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HANKEL DETERMINANT FOR ANALYTIC FUNCTIONS WITH RESPECT TO OTHER POINTS

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Abstract: This paper is concerned with the estimate of second Hankel determinant for the classes of analytic functions whose derivative has a positive real part, analytic functions with respect to conjugate points and with respect to symmetric conjugate points in the unit disc $\mathbb{U} = \{z : |z| < 1\}$.

AMS Subject Classification: 30C45
Key Words: analytic functions, starlike functions, convex functions, functions whose derivative has a positive real part, starlike functions with respect to conjugate points, starlike functions with respect to symmetric conjugate points, convex functions with respect to conjugate points, convex functions with respect to symmetric conjugate points, Hankel determinant

1. Introduction, Definitions and Preliminaries

We let $\mathcal{A}$ to denote the class of functions analytic in $\mathbb{U}$ and having the power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

in the unit disc $\mathbb{U} = \{z : |z| < 1\}$. Let $\mathcal{S}$ be the class of functions $f(z) \in A$ and

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International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 2 (2015)

univalent in $\mathbb{U}$.

In 1976, Noonan and Thomas [17] stated the q^{th} Hankel determinant of f for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by sevarl authors in the literature [17]. For example, Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions in $\mathbb{U}$ with bounded boundary. Later, Ehrenborg [3] considered the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some its properties were discussed by thoroughly by Hayman in [9].

Also, Hankel determinant was studied by various authors including Hayman [9] and Pommerenke [19]. Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $\left|a_3 - \mu a_2^2\right|$ where μ is real and $f \in \mathbb{U}$. Ali [1] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional. For our discussion in this paper, the Hankel determinant for the case $q = 2$ and $n = 2$ are being considered

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$

Janteng, Halim and Darus [10] have determined the functional $\left|a_2 a_4 - a_3^2\right|$ and found a sharp bound for the functions f in the subclass RT of $\mathbb{U}$, consisting of functions whose derivative has a positive real part studied by Mac Gregor [16]. In their work, they have shown that if $f \in RT$ then $\left|a_2 a_4 - a_3^2\right| \leq \frac{4}{9}$. The same authors[10] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses namely, starlike and convex denoted by ST and CV of $\mathbb{U}$ and have shown that $\left|a_2 a_4 - a_3^2\right| \leq 1$ and $\left|a_2 a_4 - a_3^2\right| \leq \frac{1}{8}$ respectively. R.M. El-Ashwah and D.K. Thomas[4] defined the following classes:

$$S_c^* = \left\{ f(z) \in \mathcal{A} : Re\left\{ \frac{2zf'(z)}{f(z) + \overline{f(\overline{z})}} \right\} > 0, z \in \mathbb{U} \right\},$$

$$S_{sc}^* = \left\{ f(z) \in \mathcal{A} : Re\left\{ \frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}} \right\} > 0, z \in \mathbb{U} \right\}.$$

Functions in the classes S_c^* are called starlike functions with respect to conjugate points and S_{sc}^* are called starlike functions with respect to symmetric conjugate points.

Again Janteng et al. [10] introduced the following classes:

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 2 (2015)

$$K_c = \left\{ f(z) \in \mathcal{A} : Re \left\{ \frac{2\,(zf'(z))'}{\left(f(z) + \overline{f(\overline{z})}\right)'} \right\} > 0, z \in \mathbb{U} \right\},$$

$$K_{sc} = \left\{ f(z) \in \mathcal{A} : Re \left\{ \frac{2\,(zf'(z))'}{\left(f(z) - \overline{f(-\overline{z})}\right)'} \right\} > 0, z \in \mathbb{U} \right\}.$$

Functions in the classes K_c are called convex functions with respect to conjugate points and K_{sc} are called convex functions with respect to symmetric conjugate points.

Obviously the functions in these classes are univalent. Various subclasses of analytic functions with respect to conjugate poins and with respect to symmetric conjugate points were widely investigated by various authors including Dahar[11], Selvaraj [22], Ravichandran [20] Tang and Deng [8]. Motivated by the above mentioned results obtained by different authors in this direction, we seek upper bound of the function $\left|a_2a_4 - a_3^2\right|$ for functions belonging to the classes $S_c^{*(\alpha)}$, $S_{sc}^{*(\alpha)}$, $K_c^{(\alpha)}$ and $K_{sc}^{(\alpha)}$.

