HANKEL DETERMINANT FOR ANALYTIC FUNCTIONS
WITH RESPECT TO OTHER POINTS

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Abstract: This paper is concerned with the estimate of second Hankel determinant
for the classes of analytic functions whose derivative has a positive real part, analytic
functions with respect to conjugate points and with respect to symmetric conjugate
points in the unit disc \( U = \{ z : |z| < 1 \} \).

AMS Subject Classification: 30C45
Key Words: analytic functions, starlike functions, convex functions, functions
whose derivative has a positive real part, starlike functions with respect to conjugate
points, starlike functions with respect to symmetric conjugate points, convex
functions with respect to conjugate points, convex functions with respect to sym-
meteric conjugate points, Hankel determinant

1. Introduction, Definitions and Preliminaries

We let \( \mathcal{A} \) to denote the class of functions analytic in \( U \) and having the power series expansion

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]  

in the unit disc \( U = \{ z : |z| < 1 \} \). Let \( \mathcal{S} \) be the class of functions \( f(z) \in \mathcal{A} \) and

Received: November 6, 2014 © 2015 Academic Publications, Ltd.

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In 1976, Noonan and Thomas [17] stated the $q^{th}$ Hankel determinant of $f$ for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$ 

This determinant has been considered by several authors in the literature [17]. For example, Noor [18] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions in $\mathbb{U}$ with bounded boundary. Later, Ehrenborg [3] considered the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some its properties were discussed by thoroughly by Hayman in [9].

Also, Hankel determinant was studied by various authors including Hayman [9] and Pommerenke [19]. Easily, one can observe that the Fekete-Szegö functional is $H_2(1)$. Fekete-Szegö then further generalized the estimate $|a_3 - \mu a_2^2|$ where $\mu$ is real and $f \in \mathbb{U}$. Ali [1] found sharp bounds on the first four coefficients and sharp estimate for the Fekete-Szegö functional. For our discussion in this paper, the Hankel determinant for the case $q = 2$ and $n = 2$ are being considered

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}.$$ 

Janteng, Halim and Darus [10] have determined the functional $|a_2a_4 - a_3^2|$ and found a sharp bound for the functions $f$ in the subclass RT of $\mathbb{U}$, consisting of functions whose derivative has a positive real part studied by Mac Gregor [16]. In their work, they have shown that if $f \in \text{RT}$ then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. The same authors [10] also obtained the second Hankel determinant and sharp bounds for the familiar subclasses namely, starlike and convex denoted by ST and CV of $\mathbb{U}$ and have shown that $|a_2a_4 - a_3^2| \leq 1$ and $|a_2a_4 - a_3^2| \leq \frac{1}{8}$ respectively. R.M. El-Ashwah and D.K. Thomas [4] defined the following classes:

$$S^*_c = \left\{ f(z) \in \mathcal{A} : \text{Re} \left\{ \frac{2zf'(z)}{f(z) + f(z)} \right\} > 0, z \in \mathbb{U} \right\},$$

$$S^*_sc = \left\{ f(z) \in \mathcal{A} : \text{Re} \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0, z \in \mathbb{U} \right\}.$$ 

Functions in the classes $S^*_c$ are called starlike functions with respect to conjugate points and $S^*_sc$ are called starlike functions with respect to symmetric conjugate points.

Again Janteng et al. [10] introduced the following classes:
\[ K_c = \left\{ f(z) \in A : Re \left\{ \frac{2(zf'(z))'}{f(z) + f(z)} \right\} > 0, z \in \mathbb{U} \right\}, \]

\[ K_{sc} = \left\{ f(z) \in A : Re \left\{ \frac{2(zf'(z))'}{f(z) - f(-z)} \right\} > 0, z \in \mathbb{U} \right\}. \]

Functions in the classes \( K_c \) are called convex functions with respect to conjugate points and \( K_{sc} \) are called convex functions with respect to symmetric conjugate points.

