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COMPUTATIONAL ALGORITHM FOR AN MP3 (MOORE-PENROSE 3) MATRIX RING AND FORMULATION OF MP3 EQUIVALENCE

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Abstract: The structure theorems for complete local rings are largely due to Irving S. Cohen. The beginning part of Cohen's structure theorem for the equicharacteristic case is "A complete local ring which has the same characteristic as its residue field contains a coefficient field", see [1] The second part of Cohen's structure theorem conveys essentially that in the equicharacteristic case, a complete local ring is a homormorphic image of a formal power series ring over the residue field. This paper proposes the algebraic structure of an $n \times n$ matrix ring $M_n(R)$ $(n \geq 1)$ where R is a formal power series in one variable over a field modulo a power of its maximal ideal. The structure of this special class of matrix rings is derived using the notion of a pseudoinverse and a generalization of one of the Moore-Penrose conditions. Afterwards, some theoretical properties are given to establish the equivalence of MP3 (Moore-Penrose 3) rings that are becoming more exciting in theoretical algebraic investigations for their symmetry inducing capability.

Key Words: local ring, involution, pseudoinverse, symmetry

1. Definitions

1. An involution * on a ring R, which maps R to R, has the following properties:

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(a) $(a^*)^* = a$, $\forall a \in R$;

(b) $(a+b)^* = a^* + b^*$, $\forall a, b \in R$;

(c) $(ab)^* = b^* a^*$, $\forall a, b \in R$.

An example of involution would be complex conjugation on the ring (field) **C** of complex numbers, i.e. if $z = a + bi$, then $z^* = \mathbf{z} \; = a - bi$.

There are many special sets in a ring R with involution. In particular:

2. The set $\{x \in R | x^* = x\}$ shall be called the set of symmetric elements in R and denoted by the letter S.

3. A ring R with involution * is called MP3 (Moore-Penrose 3) if for every nonzero element $a \in R$, there exists a nonzero $x \in R$ such that $(xa)^* = xa \neq 0$.

   In other words, a ring R with involution is called MP3 if the left ideal generated by each of its nonzero elements contains a nonzero symmetric element.

4. Given a ring R, its n-square matrix ring $M_n(R)$ satisfies the MP3 property if for a given nonzero n x n matrix $A \in M_n(R)$ there exists a nonzero n x n matrix X such that XA is a nonzero symmetric matrix.

5. A *coefficient field* of a local ring R is a subfield F of R which in the natural homomorphism of R onto its residue field R $/m$where $m$is a maximal ideal of R.

   It is readily noted that the mapping of the field F onto R $/m$is actually an isomorphism so that , if a coefficient field exists, then R and its residue field necessarily share the same characteristic. Hence, coefficient fields arise only in the equicharacteristic case; the common value of the characteristic is, of course, either zero or a prime number p.

2. Examples of MP3 Rings

1. In particular, it is noted that $M_2(R)$ is MP3 where R is a field F (its involution * is given by matrix transpose).

2. Consider the Galois group $G(Q(\ \mathbf{2}\ )/Q) \approx Z_2 = \{iQ\ (\ \mathbf{2}), \phi\}$. Set $* = \quad \phi$; then $S = Q$ (fixed field) under ϕ.

3. The quotient ring $Z_2[[t]]/(t^2)$ is MP3. (its involution * is given by $(a + bt + (t^2))^* = a - bt + (t^2)$ for $a, b \in Z_2$).

Note: $Z_2[[t]]$ is the formal power series ring over the two – element field Z_2. One interesting feature which $Z_2[[t]]/(t^2)$ possess, yet the former ring Z_4does not, is the containment of a coefficient field.

