff there exists a Boolean isomorphism $\varphi : B_1 \rightarrow B_2$ such that, for each $a, b \in B_1$, aC_1b iff $\varphi(a)C_2\varphi(b)$. Note that in this paper, by a “Boolean isomorphism” we understand an isomorphism in the category **BoolAlg** of Boolean algebras and Boolean homomorphisms.

A CA (B, C) is called *connected* if it satisfies the following axiom:

(CON) If $a \neq 0, 1$ then aCa^* .

A contact algebra (B, C) is called a *normal contact Boolean algebra* or, briefly, *normal contact algebra* (abbreviated as NCA or NC-algebra) ([4, 9]) if it satisfies the following axioms (we will write “ $-C$ ” for “not C ”):

(C5) If $a(-C)b$ then $a(-C)c$ and $b(-C)c^*$ for some $c \in B$;

(C6) If $a \neq 1$ then there exists $b \neq 0$ such that $b(-C)a$.

A normal CA is called a *complete normal contact Boolean algebra* or, briefly, *complete normal contact algebra* (abbreviated as CNCA or CNC-algebra) if it is a CCA. The notion of normal contact algebra was introduced by Fedorchuk [9] under the name *Boolean δ -algebra* as an equivalent expression of the notion of compingent Boolean algebra of de Vries (see its definition below). We call such algebras “normal contact algebras” because they form a subclass of the class of contact algebras and naturally arise in normal Hausdorff spaces.

For any CA (B, C) , we define a binary relation “ \ll_C ” on B (called *non-tangential inclusion*) by “ $a \ll_C b \leftrightarrow a(-C)b^*$ ”. Sometimes we will write simply

“ \ll ” instead of “ \ll_C ”. This relation is also known in the literature under the following names: “well-inside relation”, “well below”, “interior parthood”, “non-tangential proper part” or “deep inclusion”.

The relations C and \ll are inter-definable. For example, normal contact algebras could be equivalently defined (and exactly in this way they were introduced (under the name of *compingent Boolean algebras*) by de Vries in [4]) as a pair of a Boolean algebra $B = (B, 0, 1, \vee, \wedge, *)$ and a binary relation \ll on B subject to the following axioms:

- ($\ll 1$) $a \ll b$ implies $a \leq b$;
- ($\ll 2$) $0 \ll 0$;
- ($\ll 3$) $a \leq b \ll c \leq t$ implies $a \ll t$;
- ($\ll 4$) $a \ll c$ and $b \ll c$ implies $a \vee b \ll c$;
- ($\ll 5$) If $a \ll c$ then $a \ll b \ll c$ for some $b \in B$;
- ($\ll 6$) If $a \neq 0$ then there exists $b \neq 0$ such that $b \ll a$;
- ($\ll 7$) $a \ll b$ implies $b^* \ll a^*$.

The proof of the equivalence of the two definitions of normal contact algebras is straightforward and analogous to the corresponding statement for proximity spaces (see Theorems 3.9 and 3.11 in [13]). One has just to show that xCy iff $x \ll y^*$.

Obviously, contact algebras could be equivalently defined as a pair of a Boolean algebra B and a binary relation \ll on B subject to the axioms ($\ll 1$)-($\ll 4$) and ($\ll 7$).

It is easy to see that axiom (C5) (resp., (C6)) can be stated equivalently in the form of ($\ll 5$) (resp., ($\ll 6$)).

Definition 2.4. ([15, 8]) An algebraic system

$$\underline{B}_\rho = (B, 0, 1, \vee, \wedge, *, \rho, \mathbb{B})$$

is called a *local contact algebra* (abbreviated as LCA) if

$$(B, 0, 1, \vee, \wedge, *)$$

is a Boolean algebra, ρ is a binary relation on B such that (B, ρ) is a CA, and \mathbb{B} is a subset of B , satisfying the following axioms:

- (BB1) $0 \in \mathbb{B}$;
- (BB2) For $a, b \in B$, $a \leq b$ and $b \in \mathbb{B}$ implies $a \in \mathbb{B}$;
- (BB3) $a, b \in \mathbb{B}$ implies $a \vee b \in \mathbb{B}$;
- (BC1) If $a \in \mathbb{B}$, $c \in B$ and $a \ll_\rho c$ then there exists $b \in \mathbb{B}$ such that $a \ll_\rho b \ll_\rho c$ (see 2.3 for “ \ll_ρ ”);
- (BC2) If $a\rho b$ then there exists an element c of \mathbb{B} such that $a\rho(c \wedge b)$;
- (BC3) If $a \neq 0$ then there exists $b \in \mathbb{B} \setminus \{0\}$ such that $b \ll_\rho a$.

Usually, we shall write simply (B, ρ, \mathbb{B}) for a local contact algebra. We will say that the elements of \mathbb{B} are *bounded* and the elements of $B \setminus \mathbb{B}$ are *unbounded*. When B is a complete Boolean algebra, we will say that (B, ρ, \mathbb{B}) is a *complete local contact algebra* (abbreviated by CLCA).

We will say that two LCAs (B, ρ, \mathbb{B}) and $(B_1, \rho_1, \mathbb{B}_1)$ are *LCA-isomorphic* iff there exists a Boolean isomorphism $\varphi : B \rightarrow B_1$ such that, for $a, b \in B$, $a \rho b$ iff $\varphi(a) \rho_1 \varphi(b)$ and $\varphi(a) \in \mathbb{B}_1$ iff $a \in \mathbb{B}$.

An LCA (B, ρ, \mathbb{B}) is called *connected* if the CA (B, ρ) is connected.

Example 2.5. Recall that a subset F of a topological space (X, τ) is called *regular closed* if $F = \text{cl}(\text{int}(F))$. Clearly, F is regular closed iff it is the closure of an open set.

For any topological space (X, τ) , the collection $RC(X, \tau)$ (we will often write simply $RC(X)$) of all regular closed subsets of (X, τ) becomes a complete Boolean algebra $(RC(X, \tau), 0, 1, \wedge, \vee, *)$ under the following operations: $1 = X, 0 = \emptyset, F^* = \text{cl}(X \setminus F), F \vee G = F \cup G, F \wedge G = \text{cl}(\text{int}(F \cap G))$. The infinite operations are given by the formulas: $\bigvee \{F_\gamma \mid \gamma \in \Gamma\} = \text{cl}(\bigcup \{F_\gamma \mid \gamma \in \Gamma\})$, and $\bigwedge \{F_\gamma \mid \gamma \in \Gamma\} = \text{cl}(\text{int}(\bigcap \{F_\gamma \mid \gamma \in \Gamma\}))$.

It is easy to see that setting $F \rho_{(X, \tau)} G$ iff $F \cap G \neq \emptyset$, we define a contact relation $\rho_{(X, \tau)}$ on $RC(X, \tau)$; it is called a *standard contact relation*. So, $(RC(X, \tau), \rho_{(X, \tau)})$ is a CCA (it is called a *standard contact algebra*). We will often write simply ρ_X instead of $\rho_{(X, \tau)}$. Note that, for $F, G \in RC(X)$, $F \ll_{\rho_X} G$ iff $F \subseteq \text{int}_X(G)$.

Clearly, if (X, τ) is a normal Hausdorff space then the standard contact algebra $(RC(X, \tau), \rho_{(X, \tau)})$ is a complete NCA.

In [17] the following notion was introduced:

Definition 2.6. ([17]) A triple (B, \leq, \ll) is called an *MVD-algebra* if (B, \leq) is a Boolean algebra and the axioms $(\ll 1)$ - $(\ll 6)$ (see 2.3) as well as the following two axioms are satisfied:

$(\ll 4^*)$ $a \ll b$ and $a \ll c$ imply $a \ll b \wedge c$, and

(MVD) If $a \ll 1$ and $b^* \ll a^*$ then $a \ll b$.