$S_c^{*(\alpha)}$ denotes the subclass of functions $f\,(z) \in A$ and satisfying the condition

$$Re \left[\alpha f'(z) + (1-\alpha) \left(\frac{2zf'(z)}{f(z) + \overline{f(\overline{z})}} \right) \right] > 0, \tag{2}$$

The following observations are obvious:

1. $S_c^{*(1)} \equiv RT$.
2. $S_c^{*(0)} \equiv S_c^*$, the class of starlike functions with respect to conjugate points introduced by Gagandeep Singh [6].

$S_{sc}^{*(\alpha)}$ denotes the subclass of functions $f\,(z) \in A$ and satisfying the condition

$$Re \left[\alpha f'(z) + (1-\alpha) \left(\frac{2zf'(z)}{f(z) - \overline{f(-\overline{z})}} \right) \right] > 0, \tag{3}$$

The following observations are obvious:

1. $S_{sc}^{*(1)} \equiv RT$.
2. $S_{sc}^{*(0)} \equiv S_{sc}^*$, the class of starlike functions with respect to symmetric conjugate points introduced by Gagandeep Singh [6].

$K_c^{(\alpha)}$ denotes the subclass of functions $f(z) \in A$ and satisfying the condition

$$Re\left[\alpha f'(z) + (1-\alpha)\left(\frac{2\,(zf'(z))'}{\left(f(z)+\overline{f(\overline{z})}\right)'}\right)\right] > 0, \tag{4}$$

The following observations are obvious:

1. $K_c^{(1)} \equiv RT$.
2. $K_c^{(0)} \equiv K_c$, the class of convex functions with respect to conjugate points introduced by Gagandeep Singh [6].

$K_{sc}^{(\alpha)}$ denotes the subclass of functions $f(z) \in A$ and satisfying the condition

$$Re\left[\alpha f'(z) + (1-\alpha)\left(\frac{2\,(zf'(z))'}{\left(f(z)-\overline{f(-\overline{z})}\right)'}\right)\right] > 0, \tag{5}$$

The following observations are obvious:

1. $K_{sc}^{(1)} \equiv RT$.
2. $K_{sc}^{(0)} \equiv K_{sc}$, the class of convex functions with respect to symmetric conjugate points introduced by Gagandeep Singh [6].

2. Preliminary Results

Let P be the family of all functions p analytic in $\mathbb{U}$ for which $Re\,(p(z)) > 0$ and

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \forall z \in \mathbb{U}. \tag{6}$$

Lemma 2.1. (see [19]) *We have* $|p_k| \leq 2 \quad (k = 1, 2, 3, \ldots)$.

Lemma 2.2. *If* $p \in P$, *then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right) x,$$

$$4p_3 = p_1^3 + 2p_1\left(4 - p_1^2\right) x - p_1\left(4 - p_1^2\right) x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z,$$

for some x *and* z *satisfying* $|x| \leq 1$ *and* $p_1 \in [0, 2]$.

This result was proved by Libera and Zlotkiewiez [14, 15].

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 2 (2015)

3. Main Results

Theorem 3.1. *If $f(z) \in S_c^{*(\alpha)}$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{(2+\alpha)^2}. \tag{7}$$

Proof. Since $S_c^{*(\alpha)}$ denotes the subclass of functions $f(z) \in A$ and satisfying the condition, so from (3)

$$Re\left[\alpha f'(z) + (1-\alpha)\left(\frac{2zf'(z)}{f(z)+\overline{f(\overline{z})}}\right)\right] > 0. \tag{8}$$

$$\begin{cases} a_2 = \dfrac{p_1}{1+\alpha} \\ a_3 = \dfrac{p_2}{2+\alpha} + \dfrac{(1-\alpha)\,p_1^2}{(2+\alpha)(1+\alpha)^2} \\ a_4 = \dfrac{p_3}{3+\alpha} + \dfrac{(1-\alpha)\,p_1p_1}{(1+\alpha)(2+\alpha)(3+\alpha)} + \dfrac{(1-\alpha)(1-4\alpha)\,p_1^3}{(1+\alpha)^3(2+\alpha)(3+\alpha)} \end{cases} \tag{9}$$

$$a_2a_4 - a_3^2$$

$$= \frac{1}{C(\alpha)}\left[\begin{array}{l} (1+\alpha)^3(2+\alpha)^2\,p_1p_3 \\ + \left[3(1-\alpha)(1+\alpha)^2(2+\alpha) - 2(1-\alpha)(1+\alpha)^2(3+\alpha)\right]p_1^2p_2 \\ + \left[(1-\alpha)(1-4\alpha)(2+\alpha) - (1-\alpha)^2(3+\alpha)\right]p_1^4 \\ - (1+\alpha)^4(3+\alpha)\,p_2^2 \end{array}\right], \tag{10}$$

where

$$C(\alpha) = (1+\alpha)^4(2+\alpha)^2(3+\alpha)$$

Using Lemma 2.1 and Lemma 2.2 in (10).