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3. Matrix Ring Characterizations

Theorem. *$M_n(F[[t]]/(t^k))$ is MP3, k an integer greater than or equal to 1.*

Proof. Let $A \in M_n(F[[t]]/(t^k))$ be an arbitrary and $A \neq 0$. Then A is a matrix polynomial of the form

$$A_0 + A_1t + \cdots + A_{k-1}t^{k-1},$$

where at least one of the $A_i \neq 0$. To show $M_n(F[[t]]/(t^k))$ is MP3, a matrix $X \in M_n(F[[t]]/(t^k))$, $X \neq 0$ must be found such that XA ≠0 and XA is symmetric. The matrix X also has the form $X_0 + X_1t + \cdots + X_{k-1}t^{k-1}$, where at least one of the $X_i \neq 0$. Then

$$\begin{aligned} XA =&(X_0 + X_1t + \cdots + X_{k-1}t^{k-1})(A_0 + A_1t + \cdots + A_{k-1}t^{k-1}) \quad (1)\\ =&X_0A_0 + (X_0A_1 + X_1A_0)t + (X_0A_2 + X_1A_1 + X_2A_2)t^2 + \cdots \\ &+ (X_0A_{k-1} + X_1A_{k-2} + \cdots + X_{k-2}A1 + X_{k-1}A0)t^{k-1}. \quad (2)\end{aligned}$$

Since a matrix of polynomials is a polynomial of matrix coefficients over a given field, then showing $M_n(F[[t]]/(t^k))$ is MP3 (i.e. XA is symmetric) is reduced to determining unknowns X_0, X_1, X_2,..., X_{k-1} $(k \geq 1)$ such that the expressions:

$$\begin{aligned} &X_0A_0 \\ &X_1A_0 + X_0A_1 \\ &X_2A_0 + X_1A_1 + X_0A_2 \\ &\vdots \\ &X_{k-1}A_0 + ... + X_1A_{k-2} + X_0A_{k-1} \end{aligned} \quad (3)$$

are simultaneously symmetric.

Recall that if A is an $n \times m$ matrix over a field $(m \leq n)$, then $A^\dagger$ is the unique m x n matrix which has it rows and columns in the row-space and column-space of A^T and also satisfies the equations

$$AA^\dagger = I_L, \quad \text{and} \quad A^\dagger A = I_R,$$

where I_L and I_R are both symmetric and idempotent, see [2].

If X_j is defined as $X_j = ((A_0)A_{k-1-j})(A_0)$, $(0 \leq j \leq k-1)$, where $(A_0)^\dagger$ is the pseudoinverse of A_0, then it is readily verified by computation that all expressions of matrices on the previous page are symmetric. The symmetry of X_jA_{k-1-j} follows since

$$(X_jA_{k-1-j})^T = \left(\left((A_0)^\dagger A_{k-1-j}\right)^T (A_0)^T A_{k-1-j}\right)^T = (A_{k-1-j})^T A_0 (A_0)^\dagger A_{k-1-j},$$

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which is a symmetric product since $A_0(A_0)$ is an orthogonal projection, or since the product of any matrix and its pseudoinverse is symmetric. Of course any sum of symmetric matrices is symmetric. Thereby the expression:

$$X_{k-1}A_0 + \cdots + X_1 A_{k-2} + X_0 A_{k-1}$$

(using $X_j = \left((A_0^\dagger)A_{k-1-j}\right)^T (A_0)^T$ for $0 \leq j \leq k-1$) is a symmetric sum. Thus, $M_n(F[[t]]/(t^k))$ is MP3, k an integer ≥ 1.

Corollary. *$M_n(F[[t]])$ is MP3 for an arbitrary field F.*

Proof. Let $A \in M_n(F[[t]])$ be given by $A = \sum_{k=0}^{\infty} A_k t^k$ and $A \neq 0$. The algorithm of the previous Theorem can be used to show that there exists a nonzero $X \in M_n(F[[t]])$, where $X = \sum_{k=0}^{\infty} X_k t^k$ such that XA is symmetric. The coefficient of the m-th term in the formal power series matrix XA will be:

$$X_m A_0 + \cdots + X_1 A_{m-1} + X_0 A_m,$$

where the integer $m \geq 0$. Thus, setting $X_j = \left((A_0)^\dagger A_{m-j}\right)^T (A_0)^T$, where $0 \leq j \leq m$, verification as above will show that XA is symmetric. Hence, $M_n(F[[t]])$ is MP3 for an arbitrary field F.