When (B, \leq) is a complete Boolean algebra, we will say that (B, \leq, \ll) is a *complete MVD-algebra*.

It follows immediately from the corresponding definitions that normal contact algebras coincide with MVD-algebras satisfying the additional axiom

$(\ll 2')$ $1 \ll 1$.

Proposition 2.7. ([17]) *Let L be a locally compact Hausdorff space. Then*

$$(RC(L), \subseteq, \ll_L),$$

where, for all $F, G \in RC(L)$, $F \ll_L G$ iff F is compact and $F \subseteq \text{int}(G)$, is an MVD-algebra. All such MVD-algebras will be called *standard MVD-algebras*.

Theorem 2.8. ([17]) *The notions of local contact algebra and MVD-algebra are equivalent. More precisely: let κ be the correspondence which assigns to every LCA (B, ρ, \mathbb{B}) an MVD-algebra $\kappa(B, \rho, \mathbb{B}) = (B, \leq_l, \ll_l)$, where $a \leq_l b$ iff $a \wedge b = a$, and*

$$a \ll_l b \text{ iff } a \in \mathbb{B} \text{ and } a \ll_\rho b \quad (1)$$

(see 2.3 for “ \ll_ρ ”); further, let θ be the correspondence which assigns to each MVD-algebra (B, \leq, \ll) an LCA $(B, \rho_m, \mathbb{B}_m) = \theta(B, \leq, \ll)$, where

$$\mathbb{B}_m = \{a \in B \mid a \ll 1\} \quad (2)$$

and, for $a, b \in B$,

$$a \ll_{\rho_m} b \leftrightarrow (\forall c \ll 1)[(c \wedge a) \ll (c^* \vee b)]. \quad (3)$$

(or, equivalently, $a \rho_m b$ iff there exists $c \ll 1$ such that $(c \wedge a) \not\ll (c \wedge b)^*$). Then κ and θ are bijective correspondences between the classes of all LCA's and all MVD-algebras, and $\kappa = \theta^{-1}$.

The following obvious fact was noted in [2].

Fact 2.9. ([2]) Let (X, τ) be a topological space. Then the standard contact algebra $(RC(X, \tau), \rho_{(X, \tau)})$ is connected if and only if the space (X, τ) is connected.

Proposition 2.10. (a) *Every quasi-open map is skeletal.*

(b) ([5]) *Let X be a regular space and $f : X \rightarrow Y$ be a closed map. Then f is quasi-open iff f is skeletal.*

Definition 2.11. ([5]) Let (B, ρ, \mathbb{B}) be a local contact algebra. An ultrafilter u in B is called a *bounded ultrafilter* if $u \cap \mathbb{B} \neq \emptyset$.

Notation 2.12. If \mathbf{K} is a category, then by **InK** (resp., **SuK**) we will denote the category having the same objects as the category \mathbf{K} and whose morphisms are only the injective (resp., surjective) morphisms of \mathbf{K} .

Notation 2.13. If \mathbf{K} is a category whose objects form a subclass of the class of all topological spaces (resp., contact algebras) then we will denote by **KCon** the full subcategory of \mathbf{K} whose objects are all “connected” \mathbf{K} -objects, where “connected” is understood in the usual sense when the objects of \mathbf{K} are topological spaces and in the sense of 2.3 (see the condition (CON) there) when the objects of \mathbf{K} are contact algebras.

3. Isomorphism Theorems for MVD-Algebras

In [5], a category **DSkeLC** was introduced, namely, the objects of the category **DSkeLC** are all complete local contact algebras and its morphisms $\varphi : (A, \rho, \mathbb{B}) \rightarrow$

(B, η, \mathbb{B}') are all complete Boolean homomorphisms satisfying the following conditions:

- (L1) $\forall a, b \in A, \varphi(a)\eta\varphi(b)$ implies apb ;
- (L2) $b \in \mathbb{B}'$ implies $\varphi_{\Lambda}(b) \in \mathbb{B}$ (see 2.1 for φ_{Λ}).

As it was proved in [5], the category **DSkeLC** is dually equivalent to the category **SkeLC** of all locally compact Hausdorff spaces and all continuous skeletal maps between them.

Let us note that (L1) is equivalent to the following condition:

- (L1') $\forall a, b \in A, a \ll_{\rho} b$ implies $\varphi(a) \ll_{\eta} \varphi(b)$.

Definition 3.1. Let us define a category which will be denoted by **MVDSkeLC**. Its objects are all complete MVD-algebras (see 2.6). If (B, \leq, \ll) and (B', \leq', \ll') are two complete MVD-algebras then $\varphi : (B, \leq, \ll) \rightarrow (B', \leq', \ll')$ will be an **MVDSkeLC**-morphism iff $\varphi : (B, \leq) \rightarrow (B', \leq')$ is a complete Boolean homomorphism satisfying the following axioms:

- (S1) For every $a, b \in B, [(\forall c \in B \text{ with } c \ll 1), (c \wedge a \ll c^* \vee b)]$ implies $[(\forall d \in B' \text{ with } d \ll' 1), (d \wedge \varphi(a) \ll' d^* \vee \varphi(b))]$;
- (S2) For all $b \in B', b \ll' 1$ implies $\varphi_{\Lambda}(b) \ll 1$.

Let the composition of two **MVDSkeLC**-morphisms be the usual composition of functions.

It is easy to see that in such a way we have defined a category.

Theorem 3.2. *The categories **DSkeLC** and **MVDSkeLC** are isomorphic; hence the categories **SkeLC** and **MVDSkeLC** are dually equivalent.*

Proof. Let us define two covariant functors $K : \mathbf{DSkeLC} \rightarrow \mathbf{MVDSkeLC}$ and $\Theta : \mathbf{MVDSkeLC} \rightarrow \mathbf{DSkeLC}$.

For every $(B, \rho, \mathbb{B}) \in |\mathbf{DSkeLC}|$ we put $K(B, \rho, \mathbb{B}) = \kappa(B, \rho, \mathbb{B})$ (see 2.8 for κ). Then Theorem 2.8 implies that K is well-defined on the objects of the category **DSkeLC**.

Let $\varphi \in \mathbf{DSkeLC}((B, \rho, \mathbb{B}), (B', \rho', \mathbb{B}'))$. We will prove that the same function $\varphi : B \rightarrow B'$ is an **MVDSkeLC**-morphism between $K(B, \rho, \mathbb{B})$ and $K(B', \rho', \mathbb{B}')$. Since φ is a complete Boolean homomorphism between Boolean algebras B and B' , we need only to check that φ satisfies axioms (S1) and (S2). Using 2.8 and (L1'), this can be easily done. So, we can define:

$$K(\varphi) = \varphi.$$

Then, obviously, $K : \mathbf{DSkeLC} \rightarrow \mathbf{MVDSkeLC}$ is a (covariant) functor.

Let $(B, \leq, \ll) \in |\mathbf{MVDSkeLC}|$. We put

$$\Theta(B, \leq, \ll) = \theta(B, \leq, \ll)$$

(see 2.8 for θ). Then 2.8 implies that Θ is well-defined on the objects of the category **MVDSkeLC**.

Let $\varphi \in \mathbf{MVDSkeLC}((B, \leq, \ll), (B', \leq', \ll'))$. We will show that the same function $\varphi : B \rightarrow B'$ is an **DSkeLC**-morphism between $\Theta(B, \leq, \ll)$ and $\Theta(B', \leq', \ll')$. For doing this it is enough to prove that φ satisfies conditions (L1) and (L2). Using 2.8 and (L1'), this can be easily done. So, we can define:

$$\Theta(\varphi) = \varphi.$$

Then, obviously, $\Theta : \mathbf{MVDSkeLC} \rightarrow \mathbf{DSkeLC}$ is a (covariant) functor.