Obviously the functions in these classes are univalent. Various subclasses of analytic functions with respect to conjugate points and with respect to symmetric conjugate points were widely investigated by various authors including Dahar [11], Selvaraj [22], Ravichandran [20], Tang and Deng [8]. Motivated by the above mentioned results obtained by different authors in this direction, we seek upper bound of the function \( |a_2a_4 - a_3^2| \) for functions belonging to the classes \( S_c^{\alpha}, S_{sc}^{\alpha}, K_c^{\alpha} \) and \( K_{sc}^{\alpha} \).

\( S_c^{\alpha} \) denotes the subclass of functions \( f(z) \in A \) and satisfying the condition

\[ Re \left[ \alpha f'(z) + (1 - \alpha) \left( \frac{2zf'(z)}{f(z) + f(z)} \right) \right] > 0, \quad (2) \]

The following observations are obvious:

1. \( S_c^{\alpha(1)} \equiv RT \).

2. \( S_c^{\alpha(0)} \equiv S_c^{*} \), the class of starlike functions with respect to conjugate points introduced by Gagandeep Singh [6].

\( S_{sc}^{\alpha} \) denotes the subclass of functions \( f(z) \in A \) and satisfying the condition

\[ Re \left[ \alpha f'(z) + (1 - \alpha) \left( \frac{2zf'(z)}{f(z) - f(-z)} \right) \right] > 0, \quad (3) \]

The following observations are obvious:

1. \( S_{sc}^{\alpha(1)} \equiv RT \).

2. \( S_{sc}^{\alpha(0)} \equiv S_{sc}^{*} \), the class of starlike functions with respect to symmetric conjugate points introduced by Gagandeep Singh [6].
\(K_{c}^{(\alpha)}\) denotes the subclass of functions \(f(z) \in A\) and satisfying the condition
\[
\text{Re} \left[ \alpha f'(z) + (1 - \alpha) \left( \frac{2(zf'(z))'}{f(z) + f(\overline{z})} \right) \right] > 0,
\] (4)

The following observations are obvious:

1. \(K_{c}^{(1)} \equiv RT\).
2. \(K_{c}^{(0)} \equiv K_c\), the class of convex functions with respect to conjugate points introduced by Gagandeep Singh [6].

\(K_{sc}^{(\alpha)}\) denotes the subclass of functions \(f(z) \in A\) and satisfying the condition
\[
\text{Re} \left[ \alpha f'(z) + (1 - \alpha) \left( \frac{2(zf'(z))'}{f(z) - f(-\overline{z})} \right) \right] > 0,
\] (5)

The following observations are obvious:

1. \(K_{sc}^{(1)} \equiv RT\).
2. \(K_{sc}^{(0)} \equiv K_{sc}\), the class of convex functions with respect to symmetric conjugate points introduced by Gagandeep Singh [6].

2. Preliminary Results

Let \(P\) be the family of all functions \(p\) analytic in \(U\) for which \(\text{Re} (p(z)) > 0\) and
\[
p(z) = 1 + p_1 z + p_2 z^2 + \cdots, \forall z \in U.
\] (6)

**Lemma 2.1.** (see [19]) We have \(|p_k| \leq 2\) \((k = 1, 2, 3, \ldots)\).

**Lemma 2.2.** If \(p \in P\), then
\[
2p_2 = p_1^2 + (4 - p_1^2) x,
\]
\[
4p_3 = p_1^3 + 2p_1 (4 - p_1^2) x - p_1 (4 - p_1^2) x^2 + 2 (4 - p_1^2) \left( 1 - |z|^2 \right) z,
\]
for some \(x\) and \(z\) satisfying \(|x| \leq 1\) and \(p_1 \in [0, 2]\).

