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A NOTE ON THE CANTOR-SCHROEDER-BERNSTEIN THEOREM AND ITS PROOF WITHOUT WORDS

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Abstract: In this work, the Cantor-Schroeder-Bernstein theoremt is addressed. The main goal and contribution of this presentation consists in giving an easy proof which can be easily understood by students without a strong mathematical foundation. I have employed it successfully in my Discrete Mathematics course for Computer Science students at the Instituto Politecnico Nacional.

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Key Words: the Cantor-Schroeder-Bernstein theorem, proof without words, visual proof

1. Introduction

In set theory, the Cantor-Bernstein-Schroeder theorem, named after Georg Cantor, Felix Bernstein, and Ernst Schrder, states that, if there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijective function $h : A \to B$. In terms of the cardinality of the two sets, this means that if $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$; that is, A and B are equipollent. This is a useful feature in the ordering of cardinal numbers. An important feature of this theorem is that it does not rely on the axiom of choice. However, its various proofs are non-constructive, as they depend on the law of excluded middle, and therefore rejected by intuitionists.

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An earlier proof by Cantor relied, in effect, on the axiom of choice by inferring the result as a corollary of the well-ordering theorem. The main goal of the paper is to give an easy proof, which can be easily understood by students without a strong mathematical foundation. I have employed it successfully in my Discrete Mathematics course for Computer Science students at the Instituto Politecnico Nacional. The proof without words presented, is based on a non-constructive formal proof of the theorem, and its depicted by a picture. There are similar visual proofs in the literature ([1], [2]) however they are not exactly equal to what is here presented.

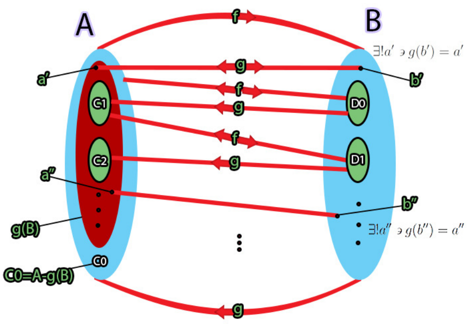
2. A Non-Constructive Formal Proof

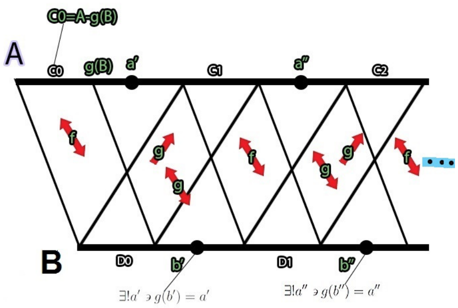
Theorem 1. *If there exist injective functions $f : A \to B$ and $g : B \to A$ between the sets A and B, then there exists a bijective function $h : A \to B$*

Proof. Let us define: $g : B \to g(B)$, $C_0 = A \setminus g(B), C_{n+1} = g(f(C_n)), D_n = f(C_n))$ for all $n \geq 0$, and $C = \bigcup_{n=0}^{\infty} C_n$, $D = \bigcup_{n=0}^{\infty} D_n$. Then, for every $a \in A$ set: $h(a) = \begin{cases} f(a) & \text{if } a \in C, \\ g^{-1}(a) & \text{if } a \notin C. \end{cases}$. It remains to check that the map $h : A \to B$ is a bijection, which follows immediately since $f : A \to B$ is injective, $g : B \to g(B)$ is a bijection and by the way as h is defined. □

3. Proof without Words

Let $h : A \to B$ be defined as above then h is a bijection:





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4. Conclusions

A visual proof of the Cantor-Schroeder-Bernstein theorem is presented. The proof can be used for teaching purposes.

References

[1] I. Stewartt, D. Tall, *The Foundations of Mathematics*, Oxford Science Publications, (1977).

[2] Proof with Pictures: Google Search.