From the definition of the functors K and Θ and the equalities $\kappa \circ \theta = id$, $\theta \circ \kappa = id$ (see 2.8), we conclude that $K \circ \Theta = Id_{\mathbf{MVDSkeLC}}$ and $\Theta \circ K = Id_{\mathbf{DSkeLC}}$. Hence, the categories **DSkeLC** and **MVDSkeLC** are isomorphic. \square

In [5] a category **DSkePerLC** was introduced, namely, the objects of the category **DSkePerLC** are all complete local contact algebras (see 2.4) and its morphisms $\varphi : (A, \rho, \mathbb{B}) \rightarrow (B, \eta, \mathbb{B}')$ are all **DSkeLC**-morphisms satisfying the following condition:

(L3) $a \in \mathbb{B}$ implies $\varphi(a) \in \mathbb{B}'$.

Obviously, the category **DSkePerLC** is a subcategory of the category **DSkeLC**.

As it was proved in [5], the category **DSkePerLC** is dually equivalent to the category **SkePerLC** of all locally compact Hausdorff spaces and all skeletal perfect maps between them.

Let us note that, by 2.10(b), the morphisms of the category **SkePerLC** are precisely the quasi-open perfect maps (because the perfect maps are closed maps).

Definition 3.3. Let's define a category which will be denoted by **MVDSkePerLC**. Its objects are all complete MVD-algebras (see 2.6). If (B, \leq, \ll) and (B', \leq', \ll') are two complete MVD-algebras then $\varphi : (B, \leq, \ll) \rightarrow (B', \leq', \ll')$ will be an **MVDSkePerLC**-morphism iff $\varphi : (B, \leq) \rightarrow (B', \leq')$ is a complete Boolean homomorphism satisfying the axiom (S2) from 3.1 and the following two additional axioms:

(ES1) For every $a, b \in B$, $a \ll b$ implies $\varphi(a) \ll' \varphi(b)$;

(S3) For all $a \in B$, $a \ll 1$ implies $\varphi(a) \ll' 1$.

Let the composition of two **MVDSkePerLC**-morphisms be the usual composition of functions.

It is easy to see that in such a way we have defined a (non-full) subcategory of the category **MVDSkeLC**.

Theorem 3.4. *The category **DSkePerLC** and the category **MVDSkePerLC** are isomorphic; hence the categories **SkePerLC** and **MVDSkePerLC** are dually equivalent.*

Proof. We will show that the restrictions

$$K_p : \mathbf{DSkePerLC} \longrightarrow \mathbf{MVDSkePerLC}$$

and

$$\Theta_p : \mathbf{MVDSkePerLC} \longrightarrow \mathbf{DSkePerLC}$$

of the functors

$$K : \mathbf{DSkeLC} \longrightarrow \mathbf{MVDSkeLC}$$

and

$$\Theta : \mathbf{MVDSkeLC} \longrightarrow \mathbf{DSkeLC}$$

defined in the proof of Theorem 3.2 are the desired isomorphism functors.

Let $\varphi \in \mathbf{DSkePerLC}((B, \rho, \mathbb{B}), (B', \rho', \mathbb{B}'))$. Then, as it was shown in the proof of 3.2, the same function $\varphi : B \longrightarrow B'$ is an $\mathbf{MVDSkeLC}$ -morphism between MVD-algebras $K(B, \rho, \mathbb{B})$ and $K(B', \rho', \mathbb{B}')$. So, we need only to check that φ satisfies axioms (ES1) and (S3).

Put $K_p(B, \rho, \mathbb{B}) = (B, \leq, \ll)$ and $K_p(B', \rho', \mathbb{B}') = (B', \leq', \ll')$. Then, by 2.8, $a \ll b$ iff $a \in \mathbb{B}$ and $a \ll_\rho b$; also $a \ll' b$ iff $a \in \mathbb{B}'$ and $a \ll_{\rho'} b$. Using (L3), we get easily that (S3) is fulfilled. We will show that (ES1) takes place. Let $a, b \in B$ and $a \ll b$. Then $a \ll_\rho b$ and $a \in \mathbb{B}$. Thus, by (L1'), $\varphi(a) \ll_{\rho'} \varphi(b)$. Since, by (L3), $\varphi(a) \in \mathbb{B}'$, we obtain that $\varphi(a) \ll' \varphi(b)$. We have established that $\varphi \in \mathbf{MVDSkePerLC}(K_p(B, \rho, \mathbb{B}), K_p(B', \rho', \mathbb{B}'))$.

Let $\varphi \in \mathbf{MVDSkePerLC}((B, \leq, \ll), (B', \leq', \ll'))$. We will show that φ satisfies condition (S1). Let $a, b \in B$ and let, for every $c \in B$ with $c \ll 1$, $c \wedge a \ll c^* \vee b$ holds. Take $d \in B'$ with $d \ll' 1$. Then, by (S2), $c = \varphi_\Lambda(d) \ll 1$. Hence $c \wedge a \ll c^* \vee b$. Using (ES1), we obtain that $\varphi(c) \wedge \varphi(a) \ll' (\varphi(c))^* \vee \varphi(b)$. Then (A1) and (\ll 3) imply that $d \wedge \varphi(a) \ll' d^* \vee \varphi(b)$. Hence (S1) is established. Now, as it was shown in the proof of 3.2, the same function $\varphi : B \longrightarrow B'$ is an \mathbf{DSkeLC} -morphism between $\Theta(B, \leq, \ll)$ and $\Theta(B', \leq', \ll')$. So, we need only to prove that φ satisfies condition (L3). This can be done readily, using (S3). The rest follows from Theorem 3.2. \square

In [5] a category \mathbf{DOpLC} was introduced, namely, the objects of the category \mathbf{DOpLC} are all complete local contact algebras (see 2.4) and its morphisms $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ are all \mathbf{DSkeLC} -morphisms satisfying the following condition:

$$(LO) \forall a \in A \text{ and } \forall b \in \mathbb{B}', \varphi_\Lambda(b)\rho a \text{ implies } b\eta\varphi(a).$$

Obviously, \mathbf{DOpLC} is a (non-full) subcategory of the category \mathbf{DSkeLC} .

As it was proved in [5], the category \mathbf{OpLC} of all locally compact Hausdorff spaces and all open maps between them and the category \mathbf{DOpLC} are dually equivalent.

Definition 3.5. Let us define a category which will be denoted by **MVDOpLC**. Its objects are all complete MVD-algebras. If (B, \leq, \ll) and (B', \leq', \ll') are two complete MVD-algebras then $\varphi : (B, \leq, \ll) \rightarrow (B', \leq', \ll')$ will be an **MVDOpLC**-morphism iff φ is an **MVDSkeLC**-morphism (see 3.1) which satisfies the following axiom:

(SO) For all $a \in B$ and $b \in B'$, $b \ll' \varphi(a)$ implies $\varphi_\Lambda(b) \ll a$ (see 2.1 for φ_Λ).

Let the composition of two **MVDOpLC**-morphisms be the usual composition of functions. It is easy to see that in such a way we have defined a category.

Theorem 3.6. *The categories **DOpLC** and **MVDOpLC** are isomorphic; hence the categories **OpLC** and **MVDOpLC** are dually equivalent.*

Proof. We will show that the restrictions $K_o : \mathbf{DOpLC} \rightarrow \mathbf{MVDOpLC}$ and $\Theta_o : \mathbf{MVDOpLC} \rightarrow \mathbf{DOpLC}$ of the functors K and Θ defined in the proof of Theorem 3.2 are the desired isomorphism functors.

