

THE ISOTROPY GRAPH OF FINITE NON-ABELIAN GROUPS

S.M.S. Omer

Department of Mathematics

Faculty of Science

University of Benghazi

Benghazi, LIBYA

Abstract: Let G be a finite non-abelian group and let Ω be a set of elements of G . Let A be the set of commuting elements in Ω , i.e $A = \{v \in \Omega : vg = gv, g \in G\}$. In this paper, we extend the work on centralizer graph by defining a new graph called the isotropy graph, denoted as Γ_{iso} . The vertices of this graph are proper isotropy groups. In other words, $|V(\Gamma_{\text{iso}})| = \text{Stab}_G(\Omega) - A$, where $\text{Stab}_G(\Omega)$ is the number of stabilizers under group action of G on Ω and A is the number of improper isotropy groups under group action on a set. Two vertices are connected by an edge if their cardinalities are identical.

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1. Introduction

The probability that a pair of elements x and y selected randomly from a group G commute, is called the commutativity degree and it was firstly introduced in 1944 [1]. The definition of commutativity degree is stated formally in the following Definition 1.

Definition 1. [1] Let G be a group. The commutativity degree is the probability that two random elements (x, y) in G commute, defined as follows:

$$P(G) = \frac{|\{(x, y) \in G \times G | xy = yx\}|}{|G|^2}.$$

The above probability is less than or equal to $5/8$ for finite non-abelian groups (see [2], [3]). The concept of commutativity degree has been extended by several authors, one of these extensions is called the probability that a group element fixes a set [4]. The stabilizers that are obtained from this probability are applied to graph theory in which a new graph called the isotropy graph is introduced. Numerous researches have been done on the commutativity degree and many results have been achieved.

In the following context, we state some basic concepts that are needed in this paper. These basic concepts can be found in one of the references (see [5], [6], [7]).

The stabilizer is a subgroup involve under group action of G on Ω . The stabilizer of $\omega \in \Omega$ is defined as follows

$$\text{Stab}_G(\omega) = \{g \in G : g\omega g^{-1} = \omega\},$$

This subgroup is often called the isotropy group which contains all elements of G that commute with ω .

A subgroup H is proper subgroup of G if $H \subset G$ and it is improper if $H \subseteq G$. In this study, the proper centralizer, denoted by $C(G)$, is a subgroup that is not equal to the whole group i.e $C(G) \subset G$.

The following theorem is one of the famous results that deals with the length of the orbits which is considered a key fact for many applications.

Theorem 2. [8] *Let G be a finite group acting on a set Ω . Then the Orbit Stabilizer Theorem is given as in the following.*

$$|cl(\omega)| = [G : \text{Stab}_G(\omega)],$$

where $cl(\omega)$ is the orbit of the element ω in Ω under group action of G on Ω , and $\text{Stab}_G(\omega)$ is its stabilizer.

Next, we provide some concepts related to graph theory.

A graph Γ is a mathematical structure consisting of two sets, namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. A connected graph is a graph in which there is a partition of vertex V into non empty subsets, V_1, V_2, \dots, V_n such that two vertices V_1 and V_2 are connected if and only if they belong to the same set V_i . Subgraphs $\Gamma(V_1), \Gamma(V_2), \dots, \Gamma(V_n)$ are all components of Γ . The graph Γ is connected if it has precisely one component. However, a graph is a complete graph if each order pair of distinct vertices is adjacent, and it is denoted by K_n , where n is the number of adjacent vertices. The graph is called empty if there is no adjacent between its vertices. In addition, a graph is called null if it has no vertices and in this paper we denote K_0 the null graph (see [9] and [10]).

This paper is divided into three sections. The first section focuses on some background on group theory and graph theory, while the second section provides some earlier and recent publications that are related to the commutativity degree and some some type of graphs. In the third section, we present our results.

2. Preliminaries

In this section, some previous works needed this paper are included.

As mentioned, the commutativity degree was extended by several authors, one of these extension was done by Omer *et al.* [4]. In 2013, Omer *et al.* [4] introduced the probability that a group element fixes a set. The probability in [4] was found for some finite non-abelian groups, including metacyclic groups [11], 2-groups [12], symmetric groups and alternating groups [13].

The work in [4] is also extended by El-sanfaz *et al.* [14] who restrict the size of the fixed set.

The commutativity degree can be obtained by find the number of conjugacy classes or by Cayley table. However, Omer *et al.* [15] introduced another method to compute the commutativity degree using the centralizers of groups. The commutativity degree in terms of centralizers was firstly computed for dihedral groups [15].

As a connection between algebraic theory and graph theory, various type of graphs have been introduced. For instance, the results obtained from conjugacy class of groups are connected to graph theory by defining the conjugacy class graph [16]. Furthermore, the results that are obtained from the probability that a group element fixes a set [4] are applied to graph theory in which a new graph called the orbit graph is introduced. However, the stabilizers under group action on a set from the same probability are also associated to graph theory by defining the isotropy graph.

In addition, the results in [15] are applied to the graph theory, more precisely to the centralizer graph [17]. The centralizer graph is a graph whose vertices are proper centralizers in which two vertices are adjacent whenever their cardinalities are identical [17].

3. Main Results

In this section, we present our results. Starting with the definition of isotropy graph.

Definition 3. Let G be a finite non-Abelian group. Let Ω be a set. If G acts on Ω , then the isotropy graph, denoted by Γ_{iso} , is a graph whose vertices are non improper isotropy groups. Two vertices ω_1 and ω_2 are connected by an edge if their cardinalities are identical, i.e $|\omega_1| = |\omega_2|$.