This result was proved by Libera and Zlotkiewicz [14, 15].
3. Main Results

**Theorem 3.1.** If \( f(z) \in S^c_\alpha \), then

\[
|a_2a_4 - a_3^2| \leq \frac{4}{(2 + \alpha)^2}. \tag{7}
\]

**Proof.** Since \( S^c_\alpha \) denotes the subclass of functions \( f(z) \in A \) and satisfying the condition, so from (3)

\[
\Re \left[ \alpha f'(z) + (1 - \alpha) \left( \frac{2zf''(z)}{f(z) + f(\overline{z})} \right) \right] > 0. \tag{8}
\]

\[
a_2 = \frac{p_1}{1 + \alpha} \quad \quad \quad a_3 = \frac{p_2}{2 + \alpha} + \frac{(1 - \alpha)p_1^2}{(2 + \alpha)(1 + \alpha)^2} \quad \quad \quad a_4 = \frac{p_3}{3 + \alpha} + \frac{(1 - \alpha)p_1p_1}{(1 + \alpha)(2 + \alpha)(3 + \alpha)} + \frac{(1 - \alpha)(1 - 4\alpha)p_1^3}{(1 + \alpha)^3(2 + \alpha)(3 + \alpha)}
\]

\[
a_2a_4 - a_3^2 = \frac{1}{C(\alpha)} \left\{ \frac{(1+\alpha)^3(2+\alpha)^2 p_1p_3}{2} + \frac{3(1-\alpha)(1+\alpha)^2(2+\alpha) - 2(1-\alpha)(1+\alpha)^2(3+\alpha)}{2} p_1^2p_2 \right\}, \tag{10}
\]

where

\[
C(\alpha) = (1+\alpha)^4(2+\alpha)^2(3+\alpha)
\]

Using Lemma 2.1 and Lemma 2.2 in (10).
\[
- \left[ (1+\alpha)^3 (2+\alpha)^2 p_1^2 + (1+\alpha)^4 (3+\alpha) (4-p_1^2) \right] (4-p_1^2) x^2 \\
+ 2 (1+\alpha)^3 (2+\alpha)^2 p_1 (4-p_1^2) \left( 1 - |x|^2 \right) z.
\]

Assume that \( p_1 = p \) and \( p \in [0, 2] \), using triangular inequality and \( |z| \leq 1 \), we have

\[
|a_2a_4 - a_3^2| \\
\leq \frac{1}{4C (\alpha)} \left[ (1+\alpha)^3 (2+\alpha)^2 + 2\alpha (1-\alpha) (1+\alpha)^2 - 4 (1-\alpha) \\
(3\alpha^2+5\alpha+1) - (1+\alpha)^4 (3+\alpha) \right] p^4 \\
+ 2 (1+\alpha)^2 p (4-p^2) + 2 (1+\alpha)^2 \\
\left[ (1+\alpha) (2+\alpha)^2 + \alpha (1-\alpha) - (1+\alpha)^2 (3+\alpha) \right] p^2 (4-p^2) \delta \\
+ (1+\alpha)^3 \left[ (2+\alpha)^2 p^2 - 2 (2+\alpha)^2 p + (1+\alpha) (3+\alpha) (4-p^2) \right] \\
(4-p^2) \delta^2 \right].
\]

Therefore

\[
|a_2a_4 - a_3^2| \leq \frac{1}{4C (\alpha)} F (\delta)
\]

where \( \delta = |x| \leq 1 \) and

\[
F (\delta) = \left[ (1+\alpha)^3 (2+\alpha)^2 + 2\alpha (1-\alpha) (1+\alpha)^2 - 4 (1-\alpha) \\
(3\alpha^2+5\alpha+1) - (1+\alpha)^4 (3+\alpha) \right] p^4 + 2 (1+\alpha)^3 \\
(2+\alpha)^2 p (4-p^2) + 2 (1+\alpha)^2 \\
\left[ (1+\alpha) (2+\alpha)^2 + \alpha (1-\alpha) - (1+\alpha)^2 (3+\alpha) \right] p^2 (4-p^2) \delta
\]

(1+\alpha) (2+\alpha)^2 + \alpha (1-\alpha) - (1+\alpha)^2 (3+\alpha) \right] p^2 (4-p^2) \delta
\]
$\left[ (2+\alpha)^2 p^2 - 2 (2+\alpha)^2 p + (1+\alpha) (3+\alpha) (4-p^2) \right]$

$(4-p^2) \delta^2$.

is an increasing function. Therefore $\text{Max} \ F(\delta) = F(1)$.