Let $\varphi \in \mathbf{DOpLC}((B, \rho, \mathbb{B}), (B', \rho', \mathbb{B}'))$. We will prove that the same function $\varphi : B \rightarrow B'$ is an **MVDOpLC**-morphism between $K_o(B, \rho, \mathbb{B})$ and $K_o(B', \rho', \mathbb{B}')$. Since φ is an **MVDSkeLC**-morphism (see the proof of Theorem 3.2), we need only to check that φ satisfies the axiom (SO).

Put $K(B, \rho, \mathbb{B}) = (B, \leq, \ll)$ and $K(B', \rho', \mathbb{B}') = (B', \leq', \ll')$. Then, by 2.8, $a \ll b$ iff $a \in \mathbb{B}$ and $a \ll_\rho b$; also $a \ll' b$ iff $a \in \mathbb{B}'$ and $a \ll_{\rho'} b$.

For verifying (SO), note first that (LO) can be formulated equivalently as:

$$\forall a \in A \text{ and } \forall b \in \mathbb{B}', b \ll_{\rho'} \varphi(a) \text{ implies } \varphi_\Lambda(b) \ll_\rho a. \quad (4)$$

Let now $a \in B$, $b \in B'$ and $b \ll' \varphi(a)$. Then $b \ll_{\rho'} \varphi(a)$ and $b \in \mathbb{B}'$. Hence, by (LO), $\varphi_\Lambda(b) \ll_\rho a$. Since, by (L2), $\varphi_\Lambda(b) \in \mathbb{B}$, we obtain that $\varphi_\Lambda(b) \ll a$.

Therefore, the functor K_o is well-defined.

Let $\varphi \in \mathbf{MVDOpLC}((B, \leq, \ll), (B', \leq', \ll'))$. We will show that the same function $\varphi : B \rightarrow B'$ is an **DOpLC**-morphism between $\Theta_o(B, \leq, \ll)$ and $\Theta_o(B', \leq', \ll')$. Since, by the proof of Theorem 3.2, φ is an **DSkeLC**-morphism, it is enough to show that φ satisfies condition (LO).

Put $\Theta(B, \leq, \ll) = (B, \rho, \mathbb{B})$ and $\Theta(B', \leq', \ll') = (B', \rho', \mathbb{B}')$.

Let $a \in B$, $b \in \mathbb{B}'$ and $b \ll_{\rho'} \varphi(a)$. Then, by Theorem (2.8), $b \ll' \varphi(a)$. Hence, by (SO), $\varphi_\Lambda(b) \ll a$. This implies that $\varphi_\Lambda(b) \ll_\rho a$. Hence, φ satisfies condition (LO). So, Θ_o is well-defined.

The rest follows from Theorem 3.2. □

In [5], a category **DOpPerLC** was introduced, namely, the objects of the category **DOpPerLC** are all complete local contact algebras (see 2.4) and its morphisms $\varphi : (A, \rho, \mathbb{B}) \rightarrow (B, \eta, \mathbb{B}')$ are all **DSkePerLC**-morphisms satisfying condition (LO).

Clearly, the category **DOpPerLC** is a subcategory of the category **DSkePerLC**.

As it was proved in [5], the category **OpPerLC** all locally compact Hausdorff spaces and all open perfect maps between them and the category **DOpPerLC** are dually equivalent.

Definition 3.7. Let us define a subcategory **MVDOpPerLC** of the category **MVDSkePerLC**. Its objects are all complete MVD-algebras. If (B, \leq, \ll) and (B', \leq', \ll') are two complete MVD-algebras then a **MVDSkePerLC**-morphism

$$\varphi : (B, \leq, \ll) \longrightarrow (B', \leq', \ll')$$

(see 3.3) will be a **MVDOpPerLC**-morphism if it satisfies the axiom (SO) (see 3.5).

It is easy to see that in such a way we have defined a category.

Theorem 3.8. *The category **DOpPerLC** and the category **MVDOpPerLC** are isomorphic; hence the categories **OpPerLC** and **MVDOpPerLC** are dually equivalent.*

Proof. It follows from the proofs of Theorem 3.4 and Theorem 3.6. □

In [7], a category **DInSkeLC** was introduced, namely, the objects of the category **DInSkeLC** are all complete local contact algebras (see 2.4) and, for any two CLCA's (A, ρ, \mathbb{B}) and (B, η, \mathbb{B}') , $\varphi : (A, \rho, \mathbb{B}) \longrightarrow (B, \eta, \mathbb{B}')$ is an **DInSkeLC**-morphism iff φ is an **DSkeLC**-morphism which satisfies the following condition:

$$(LS) \forall a, b \in \mathbb{B}', \varphi_{\Lambda}(a)\rho\varphi_{\Lambda}(b) \text{ implies } a\eta b \text{ (see 2.1 for } \varphi_{\Lambda}).$$

As it was shown in [7], the category **DInSkeLC** is dually equivalent to the category **InSkeLC**.

Note that condition (LS) is equivalent to the condition below:

$$(LS') \forall a, b \in \mathbb{B}', a \ll_{\eta} b \text{ implies } \varphi_{\Lambda}(a) \ll_{\rho} (\varphi_{\Lambda}(b^*))^*.$$

Definition 3.9. Let **MVDInSkeLC** be the category having as objects all complete MVD-algebras and let for any two complete MVD-algebras (B, \leq, \ll) and (B', \leq', \ll') ,

$$\varphi : (B, \leq, \ll) \longrightarrow (B', \leq', \ll')$$

be an **MVDInSkeLC**-morphism iff φ is an **MVDSkeLC**-morphism (see 3.1) which satisfies the following condition:

$$(LS'') \forall a, b \in B' \text{ such that } a, b \ll' 1, a \ll' b \text{ implies } \varphi_{\Lambda}(a) \ll (\varphi_{\Lambda}(b^*))^*.$$

Theorem 3.10. *The category **DInSkeLC** and the category **MVDInSkeLC** are isomorphic; hence the categories **InSkeLC** and **MVDInSkeLC** are dually equivalent.*

Proof. We will show that the restrictions

$$K_r : \mathbf{DInSkeLC} \longrightarrow \mathbf{MVDInSkeLC}$$

and

$$\Theta_r : \mathbf{MVDInSkeLC} \longrightarrow \mathbf{DInSkeLC}$$

of the functors

$$K : \mathbf{DSkeLC} \longrightarrow \mathbf{MVDSkeLC}$$

and

$$\Theta : \mathbf{MVDSkeLC} \longrightarrow \mathbf{DSkeLC}$$

defined in the proof of Theorem 3.2 are the desired isomorphism functors.

Let $\varphi \in \mathbf{DInSkeLC}((B, \rho, \mathbb{B}), (B', \rho', \mathbb{B}'))$. We will show that $\varphi : B \rightarrow B'$ is an $\mathbf{MVDInSkeLC}$ -morphism between MVD -algebras $K(B, \rho, \mathbb{B})$ and $K(B', \rho', \mathbb{B}')$. We need only to check that φ satisfies (LS'') . Let $a, b \in B'$, $a, b \ll' 1$ and $a \ll' b$. Then by 2.8, $a, b \in \mathbb{B}'$ and $a \ll_\rho b$. Since φ satisfy (LS') , $\varphi_\Lambda(a) \ll_\rho (\varphi_\Lambda(b^*))^*$. It follows from (S2), that $\varphi_\Lambda(a) \ll 1$, i.e. $\varphi_\Lambda(a) \ll (\varphi_\Lambda(b^*))^*$. Hence φ is an $\mathbf{MVDInSkeLC}$ -morphism.

