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PERIODIC MOTION AT RESONANCE IN SEMICONDUCTOR LASER EQUATIONS

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Abstract: A weakly nonlinear third-order equation describing the behavior of the phase of the electric field of a single-mode semiconductor laser subject to optical injection and detuning is analyzed. The method of averaging is used to explore the creation of periodic orbits in the system at resonance, when the detuning frequency is a rational multiple of the laser's natural frequency.

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1. Introduction

The behavior of a single-mode semiconductor laser diode subject to injection from an external beam is governed by the rate equations

$$\begin{aligned} \dot{E} &= NE + \eta\cos(\psi + \Omega s) \\ T\dot{N} &= P - N - P(1+2N)E^2 \\ \dot{\psi} &= -bN - \eta\frac{\sin(\psi+\Omega s)}{E}, \end{aligned} \tag{1}$$

where E and ψ are the amplitude and phase of the complex electric field inside the laser cavity and N, the number of carriers (electrons and holes) above threshold. The dots symbolize differentiation with respect to the time variable s.

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The parameters T and b represent quantities associated with the laser's material characteristics, and P corresponds to the intensity $(\frac{J}{J_{th}} - 1)$ of the pumping current above threshold. The external beam's strength is denoted by η, and Ω is the difference between the beam's frequency and the laser's natural frequency.

The rate equations can be averaged and the averaged equations have been studied [7], [12], [14], [6], [8] using the tools of bifurcation theory.

Kovacic et al. [5] proved the existence of homoclinic orbits for this system of differential equations. Evidence of anomalous pushing of the laser's frequency away from the external reference frequency was presented by Politi et al. [10]. Numerical investigations by several authors [11], [6], [2], [9], [1], [13] have shed light on such phenomena as the coexistence of conservative and dissipative behavior in the same phase space, periodic and quasi-periodic motions on a torus born of a Hopf bifurcation, period-doubling cascading, to name a few.

In the absence of injection, i.e., when $\eta = 0$, the equations for E and N decouple from the equation for the phase ψ, and the system is thus easy to analyze, being essentially two-dimensional. The unperturbed laser is governed by the subsystem

$$\dot{E} = NE,$$
$$T\dot{N} = P - N - P(1 + 2N)E^2,$$

which admits a circle of degenerate fixed points given by $(E, N) = (1, 0)$ in the complex $Z-$plane. Linearization about $(E, N) = (1, 0)$ reveals that the unit circle is attracting with nearby motion characterized by damped oscillation with frequency $\omega = \sqrt{\frac{2P}{T}}$ in the direction perpendicular to the $Z-$plane. What becomes of this circle of degenerate fixed points and the structure of the flow in a small neighborhood of it once we restore the injection is the thrust of our investigation.

Our focus will be on how resonance between the relative frequency of the injected beam and the frequency of the free-running laser influences the organization of the periodic orbits in the three-dimensional phase space.

This paper is organized as follows: In Section 2, under appropriate assumptions on the parameters of the system, we derive a third-order differential equation for the phase of the electric field, the so-called phase equation. In Section 3, we use an averaging approach [15] to explore the occurence of periodic solutions for the phase equation for rational values of the frequency ratio $\frac{\Omega}{\omega}$. Our results are summarized in the last section of the paper.

2. Phase Equation

Kovanis, Erneux et al. [2], [3] use the fact that the linewidth enhancement factor b is large to come up with an equation for ψ that governs the dynamics in a small neighborhood of the unit circle.

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In [3] the scaling

$$t = s\sqrt{2P/T}, \;\; E = 1 + a/b, \;\; N = n\sqrt{2P/T}\frac{1}{b}$$

is used to turn (1) into

$$\begin{aligned} a' &= n + \Lambda\cos(\psi + \Delta t) + O(\frac{1}{b}), \\ n' &= -a - \zeta n + O(\frac{1}{b}), \\ \psi' &= -n + O(\frac{1}{b}). \end{aligned} \tag{2}$$

The prime symbol stands for differentiation with respect to the re-scaled time parameter t. The new parameters are related to η, P, b, T and Ω as follows:

$$\begin{aligned} \omega &\equiv \sqrt{2P/T}, \\ \Lambda &= \eta b/\omega, \\ \Delta &\equiv \Omega/\omega, \\ \zeta &\equiv \omega\frac{1+2P}{2P}. \end{aligned}$$

We have control over the parameter Λ, since the injection strength η can be made as small as desired. The value of the pumping current P lies in the interval $[0, 1)$. The magnitude for T is around 10^3 and b usually varies between 3 and 14. As a result, the new parameters ω and ζ are small and we can adjust the new detuning parameter Δ by choosing Ω appropriately.

An inspection of Equation (2) suggests that, for Λ nonzero but small, the new variables a and n oscillate with frequency approximately equal to 1. This means that the E and N components of (1) continue to oscillate with frequency near ω, and a qualitative description of the flow only requires an understanding of how the electric field vector rotates in the complex $Z-$plane. We will henceforth focus our attention on the behavior of the phase ψ.