$$\left|a_2a_4 - a_3^2\right| = \frac{1}{4C(\alpha)}$$

$$\Big|\Big[(1+\alpha)^3(2+\alpha)^2 + 2\alpha(1-\alpha)(1+\alpha)^2 - 4(1-\alpha)$$

$$\left(3\alpha^2+5\alpha+1\right) - (1+\alpha)^4(3+\alpha)\Big]p_1^4 + \Big[2(1+\alpha)^3(2+\alpha)^2$$

$$+2\alpha(1-\alpha)(1+\alpha)^2 - 2(1+\alpha)^4(3+\alpha)\Big]p_1^2\left(4-p_1^2\right)x$$

$$-\left[(1+\alpha)^3(2+\alpha)^2p_1^2+(1+\alpha)^4(3+\alpha)\left(4-p_1^2\right)\right]\left(4-p_1^2\right)x^2$$
$$+2(1+\alpha)^3(2+\alpha)^2p_1\left(4-p_1^2\right)\left(1-|x|^2\right)z\Big|.$$

Assume that $p_1=p$ and $p\in[0,2]$, using triangular inequality and $|z|\leq 1$, we have

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\ &\leq\frac{1}{4C(\alpha)}\Big[\Big[(1+\alpha)^3(2+\alpha)^2+2\alpha(1-\alpha)(1+\alpha)^2-4(1-\alpha)\\ &\left(3\alpha^2+5\alpha+1\right)-(1+\alpha)^4(3+\alpha)\Big]p^4+2(1+\alpha)^2\\ &\left[(1+\alpha)(2+\alpha)^2+\alpha(1-\alpha)-(1+\alpha)^2(3+\alpha)\right]p^2\left(4-p^2\right)x\\ &+(1+\alpha)^3\left[(2+\alpha)^2p^2+(1+\alpha)(3+\alpha)\left(4-p^2\right)\right]\left(4-p^2\right)x^2\\ &+2(1+\alpha)^3(2+\alpha)^2p\left(4-p^2\right)\left(1-|x|^2\right)\Big].\end{aligned}$$

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\ &\leq\frac{1}{4C(\alpha)}\Big[\Big[(1+\alpha)^3(2+\alpha)^2+2\alpha(1-\alpha)(1+\alpha)^2-4(1-\alpha)\\ &\left(3\alpha^2+5\alpha+1\right)-(1+\alpha)^4(3+\alpha)\Big]p^4\\ &+2(1+\alpha)^3(2+\alpha)^2p\left(4-p^2\right)+2(1+\alpha)^2\\ &\left[(1+\alpha)(2+\alpha)^2+\alpha(1-\alpha)-(1+\alpha)^2(3+\alpha)\right]p^2\left(4-p^2\right)\delta\\ &+(1+\alpha)^3\left[(2+\alpha)^2p^2-2(2+\alpha)^2p+(1+\alpha)(3+\alpha)\left(4-p^2\right)\right]\\ &\left(4-p^2\right)\delta^2\Big].\end{aligned}$$

Therefore

$$\left|a_2a_4-a_3^2\right|\leq\frac{1}{4C(\alpha)}F(\delta)$$

where $\delta=|x|\leq 1$ and

$$\begin{aligned}F(\delta)=&\Big[(1+\alpha)^3(2+\alpha)^2+2\alpha(1-\alpha)(1+\alpha)^2-4(1-\alpha)\\ &\left(3\alpha^2+5\alpha+1\right)-(1+\alpha)^4(3+\alpha)\Big]p^4+2(1+\alpha)^3\\ &(2+\alpha)^2p\left(4-p^2\right)+2(1+\alpha)^2\\ &\left[(1+\alpha)(2+\alpha)^2+\alpha(1-\alpha)-(1+\alpha)^2(3+\alpha)\right]p^2\left(4-p^2\right)\delta\end{aligned}$$