Let $\varphi \in \mathbf{MVDInSkeLC}((B, \leq, \ll), (B', \leq', \ll'))$. We will show that the same function $\varphi : B \rightarrow B'$ is a $\mathbf{DInSkeLC}$ -morphism between $\Theta(B, \leq, \ll)$ and $\Theta(B', \leq', \ll')$. Since, by 3.2, we have that φ is an \mathbf{DSkeLC} -morphism, we have only to show that φ satisfies (LS') . Let $a, b \in \mathbb{B}'$ and $a \ll_\eta b$. By, 2.8 $a, b \ll' 1$ and $a \ll' b$. Since φ satisfies (LS'') , we get that $\varphi_\Lambda(a) \ll' (\varphi_\Lambda(b^*))^*$, i.e. $\varphi_\Lambda(a) \in \mathbb{B}$ and $\varphi_\Lambda(a) \ll_\rho (\varphi_\Lambda(b^*))^*$. Then φ satisfy (LS') . Thus φ is a $\mathbf{DInSkeLC}$ -morphism. \square

In [7], a category $\mathbf{DSuSkeLC}$ was introduced, namely, the objects of the category $\mathbf{DSuSkeLC}$ are all complete local contact algebras (see 2.4) and its morphisms $\varphi : (A, \rho, \mathbb{B}) \rightarrow (B, \eta, \mathbb{B}')$, where (A, ρ, \mathbb{B}) and (B, η, \mathbb{B}') are CLCA's, are all \mathbf{DSkeLC} -morphism which satisfy the following condition:

(IS) For every bounded ultrafilter u in (A, ρ, \mathbb{B}) there exists a bounded ultrafilter v in (B, η, \mathbb{B}') such that $\varphi_\Lambda(v)\rho u$ (see 2.11, 2.1 and 2.3 for the notations).

As it was proved in [7], the category $\mathbf{SuSkeLC}$ and the category $\mathbf{DSuSkeLC}$ are dually equivalent.

Definition 3.11. Let's define a category which will be denoted by $\mathbf{MVDSuSkeLC}$. Its objects are all complete MVD -algebras. If (B, \leq, \ll) and (B', \leq', \ll') are two complete MVD -algebras then $\varphi \in \mathbf{MVDSuSkeLC}((B, \leq, \ll), (B', \leq', \ll'))$ iff φ is an $\mathbf{MVDSkeLC}$ -morphism (see 3.1) which satisfies the following axiom:

(IS') For every ultrafilter u in (B, \leq, \ll) such that $\exists c \in u, c \ll 1$, there exists a ultrafilter v in (B', \leq', \ll') such that $\exists c' \in v, c' \ll' 1$, and $\forall a \in u$ and $\forall b \in v$ there exists a $c''_{ab} \in B$ such that $c''_{ab} \ll 1$ and $\varphi_\Lambda(b) \wedge c''_{ab} \not\ll (a \wedge c''_{ab})^*$.

Theorem 3.12. *The category $\mathbf{DSuSkeLC}$ and the category $\mathbf{MVDSuSkeLC}$ are isomorphic; hence the categories $\mathbf{SuSkeLC}$ and $\mathbf{MVDSuSkeLC}$ are dually equivalent.*

Proof. We will show that the restrictions $K_q : \mathbf{DSuSkeLC} \rightarrow \mathbf{MVDSuSkeLC}$ and $\Theta_q : \mathbf{MVDSuSkeLC} \rightarrow \mathbf{DSuSkeLC}$ of the functors

$$K : \mathbf{DSkeLC} \rightarrow \mathbf{MVDSkeLC}$$

and

$$\Theta : \mathbf{MVDSkeLC} \rightarrow \mathbf{DSkeLC}$$

defined in the proof of Theorem 3.2 are the desired isomorphism functors.

Let $\varphi \in \mathbf{DSuSkeLC}((B, \rho, \mathbb{B}), (B', \rho', \mathbb{B}'))$. Since, by the proof of Theorem 3.2, φ is an $\mathbf{MVDSkeLC}$ -morphism, we need only to check that φ satisfies the axiom (IS') . Let u be an ultrafilter in (B, \leq, \ll) such that $\exists c \in u, c \ll 1$. Then, by 2.8, $c \in \mathbb{B}$ and hence u is a bounded ultrafilter in (B, ρ, \mathbb{B}) . Since φ satisfies (IS) , there exists a bounded ultrafilter v in (B', ρ', \mathbb{B}') , such that $\varphi_\Lambda(v)\rho u$, i.e. $\exists c' \in v$, such that $c' \ll' 1$ and $\forall a \in u$ and $\forall b \in v$ there exists a $c''_{ab} \in B$ such that $c''_{ab} \ll 1$ and $\varphi_\Lambda(b) \wedge c''_{ab} \ll (a \wedge c''_{ab})^*$. Hence, φ is an $\mathbf{MVDSuSkeLC}$ -morphism.

Let $\varphi \in \mathbf{MVDSuSkeLC}((B, \leq, \ll), (B', \leq', \ll'))$. We will show that the same function $\varphi : B \rightarrow B'$ is a $\mathbf{DSuSkeLC}$ -morphism between $\Theta_q(B, \leq, \ll)$ and $\Theta_q(B', \leq', \ll')$. We have by 3.2 that φ is an \mathbf{DSkeLC} -morphism. We need to check only that φ satisfies (IS) . Let u be a bounded ultrafilter in (B, ρ, \mathbb{B}) . Then, by 2.8, $\exists c \in u, c \ll 1$. Since φ satisfies (IS') , there exists an ultrafilter v in (B', \leq', \ll') such that $\exists c' \in v, c' \ll' 1$ and $\forall a \in u$ and $\forall b \in v$ there exists an $c''_{ab} \in B$ such that $c''_{ab} \ll 1$ and $\varphi_\Lambda(b) \wedge c''_{ab} \ll (a \wedge c''_{ab})^*$. Hence v is a bounded ultrafilter in (B', ρ', \mathbb{B}') and $\varphi_\Lambda(v)\rho u$. Thus φ is a $\mathbf{DSuSkeLC}$ -morphism. \square

In [7], a category $\mathbf{DSuSkePerLC}$ is introduced, namely, the objects of the category $\mathbf{DSuSkePerLC}$ are all CLCA's (see 2.4) and its morphisms are all injective complete Boolean homomorphisms $\varphi : (A, \rho, \mathbb{B}) \rightarrow (B, \eta, \mathbb{B}')$ satisfying axioms (L1)-(L3).

In [7], a category $\mathbf{DInSkePerLC}$ is introduced, namely, the objects of the category $\mathbf{DInSkePerLC}$ are all CLC-algebras and its morphisms are all $\mathbf{DSkePerLC}$ -morphisms $\varphi : (A, \rho, \mathbb{B}) \rightarrow (B, \eta, \mathbb{B}')$ which satisfy condition (LS);

As it was proved in [7], the category $\mathbf{SuSkePerLC}$ and the category $\mathbf{DSuSkePerLC}$ are dually equivalent. Also, in [7] it was proved that the categories $\mathbf{InSkePerLC}$ and $\mathbf{DInSkePerLC}$ are dually equivalent.

Definition 3.13. Let $\mathbf{MVDInSkePerLC}$ be the category of all complete MVD-algebras and if (B, \leq, \ll) and (B', \leq', \ll') are two complete MVD-algebras then $\varphi : (B, \leq, \ll) \rightarrow (B', \leq', \ll')$ will be an $\mathbf{MVDInSkePerLC}$ -morphism if and only if φ is an $\mathbf{MVDSkePerLC}$ -morphism (see 3.3) which satisfies the following axiom:

$$(CS) \forall a, b \in B', a \ll' b \text{ implies } \varphi_\Lambda(a) \ll (\varphi_\Lambda(b^*))^*.$$

Definition 3.14. We will denote by **MVDSuSkePerLC** the category of all complete MVD-algebras and all injective complete Boolean homomorphisms between them satisfying axioms (ES1), (S2) and (S3) (see 3.1 and 3.3).