After dropping the small $O(\frac{1}{b})$ terms, differentiation of the ψ component of (2) yields the third-order differential equation

$$\psi''' + \zeta\psi'' + \psi' = \Lambda\cos(\psi + \Delta t). \tag{3}$$

Notice that $\Delta = \frac{m}{n}$, m and n relatively prime integers, corresponds to the resonance relation $n\Omega = m\omega$ between the natural frequency of the free-running laser and the frequency of the beam injected into the laser cavity.

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3. Averaged Phase Equation

We rewrite Equation (3) as the 3-dimensional system

$$\begin{aligned} u' &= v, \\ v' &= -u - \zeta v + \Lambda \cos(\psi + \Delta t), \\ \psi' &= u. \end{aligned}$$

The change of variables

$$u = \sigma, v = \rho, \psi = \gamma - \rho,$$

casts this system in the form

$$\begin{aligned} \sigma' &= \rho, \\ \rho' &= -\sigma - \zeta\rho + \Lambda \cos(\gamma - \rho + \Delta t), \\ \gamma' &= -\zeta\rho + \Lambda \cos(\gamma - \rho + \Delta t). \end{aligned}$$

Using $\zeta = \delta\Lambda$ and polar coordinates

$$\sigma = r\cos\theta, \quad \rho = r\sin\theta,$$

we get

$$\begin{aligned} r' &= \Lambda[\sin\theta\cos(\gamma - r\sin\theta + \Delta t) - \delta r \sin^2\theta], \\ \gamma' &= \Lambda[-\delta r\sin\theta + \cos(\gamma - r\sin\theta + \Delta t)], \\ \theta' &= -1 - \Lambda\delta\sin\theta\cos\theta + \frac{\Lambda}{r}\cos\theta\cos(\gamma - r\sin\theta + \Delta t), \end{aligned} \tag{4}$$

a nonautonomous vector field.

The introduction of a new variable $\tau = \gamma + \Delta\theta + \Delta t$ leads to the autonomous system

$$\begin{aligned} r' &= \Lambda[-\delta r\sin^2\theta + \sin\theta\cos(\tau - r\sin\theta - \Delta\theta)], \\ \tau' &= \Lambda[-\delta r\sin\theta + \cos(\tau - r\sin\theta - \Delta\theta), \\ &\quad - \delta\Delta\sin\theta\cos\theta + \Delta\frac{\cos\theta}{r}\cos(\tau - r\sin\theta - \Delta\theta)], \\ \theta' &= -1 + \Lambda[-\delta\sin\theta\cos\theta + \frac{1}{r}\cos\theta\cos(\tau - r\sin\theta - \Delta\theta)]. \end{aligned} \tag{5}$$

The vector field in (5) is periodic in θ when Δ satisfies the resonance relation

$$n\Delta = m, \; m \text{ and } n \text{ integers}.$$

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With this resonance condition in place, averaging over θ for a period $T = 2n\pi$ yields the averaged system

$$r' = \Lambda[-\delta r - \sin\tau j'_{-\frac{m}{n}}(r)],$$
$$\tau' = \Lambda\cos\tau[j_{-\frac{m}{n}}(r) - \frac{m^2}{n^2r^2}j_{-\frac{m}{n}}(r) + \frac{m}{nr^2}\frac{\sin(\frac{m\pi}{n})}{\pi}], \tag{6}$$

where $j_w(r)$ is the Anger function of order w and is equal to the Bessel function $J_w(r)$ when w is an integer [4], and $j'_w(r)$ is the derivative of $j_w(r)$ with respect to r. The fixed points of this averaged system correspond to periodic orbits of (3).

When $\delta = 0$, it is easy to visualize the phase portrait for the averaged system. The zeros of the function $j'_{-\frac{m}{n}}$ form one-dimensional invariant manifolds that run perpendicular to the invariant manifolds given by the lines $\tau = (2n+1)\frac{\pi}{2}$ in the $r-\tau$ plane. Thus the flow follows a simple rectangular grid pattern where horizontal and vertical invariant lines intersect to form heteroclinic 4-cycles. Inside each heteroclinic 4-cycle is an elliptic fixed point, surrounded by a continuous band of periodic orbits; there cannot be any foci inside the rectangular shaped heteroclinic cycles, because the undamped vector field in (6) is invariant with respect to the transformation $\{\tau \to -\tau,\, t \to -t\}$. This symmetry dictates that any arc connecting two points on the $r-$axis be reflected across the line $\{\tau = 0\}$, resulting in a closed trajectory.

The center fixed points become attracting foci once δ is set to a small positive value. These foci represent the stable periodic orbits of (3), the stable states that the laser will approach and eventually operate in.

4. Conclusion

Using a third-order equation for the phase of the electric field, we were able to study the dynamics of a semiconductor laser subject to optical injection and detuning. This weakly nonlinear third-order phase equation describes the dynamics of the device in a neighborhood of the unit circle $\{E = 1\}$. Its derivation is based on assumptions about the linewidth enhancement factor and other parameters of the system. Through the method of averaging, we found that the phase equation admits periodic solutions of period $2n\pi$ for rational values $\frac{m}{n}$ of the ratio $\Delta = \frac{\Omega}{\omega}$. Some of those periodic orbits are unstable, as they appear as saddle fixed points of the averaged equations. The others appear as stable foci in the averaged system and, therefore, correspond to stable operating states of the injected laser system.

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