Consequently

$$|a_2a_4 - a_3^2| \leq \frac{1}{4C(\alpha)} G(p),$$

(11)

where $G(p) = F(1)$.

So

$$G(p) = A(\alpha) p^4 - 4B(\alpha) p^2 + 16 (1+\alpha)^4 (3+\alpha),$$

where

$$A(\alpha) = 2 \left( 7\alpha^3 - 7\alpha^2 + 5\alpha + 1 \right),$$

and

$$B(\alpha) = (\alpha^5 + 9\alpha^4 + 2\alpha^3 + 11\alpha^2 + 2\alpha),$$

Now

$$G'(p) = 4A(\alpha) p^3 - 8B(\alpha) p.$$ 

and

$$G''(p) = 12A(\alpha) p^2 - 8B(\alpha).$$

$G'(p) = 0$ gives

$$4p \left[ A(\alpha) p^2 - 2B(\alpha) \right] = 0.$$ 

$G''(p)$ is negative at

$$p = \sqrt{\frac{\alpha^5 + 9\alpha^4 + 2\alpha^3 + 11\alpha^2 + 2\alpha}{7\alpha^3 - 7\alpha^2 + 5\alpha + 1}} = p'.$$

So $\text{Max} G(p) = G(p')$. Hence from (11), we obtain (3.1).

The result is sharp for $p_1 = p', p_2 = p_1^2 - 2$ and $p_3 = p_1 (p_1^2 - 3)$.

For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

**Corollary 3.2.** If $f(z) \in RT$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$ 

**Remark 3.3.** For the choice of $\alpha = 1$, The result coincide with those of A.Janteng, S.A.Halim and M.Darus [12].

**Corollary 3.4.** If $f(z) \in S^*_c$, then

$$|a_2a_4 - a_3^2| \leq 1.$$
Remark 3.5. For the choice of $\alpha = 0$, The result coincide with those of Gagandeep Singh [6].

Theorem 3.6. If $f \in S^{sc}_{\alpha}$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{(2 + \alpha)^2}. \quad (12)$$

The result is sharp for $p_1 = p', p_2 = p_1^2 - 2$ and $p_3 = p_1 (p_1^2 - 3)$.

For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 3.7. If $f (z) \in RT$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}. $$

Corollary 3.8. If $f (z) \in S^{sc}_{\alpha}$, then

$$|a_2a_4 - a_3^2| \leq 1. $$

Theorem 3.9. If $f \in K^{(\alpha)}_c$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9 (2 - \alpha)^2}. \quad (13)$$

The result is sharp for $p_1 = p', p_2 = p_1^2 - 2$ and $p_3 = p_1 (p_1^2 - 3)$.

For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 3.10. If $f (z) \in RT$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}. $$

Corollary 3.11. If $f (z) \in K_c$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{9}. $$

Theorem 3.12. If $f \in K^{(\alpha)}_{sc}$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9 (2 - \alpha)^2}. \quad (14)$$

The result is sharp for $p_1 = p', p_2 = p_1^2 - 2$ and $p_3 = p_1 (p_1^2 - 3)$.

For $\alpha = 1$ and $\alpha = 0$ respectively, we obtain the following results:

Corollary 3.13. If $f (z) \in RT$, then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}. $$
Corollary 3.14. If \( f(z) \in K_{sc} \), then
\[
|a_2a_4 - a_3^2| \leq \frac{1}{9}.
\]

References