Theorem 3.15. (i) *The category **DSuSkePerLC** and the category **MVDSuSkePerLC** are isomorphic; hence the categories **SuSkePerLC** and **MVDSuSkePerLC** are dually equivalent.*

(ii) *The categories **DInSkePerLC** and **MVDInSkePerLC** are isomorphic; hence the category **InSkePerLC** and the category **MVDInSkePerLC** are dually equivalent.*

Proof. (i) It follows immediately from Theorem 3.4.

(ii) Let $\varphi \in \mathbf{DInSkePerLC}((B, \rho, \mathbb{B}), (B', \rho', \mathbb{B}'))$. We will prove that the same function $\varphi : B \rightarrow B'$ is an **MVDInSkePerLC**-morphism between $K(B, \rho, \mathbb{B})$ and $K(B', \rho', \mathbb{B}')$ (see the proof of 3.2 for K). In Theorem 3.4 we have seen that φ is an **MVDSkePerLC**-morphism. Thus we need only to show that φ satisfies condition (CS) of 3.13. Put $K(B, \rho, \mathbb{B}) = (B, \leq, \ll)$ and $K(B', \rho', \mathbb{B}') = (B', \leq', \ll')$ (see 3.4 and 2.8 for the corresponding definitions). Let $a, b \in B'$ and $a \ll' b$. Then, by (1), $a(-\rho')b^*$. Hence, by (LS), $\varphi_\Lambda(a)(-\rho)\varphi_\Lambda(b^*)$, i.e.

$$\varphi_\Lambda(a) \ll_\rho (\varphi_\Lambda(b^*))^*.$$

Since, by (S2), $\varphi_\Lambda(a) \ll 1$ (because $a \ll' 1$), we obtain, using twice (1), that $\varphi_\Lambda(a) \in \mathbb{B}$ and $\varphi_\Lambda(a) \ll (\varphi_\Lambda(b^*))^*$. So, φ is an **MVDInSkePerLC**-morphism.

Let $\varphi \in \mathbf{MVDInSkePerLC}((B, \leq, \ll), (B', \leq', \ll'))$. We will show that the same function $\varphi : B \rightarrow B'$ is an **DInSkePerLC**-morphism between $\Theta(B, \leq, \ll)$ and $\Theta(B', \leq', \ll')$ (see the proof of 3.2 for Θ). For doing this it is enough (by Theorem 3.4) to prove that φ satisfies condition (LS). We will show that φ satisfies condition (LS') which, as we know, is equivalent to condition (LS).

Put $\Theta(B, \leq, \ll) = (B, \rho, \mathbb{B})$ and $\Theta(B', \leq', \ll') = (B', \rho', \mathbb{B}')$.

Let $a, b \in B'$ and $a \ll_{\rho'} b$. Then, using 2.8, (S3), (2.2) and (CS), we obtain that

$$\begin{aligned} (a \ll_{\rho'} b) &\Rightarrow (\forall c \in B \text{ such that } c \ll 1, \varphi(c) \wedge a \ll' \varphi(c^*) \vee b) \Rightarrow \\ &(\forall c \in B \text{ such that } c \ll 1, \varphi_\Lambda(\varphi(c) \wedge a) \ll (\varphi_\Lambda((\varphi(c^*) \vee b^*))^*)) \Leftrightarrow \\ &(\forall c \in B \text{ such that } c \ll 1, c \wedge \varphi_\Lambda(a) \ll (\varphi_\Lambda(\varphi(c) \wedge b^*))^*) \Leftrightarrow \\ &(\forall c \in B \text{ such that } c \ll 1, c \wedge \varphi_\Lambda(a) \ll (c \wedge \varphi_\Lambda(b^*))^*) \Leftrightarrow \\ &(\forall c \in B \text{ such that } c \ll 1, c \wedge \varphi_\Lambda(a) \ll c^* \vee (\varphi_\Lambda(b^*))^*) \Leftrightarrow \\ &(\varphi_\Lambda(a) \ll_\rho (\varphi_\Lambda(b^*))^*). \end{aligned}$$

Therefore, φ satisfies condition (LS). The rest follows from Theorem 3.4. □

In [7], a category **DSuOpPerLC** was introduced, namely, the objects of the category **DSuOpPerLC** are all CLCA's and its morphisms are all injective complete Boolean homomorphisms between them satisfying axioms (L1)-(L3) and (LO). Also, in [7], a category **DInOpPerLC** was introduced, namely, the objects of the category **DInOpPerLC** are all CLCA's and its morphisms are all surjective complete Boolean homomorphisms between them satisfying axioms (L1)-(L3) and (LO). As it was proved in [7], the categories **DSuOpPerLC** and **DInOpPerLC** are dually equivalent to the categories **SuOpPerLC** and **InOpPerLC**, respectively.

Notation 3.16. We will denote by:

- **MVDSuOpPerLC** the category of all complete MVD-algebras and all injective complete Boolean homomorphisms between them satisfying axioms (ES1), (S2), (S3) and (SO) (see 3.1, 3.3 and 3.5).
- **MVDInOpPerLC** the category of all complete MVD-algebras and all surjective complete Boolean homomorphisms between them satisfying axioms (ES1), (S2), (S3) and (SO) (see 3.1, 3.3 and 3.5).

Theorem 3.17. (i) *The category **DSuOpPerLC** and the category **MVDSuOpPerLC** are isomorphic; hence the categories **SuOpPerLC** and **MVDSuOpPerLC** are dually equivalent.*

(ii) *The categories **DInOpPerLC** and **MVDInOpPerLC** are isomorphic; therefore the category **InOpPerLC** and the category **MVDInOpPerLC** are dually equivalent.*

Proof. It follows immediately from Theorems 3.8 and 3.15. □

In [7], a category **DSuOpLC** was introduced, namely, the objects of the category **DSuOpLC** are all CLCA's and its morphisms are all complete Boolean homomorphisms between them satisfying axioms (L1), (L2), (IS) and (LO); Also, in [7], a category **DInOpLC** was introduced, namely, the objects of the category **DInOpLC** are all CLCA's and its morphisms are all surjective complete Boolean homomorphisms between them satisfying axioms (L1), (L2) and (LO).

As it was proved in [7], the categories **InOpLC** and **DInOpLC** are dually equivalent; also, the category **SuOpLC** and the category **DSuOpLC** are dually equivalent.

Notation 3.18. We will denote by:

- **MVDSuOpLC** the category of all complete MVD-algebras and all complete Boolean homomorphisms between them satisfying the axioms (S1), (S2), (IS') and (SO) (see 3.1, 3.11 and 3.5).
- **MVDInOpLC** the category of all complete MVD-algebras and all surjective complete Boolean homomorphisms between them satisfying the axioms (S1), (S2) (SO) (see 3.1 and 3.5).

The next theorem follows immediately from Theorem 3.2, Theorem 3.6 and Theorem 3.12:

Theorem 3.19. *The category $\mathbf{DSuOpLC}$ and the category $\mathbf{MVDSuOpLC}$ are isomorphic; hence the categories \mathbf{SuOpLC} and $\mathbf{MVDSuOpLC}$ are dually equivalent.*

The next theorem follows immediately from Theorem 3.2 and Theorem 3.6:

Theorem 3.20. *The category $\mathbf{DInOpLC}$ and the category $\mathbf{MVDInOpLC}$ are isomorphic; hence the categories \mathbf{InOpLC} and $\mathbf{MVDInOpLC}$ are dually equivalent.*

Definition 3.21. An MVD-algebra (B, \leq, \ll) is called *connected* if it satisfies the following axiom:

(CONA) If $a \neq 0, 1$ then there exists $c \ll 1$ such that $c \wedge a \not\ll a \vee c^*$.