4. Formulating Equivalence of MP3 Rings

The most important subset of a ring R with involution * is its set S = {a|a = a*} of symmetric elements. Since the MP3 property is directly tied to this special set, it makes sense to establish the equivalence of MP3 rings through the manipulation of their elements of symmetry. One special class of rings to approach this concept is through field extensions with a fixed subfield. For instance, let C be the field of complex numbers, i.e., C = {a + bi|a ,b∈R, the real field; i^2 = 0}. One involution on C which fixes the set of real numbers R is * = complex conjugation given by $\overline{z} = \overline{a+bi} = a - bi$. Obviously complex conjugation is an involution on C with corresponding set of symmetric elements being S = R, the set of real numbers. Of course C with involution* being complex conjugation is a trivial case of an MP3 ring, i.e., given z ∈ C (z ≠ 0) with involution complex conjugation, then z z ∈ S = R, the field of real numbers and z z ≠0 .

Another involution ∧: C → C is given by ∧(a + bi) = (b + ai) ; once again, this involution on C makes it an MP3 ring, as well as sharing the same symmetric set R, as was the case with complex conjugation being the involution.

Theorem. *Let E be a normal extension of the field F and G(E/F) denote the Galois group of E over F. If G(E/F)* $= \{\sigma_0, \sigma_1\}$ *where* σ_0 *is the identity automorphism on E, then E is an MP3 ring with involution* σ_1*. Further, F is the set of symmetric elements of E under* σ_1*.*

Proof. Since G(E/F) is a multiplicative group, obviously ${\sigma_1}^2 = \sigma 0$ the identity automorphism. Hence, if $\alpha \in$ E, then $\sigma_1(\sigma_1(\alpha)) = \alpha$. Also, given α,$\beta \in$ E then $\sigma_1(\alpha + \beta) = \sigma_1(\alpha) + \sigma_1(\beta)$ since σ_1 is an automorphism of E and preserves its additive structure. Now $\sigma_1(\alpha \cdot \beta) = \sigma_1(\alpha) \cdot \sigma_1(\beta) = \sigma_1(\beta) \cdot \sigma_1(\alpha)$. Hence, σ_1 is an involution. Naturally, $\sigma_1(\alpha) = \alpha$ if and only if $\alpha \in$F. Thus, F = S, the set of symmetric elements of E under σ_1 . To show E is MP3, let $\alpha \in$E and $\alpha \neq 0$. Set $\gamma = \sigma_1(\alpha) \cdot \alpha$. Then $\gamma \neq 0$ since $\alpha \neq 0$. Accordingly, $\sigma 1(\sigma_1(\alpha) \cdot \alpha) = \alpha \cdot \sigma_1(\alpha) = \sigma_1(\alpha) \cdot \alpha$. Hence, E is MP3.

Corollary. *If G(E/F) denote the Galois group of the field E over F, and G(E′/F′) denote the Galois group of the field E′ over F′, both satisfying the conditions of the previous theorem, and if G(E/F) and G(E′/F′)* $= \{{\sigma_0}', {\sigma_1}'\}$*are such that* $\sigma_1(E) \approx {\sigma_1}'(E')$ *then E and E′ are MP3- equivalent in the sense that they share the same set of symmetric elements.*

Proof. Let $\alpha \in$F. Then $\sigma_1(\alpha) = \alpha$ and $({\sigma_1}')^{-1}(\sigma_1(\alpha)) = ({\sigma_1}')^{-1}(\alpha)$. Thereby $\alpha = ({\sigma_1}')\ ^{-1}(\alpha)$. Hence, from having ${\sigma_1}'(\alpha) = \alpha$ it is apparent that σ_1 and ${\sigma_1}'$ fix the same subfield of symmetric elements. Consequently, E and E′ are MP3 – equivalent.

References

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