$$+\left(1+\alpha\right)^{3}\left[\left(2+\alpha\right)^{2}p^{2}-2\left(2+\alpha\right)^{2}p+\left(1+\alpha\right)\left(3+\alpha\right)\left(4-p^{2}\right)\right]$$
$$\left(4-p^{2}\right)\delta^{2}.$$

is an increasing function. Therefore Max F(δ)= F(1).
Consequently

$$\left|a_2a_4-a_3^2\right|\leq\frac{1}{4C\left(\alpha\right)}G\left(p\right), \tag{11}$$

where $G\left(p\right)=F\left(1\right)$.
So

$$G\left(p\right)=A\left(\alpha\right)p^{4}-4B\left(\alpha\right)p^{2}+16\left(1+\alpha\right)^{4}\left(3+\alpha\right),$$

where

$$A\left(\alpha\right)=2\left(7\alpha^{3}-7\alpha^{2}+5\alpha+1\right),$$

and

$$B\left(\alpha\right)=\left(\alpha^{5}+9\alpha^{4}+2\alpha^{3}+11\alpha^{2}+2\alpha\right),$$

Now

$$G'\left(p\right)=4A\left(\alpha\right)p^{3}-8B\left(\alpha\right)p.$$

and

$$G''\left(p\right)=12A\left(\alpha\right)p^{2}-8B\left(\alpha\right).$$

$G'\left(p\right)=0$ gives

$$4p\left[A\left(\alpha\right)p^{2}-2B\left(\alpha\right)\right]=0.$$

$G''\left(p\right)$ is negative at

$$p=\sqrt{\frac{\alpha^{5}+9\alpha^{4}+2\alpha^{3}+11\alpha^{2}+2\alpha}{7\alpha^{3}-7\alpha^{2}+5\alpha+1}}=p'.$$

So $MaxG\left(p\right)=G\left(p'\right)$.Hence from (11), we obtain (3.1).

□

The result is sharp for $p_1=p', p_2=p_1^2-2$ and $p_3=p_1\left(p_1^2-3\right)$.
For $\alpha=1$ and $\alpha=0$ respectively, we obtain the following results:

Corollary 3.2. *If $f\left(z\right)\in RT$,then*

$$\left|a_2a_4-a_3^2\right|\leq\frac{4}{9}.$$

Remark 3.3. For the choice of $\alpha=1$, The result coincide with those of A.Janteng, S.A.Halim and M.Darus [12].

Corollary 3.4. *If $f\left(z\right)\in S_c^*$,then*

$$\left|a_2a_4-a_3^2\right|\leq 1.$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 9, No. 2 (2015)

Remark 3.5. For the choice of $\alpha = 0$, The result coincide with those of Gagandeep Singh [6].

Theorem 3.6. *If $f \in S_{sc}^{*(\alpha)}$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{(2+\alpha)^2}. \tag{12}$$

The result is sharp for $p_1 = p', p_2 = p_1^2 - 2$ and $p_3 = p_1\left(p_1^2 - 3\right)$. For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 3.7. *If $f(z) \in RT$,then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9}.$$

Corollary 3.8. *If $f(z) \in S_{sc}^{*}$,then*

$$\left|a_2a_4 - a_3^2\right| \leq 1.$$

Theorem 3.9. *If $f \in K_c^{(\alpha)}$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9(2-\alpha)^2}. \tag{13}$$

The result is sharp for $p_1 = p', p_2 = p_1^2 - 2$ and $p_3 = p_1\left(p_1^2 - 3\right)$. For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 3.10. *If $f(z) \in RT$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9}.$$

Corollary 3.11. *If $f(z) \in K_c$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{9}.$$

Theorem 3.12. *If $f \in K_{sc}^{(\alpha)}$, then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9(2-\alpha)^2}. \tag{14}$$

The result is sharp for $p_1 = p', p_2 = p_1^2 - 2$ and $p_3 = p_1\left(p_1^2 - 3\right)$. For $\alpha = 1$ and $\alpha = 0$ respectively,we obtain the following results:

Corollary 3.13. *If $f(z) \in RT$,then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{4}{9}.$$

Corollary 3.14. *If* $f(z) \in K_{sc}$,*then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{9}.$$

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