Fact 3.22. Let (L, τ) be a locally compact Hausdorff space. Then the standard MVD-algebra $(RC(L), \subseteq, \ll_L)$ is connected iff the space (L, τ) is connected.

Proof. Let's note that $\forall F, G \in RC(L)$, $F \ll_L G$ iff $F \in CR(L)$ and $F \ll_{\rho_L} G$. Hence $k(RC(L), \rho_L, CR(L)) = (RC(L), \subseteq, \ll_L)$. Let (L, τ) be connected. Then, from 2.9, it follows that $(RC(L), \rho_L)$ is connected. Let $a \in RC(L)$, $a \neq 0, 1$. Then, from (CON), $a \rho_L a^*$. It follows from 2.8 that there exists $c \ll_L 1$ such that $c \wedge a \not\ll_L a \vee c^*$, i.e. $(RC(L), \subseteq, \ll_L)$ is connected.

Let now $(RC(L), \subseteq, \ll_L)$ be connected. Then for every $a \in RC(L)$ such that $a \neq 0, 1$, there exists an $c \ll_L 1$ such that

$$c \wedge a \not\ll_L a \vee c^*.$$

It follows from 2.8 that $a \rho_L a^*$. Hence $(RC(L), \rho_L)$ is connected. Then, it follows from 2.9 that (L, τ) is connected. \square

Notation 3.23. We will denote by:

- $\mathbf{MVDSkePerLCCCon}$ the category of all connected complete MVD-algebras and all complete Boolean homomorphisms between them satisfying axioms (ES1), (S2), (S3) (see 3.3).
- $\mathbf{MVDOpPerLCCCon}$ the category of all connected complete MVD-algebras and all complete Boolean homomorphisms between them satisfying axioms (ES1), (S2), (S3) and (SO) (see 3.1, 3.3 and 3.5).

The next theorem follows immediately from 2.9, 3.22 and 3.4:

Theorem 3.24. *The category $\mathbf{DSkePerLCCCon}$ and the category $\mathbf{MVDSkePerLCCCon}$ are isomorphic; hence the categories $\mathbf{SkePerLCCCon}$ and $\mathbf{MVDSkePerLCCCon}$ are dually equivalent.*

The next theorem follows immediately from 2.9, 3.22 and 3.8:

Theorem 3.25. *The category **DOPPerLCCCon** and the category **MVDOPPerLCCCon** are isomorphic; hence the categories **OPPerLCCCon** and **MVDOPPerLCCCon** are dually equivalent.*

Analogously one can formulate and prove the connected versions of Theorems 3.2 and 3.6.

In [6], a category **DHLC** was introduced, namely, the objects of the category **DHLC** are all complete LC-algebras and its morphisms are all functions $\psi : (A, \rho, \mathbb{B}) \rightarrow (B, \eta, \mathbb{B}')$ between the objects of **DHLC** satisfying the conditions

- (DLC1) $\psi(0) = 0$.
- (DLC2) $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$ for all $a, b \in A$.
- (DLC3) If $a \in \mathbb{B}, b \in A$ and $a \ll_\rho b$, then $(\psi(a^*))^* \ll_\eta \psi(b)$.
- (DLC4) For every $b \in \mathbb{B}'$ there exists $a \in \mathbb{B}$ such that $b \leq \psi(a)$.
- (DLC5) $\psi(a) = \bigvee \{\psi(b) \mid b \in \mathbb{B}, b \ll_\rho a\}$, for every $a \in A$.

Let the composition “ \odot ” of two morphisms $\psi_1 : (A_1, \rho_1, \mathbb{B}_1) \rightarrow (A_2, \rho_2, \mathbb{B}_2)$ and $\psi_2 : (A_2, \rho_2, \mathbb{B}_2) \rightarrow (A_3, \rho_3, \mathbb{B}_3)$ of **DHLC** be defined by the formula

$$\psi_2 \odot \psi_1(a) = \bigvee \{(\psi_2 \circ \psi_1)(b) \mid b \in \mathbb{B}, b \ll_\rho a\}, \tag{5}$$

for every $a \in A$.

As it was proved in [6], the category **DHLC** is dually equivalent to the category **HLC** of all locally compact Hausdorff spaces and all continuous mappings between them.

Definition 3.26. Let **MVDHLC** be the category whose objects are all complete MVD-algebras and whose morphisms are all functions $\psi : (A, \leq, \ll) \rightarrow (B, \leq', \ll')$ between the objects of **MVDHLC** satisfying the conditions

- (MVDLC1) $\psi(0) = 0$.
- (MVDLC2) $\psi(a \wedge b) = \psi(a) \wedge \psi(b)$ for all $a, b \in A$.
- (MVDLC3) If $a, b \in A$ and $a \ll b$, then $\forall c \in B$ with $c \ll' 1$, $(\psi(a^*))^* \wedge c \ll' \psi(b) \vee c^*$.
- (MVDLC4) For every $b \in B$ with $b \ll' 1$ there exists $a \in A$ with $a \ll 1$ such that $b \leq \psi(a)$.
- (MVDLC5) $\psi(a) = \bigvee \{\psi(b) \mid b \ll a\}$, for every $a \in A$.

Let the composition “ \odot ” of two **MVDHLC**-morphisms

$$\psi_i : (A_i, \leq_i, \ll_i) \rightarrow (A_{i+1}, \leq_{i+1}, \ll_{i+1}), \quad i = 1, 2$$

be defined by the formula

$$\psi_2 \odot \psi_1(a) = \bigvee \{\psi_2 \circ \psi_1(b) \mid b \ll_1 a\}, \quad \forall a \in A_1. \tag{6}$$

Theorem 3.27. *The categories **DHLC** and **MVDHLC** are isomorphic; hence the categories **MVDHLC** and **HLC** are dually equivalent.*

Proof. Let us define two covariant functors $P : \mathbf{DHLC} \rightarrow \mathbf{MVDHLC}$ and $Q : \mathbf{MVDHLC} \rightarrow \mathbf{DHLC}$.

For every $(B, \rho, \mathbb{B}) \in |\mathbf{DHLC}|$ we put $P(B, \rho, \mathbb{B}) = k(B, \rho, \mathbb{B})$ (see 2.8 for κ). Then Theorem 2.8 implies that P is well-defined on the objects of the category \mathbf{DHLC} .

Let $\psi \in \mathbf{DHLC}((B, \rho, \mathbb{B}), (B', \rho', \mathbb{B}'))$. We will prove that ψ is a \mathbf{MVDHLC} -morphism between $P(B, \rho, \mathbb{B}) = (B, \leq, \ll)$ and $P(B', \rho', \mathbb{B}') = (B', \leq', \ll')$. It is obvious that ψ satisfies axioms (MVDLC1) and (MVDLC2). Let $a \ll b$. Then $a \in \mathbb{B}$ and $a \ll_{\rho} b$. It follows from (DLC3) that $(\psi(a^*))^* \ll_{\rho'} \psi(b)$. Then, from 2.8 it follows that $\forall c \ll' 1, (\psi(a^*))^* \wedge c \ll' \psi(b) \vee c^*$. Hence ψ satisfies (MVDLC3).

Let $b \ll' 1$. From (DLC4) it follows that there exists an $a \in \mathbb{B}$ such that $b \leq \psi(a)$. Hence $a \ll 1$ and $b \leq \psi(a)$, i.e. ψ satisfies (MVDLC4).