In this section, the isotropy graph is found when a group acts on a set of all subsets of commuting elements of size two, more specifically on Ω .

The following proposition illustrates the case that the isotropy graph is tend to be null.

Proposition 4. *Let G be a finite non-Abelian group. Let Ω be the set of all subsets of commuting elements of G of size two. If G acts on Ω by conjugation and $P_G(\Omega) = 1$, then the isotropy graph Γ_{iso} is null.*

Proof. If $P_G(\Omega) = 1$, then $|K(\Omega)| = |A|$ thus for all $\omega \in \Omega$ the orbit $cl(\omega) = \omega$. Using Theorem 2, thus $Stab_G(\omega) = G$ for all $\omega \in \Omega$. By vertices adjacency of Γ_{iso} , the graph is null. □

In the next theorems, let S be a set of elements of G of size two of the form (a, b) where a and b commute. Let Ω be the set of all subsets of commuting elements of G of size two.

In the following, the isotropy graph of dihedral groups, quaternion groups, dicyclic groups, semi-dihedral groups and quasi-dihedral groups, is found. Starting with the isotropy graph of dihedral groups.

Theorem 5. *Let G be a dihedral group of order $2n$. If G acts on Ω by conjugation, then*

$$\Gamma_{iso} = \begin{cases} K_{\frac{n}{2}}, & \text{if } n \text{ is even and } \frac{n}{2} \text{ is odd,} \\ K_{\frac{n}{2}}, & \text{with one isolated vertex, if } n \text{ and } \frac{n}{2} \text{ are even,} \\ K_2, & \text{if } n = 4, \\ K_n, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. In the first case when n is even and $\frac{n}{2}$ is odd. Based on Theorem 3.1 in [4] and Theorem 2, all the stabilizers have of size four. Since $|Z(G)| = 2$, thus the number of stabilizers under group action on Ω is $\frac{n}{2}$. According to Definition 3, the number of vertices in Γ_{iso} is $\frac{n}{2}$. By vertices adjacency of Γ_{iso} , two vertices are identical if their sizes are the same thus, there are $\frac{n}{2}$ vertices have the same size. Thus Γ_{iso} consists a complete graph of $K_{\frac{n}{2}}$. In the second case when n and $\frac{n}{2}$ are even, the stabilizers are of size four and eight. If the stabilizers have of size four, the graph is similar to the first case. Now, if the stabilizers have size eight, there is only one vertex of size eight, namely $Stab_G(\omega) = \{ \langle a^{\frac{n}{4}}, a^{\frac{n}{4}i}b \rangle, 0 \leq i \leq 2n \}$ this occurs only when $n \neq 4$, if $n = 4$ there is one one complete graph of K_2 . Lastly, when n is odd, the number of vertices is n . Using Theorem 3.1 in [4] and the Orbit Stabilizer Theorem, the isotropy subgroups are of size two. Since $|V(\Gamma_{iso})| = n$, thus Γ_{iso} consists of complete graph of K_n . The result then follows. □

Next, the isotropy graph of quaternion groups and dicyclic groups, is determined.

Theorem 6. *Let G be a quaternion group $Q_{2^{n+1}}$. If G acts on Ω by conjugation, then $\Gamma_{iso} = K_0$.*

Proof. Based on Theorem 3.2 in [4], the probability that a group element fixes a set, $P_G(\Omega) = 1$. Using Proposition 4, Γ_{iso} is null. □

Theorem 7. *Let G be a dicyclic group, $G \cong \langle a, b : a^{2\beta} = b^4 = e, b^{-1}ab = a^{-1}, a^\beta = b^2 \rangle$. If G acts on Ω by conjugation, then $\Gamma_{iso} = K_0$.*

Proof. The proof is similar to the proof of Theorem 6 since the number of stabilizer in Theorem 3.2 and Theorem 3.3 in [4] are the same. \square

Next, the isotropy graph of semi-dihedral groups and quasi-dihedral groups is found, starting with semi-dihedral groups.

Theorem 8. *Let G be a semi-dihedral group, $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}-1} \rangle$, where $n \geq 3$. If G acts on Ω by conjugation, then $\Gamma_{iso} = K_2$.*

Proof. Based on Theorem 3.1 in [12], there are four orbits under group action of G on Ω , two of them are of size 2^{n-1} , one is of size 2^{n-2} and one of size 2^{n+1} . Using Theorem 2, there are two stabilizers of size four and one is of size eight. By vertices adjacency of isotropy graph, there is one complete component of K_2 and one isolated vertex. The proof then follows. \square

Theorem 9. *Let G be a quasi-dihedral group, $G \cong \langle a, b : a^{2^n} = b^2 = e, ab = ba^{2^{n-1}+1} \rangle$, where $n \geq 3$. If G acts on Ω by conjugation, then Γ_{iso} is an empty graph.*

Proof. According to Theorem 3.2 in [12], there are two orbits of size two. Using Theorem 2, the stabilizers are of size 2^n . Since there are only six elements in Ω , thus $\text{Stab}_G(\Omega) = 1$. By Definition 3, the number of vertices in Γ_{iso} is one. Therefore, the graph is empty. \square

4. Conclusion

In this paper, a new graph was introduced called, the isotropy graph. It is proven that the isotropy graph is null whenever the action on the set is by conjugation and $P_G(\Omega) = 1$. The isotropy graph is determined for some finite non-abelian groups including dihedral groups, quaternion groups, semi-dihedral groups and quasi-dihedral groups.

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