Let $a \in B$. Then

$$\psi(a) = \bigvee \{ \psi(b) \mid b \in \mathbb{B}, b \ll_{\rho} a \} = \bigvee \{ \psi(b) \mid b \ll a \}.$$

Hence ψ satisfy (MVDLC5). Therefore

$$\psi \in \mathbf{MVDHLC}((B, \leq, \ll), (B', \leq', \ll')).$$

So, we can define $P(\psi) = \psi$.

Let $\psi_i \in \mathbf{DHLC}((B_i, \rho_i, \mathbb{B}_i), (B_{i+1}, \rho_{i+1}, \mathbb{B}_{i+1}))$ and $P(\psi_i) = \varphi_i, i = 1, 2$. We have that $\forall a \in B_1$,

$$\begin{aligned} (\varphi_2 \odot \varphi_1)(a) &= \bigvee \{ (\varphi_2 \circ \varphi_1)(b) \mid b \ll_1 a \} = \\ &= \bigvee \{ (\psi_2 \circ \psi_1)(b) \mid b \in \mathbb{B}_1, b \ll_{\rho_1} a \} = (\psi_2 \odot \psi_1)(a) = \\ &= (P(\psi_2 \odot \psi_1))(a). \end{aligned}$$

Since, obviously, P preserves the identities, we get that

$$P : \mathbf{DHLC} \rightarrow \mathbf{MVDHLC}$$

is a (covariant) functor.

Let $(B, \leq, \ll) \in |\mathbf{MVDHLC}|$. We put

$$Q(B, \leq, \ll) = \theta(B, \leq, \ll)$$

(see 2.8 for θ). Then Theorem 2.8 implies that Q is well-defined on the objects of the category \mathbf{MVDHLC} .

Let $\psi \in \mathbf{MVDHLC}((B, \leq, \ll), (B', \leq', \ll'))$. We will prove that ψ is a \mathbf{DHLC} -morphism between $Q(B, \leq, \ll) = (B, \rho, \mathbb{B})$ and $Q(B', \leq', \ll') = (B', \rho', \mathbb{B}')$. It is obvious that ψ satisfies axioms (DLC1) and (DLC2).

Let $a \in \mathbb{B}$, $b \in B$ and $a \ll_{\rho} b$. Hence $a \ll b$. It follows from (MVDLC3) that $\forall c \ll' 1$, $(\psi(a^*))^* \wedge c \ll' \psi(b) \vee c^*$. By 2.8, we get that $(\psi(a^*))^* \ll_{\rho'} \psi(b)$. Then ψ satisfies (DLC3).

Let $b \in \mathbb{B}'$. Then $b \ll' 1$. It follows from (MVDLC4) that $\exists a \ll 1$, such that $b \leq' \psi(a)$. Hence $a \in \mathbb{B}$ and $b \leq' \psi(a)$. Then ψ satisfies (DLC4).

Let $a \in B$. Then from (MVDLC5) we get that

$$\psi(a) = \bigvee \{ \psi(b) \mid b \ll a \} = \bigvee \{ \psi(b) \mid b \in \mathbb{B}, b \ll_{\rho} a \}.$$

Therefore $\psi \in \mathbf{DHLC}((B, \rho, \mathbb{B}), (B', \rho', \mathbb{B}'))$. So, we can define $Q(\psi) = \psi$.

Let $\varphi_i \in \mathbf{MVDHLC}((B_i, \leq_i, \ll_i), (B_{i+1}, \leq_{i+1}, \ll_{i+1}))$, $Q(\varphi_i) = \psi_i$, $i = 1, 2$. We have that $\forall a \in B_1$,

$$(\psi_2 \odot \psi_1)(a) = \bigvee \{ (\psi_2 \circ \psi_1)(b) \mid b \in \mathbb{B}_1, b \ll_{\rho_1} a \} =$$

$$\bigvee \{ (\varphi_2 \circ \varphi_1)(b) \mid b \ll_1 a \} = (\varphi_2 \odot \varphi_1)(a) = Q(\varphi_2 \odot \varphi_1)(a).$$

Since, obviously, Q preserves the identities, we get that

$$Q : \mathbf{MVDHLC} \longrightarrow \mathbf{DHLC}$$

is a (covariant) functor.

From the definition of the functors P and Q and the equalities $\kappa \circ \theta = id$, $\theta \circ \kappa = id$ (see 2.8), we conclude that $P \circ Q = Id_{\mathbf{MVDHLC}}$ and $Q \circ P = Id_{\mathbf{DHLC}}$. Hence, the categories \mathbf{DHLC} and \mathbf{MVDHLC} are isomorphic. \square

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References

- [1] Bennett, B. and Düntsch, I., Axioms, Algebras and Topology, Aiello, M. and Pratt-Hartmann, I. and van Benthem, J., *Handbook of Spatial Logics*, Springer-Verlag (Berlin Heidelberg), 2007, 99–160.
- [2] L. Biacino, G. Gerla, Connection structures: Grzegorzczuk’s and Whitehead’s definition of point, *Notre Dame Journal of Formal Logic*, 37:431–439, 1996.
- [3] T. de Laguna, Point, line and surface as sets of solids, *The Journal of Philosophy*, 19:449–461, 1922.

- [4] H. de Vries, Compact spaces and compactifications, an algebraic approach, *Van Gorcum*, The Netherlands, 1962.
- [5] Dimov, G., Some generalizations of the Fedorchuk Duality Theorem - I *Topology Appl.*, 156:728–746, 2009.
- [6] Dimov, G., A de Vries-type duality theorem for the category of locally compact spaces and continuous maps - I, *Acta Math. Hungarica*, 129:314–349, 2010.
- [7] Dimov, G., A de Vries-type duality theorem for the category of locally compact spaces and continuous maps - II, *Acta Math. Hungarica*, 130:50–77, 2011.
- [8] G. Dimov and D. Vakarelov, Contact algebras and region-based theory of space: a proximity approach – I *Fund. Informaticae*, 74(2-3):209–249, 2006.
- [9] V. V. Fedorchuk, Boolean δ -algebras and quasi-open mappings, *Sibirsk. Mat. Ž.* 14 (5) (1973) 1088–1099: = *Siberian Math. J.* 14 (1973), 759–767 (1974).
- [10] Gerla, G., Pointless geometries, Buekenhout, F., *Handbook of Incidence Geometry*, 1015–1031, Elsevier Science B.V., 1995.
- [11] S. Mardešić, P. Papic, Continuous images of ordered compacta, the Suslin property and dyadic compacta, *Glasnik mat.-fis. i astronom.* 17 (1-2):3–25, 1962.
- [12] T. Mormann, Continuous lattices and Whiteheadian theory of space, *Logic and Logical Philosophy* 6:35–54, 1998.
- [13] S. A. Naimpally, B. D. Warrack, Proximity Spaces, *Cambridge University Press*, Cambridge, 1970.
- [14] Pratt-Hartmann, I., First-Order Mereotopology, Aiello, M. and Pratt-Hartmann I. and van Benthem, J., *Handbook of Spatial Logics*, Springer-Verlag (Berlin Heidelberg), 13–97, 2007.
- [15] P. Roeper, Region-based topology, *Journal of Philosophical Logic* 26:251–309, 1997.
- [16] Vakarelov, D. Region-Based Theory of Space: Algebras of Regions, Representation Theory and Logics, Dov Gabbay et al., *Mathematical Problems from Applied Logics. New Logics for the XXIst Century. II*, Springer-Verlag (Berlin Heidelberg), 267–348, 2007.
- [17] D. Vakarelov, G. Dimov, I. Düntsch, B. Bennett, A proximity approach to some region-based theories of space, *J. Applied Non-Classical Logics* 12 (3-4):527–559, 2002.

- [18] A. N. Whitehead, *Process and reality*, MacMillan, New York, 1929.

