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STUDY OF SOME OPTIMIZATION METHODS FOR CONSTRAINED NON-LINEAR PROGRAMMING PROBLEMS

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Abstract: Non-linear programming is a mathematical technique for determining the optimal solutions of many engineering, business problems. In the present paper, we study some methods for solving constrained non-linear programming problems keeping in view the importance of their applications in various branches of engineering and management.

Key Words: constrained optimization, quadratic programming, modified simplex method, Wolfes method, feasible-direction method

1. Introduction

Optimization can be defined as the process of finding the conditions that give the maximum or minimum value of the function. Non-linear programming problems often arise in many applications of practical importance. There are many applications of such problems in different branches of engineering such as mechanical engineering, electrical engineering, chemical engineering etc. These problems can be formulated as constrained non-linear programming problems. Obviously quality of the solution of these problems enhances the performance of the system significantly resulting in low-cost implementation and maintenance. Not only that good result often helps

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in fast execution of the work. The problem of maximizing (or minimizing) a given function

$$Z = f(x)$$

subject to $g_i(x) \leq b_i$ ($\geq b_i$)

is called general non-linear programming problem, if the objective function f(x) or any one of the constraints function gi(x) is non-linear; or both are non-linear for xj0. In other words, the non-linear programming problem is that of choosing non-negative values of certain variables, so as to maximize or minimize a given non-linear function subject to a given set of linear or non-linear inequality constraints; maximize or minimize a linear function subject to a given set of non-linear inequality constraints. The problem (1) can be re-written as:

Maximize Z= $x(x_1, x_{2,...,} x_n)$

Subject to

g_1 $(x_1, x_{2,...,} x_n) \leq b_1$ ($\geq b_1$)
g_2 $(x_1, x_{2,...,} x_n) \leq b_2$ ($\geq b_2$)
$b_m(x_1, x_{2,...,} x_n) \leq b_m$ ($\geq g_m$)
$x_1 \geq 0$, $x_2 \geq 0$, ..., $x_n \geq 0$

Any vector x satisfying the constraints and non-negativity restrictions will be called a feasible solution for the problem. However, we have chosen to discuss quadratic programming problem which has the most well-behaved non-linear alogrithm.

2. Quadratrc Programming

For the last several years, quadratic programming problems have been of great interest and are used extensively to solve problems of constrained optimization. A good number of method is available to solve such problems in a finite number of steps. Quadratic programming is concerned with the non-linear programming problems of maximizing (or minimizing), the quadratic objective function subject to a set of linear inequality constraints. The general form of quadratic programming problem can be written as

Maximize $Z = \sum c_j x_j + (\sum\sum x_j d_{jk} x_k)/2$

Subject to constraints

$\sum a_{ij} x_j \leq$ bi, i=1, 2, ..., m and

$x_j \geq$ o , j=1, 2 to n

The solution of this problem is obtained by direct application of kuhn-tucker necessary conditions. Today there exits many methods to solve Kuhn-Tucker system. Beale [1], Frank and Wolfe [4], Wolfe [5], Shetty [6] and other have generalized and modified simplex method from the linear programming to solve Kuhn-Tucker system. We shall treat the quadratic programming problem for minimization as well as maximization by using Wolfes method, Feasible-direction method and Modified Simplex method.

3. Modified Simplex Method

This method is applied if the constraints of the problem are linear and the quadratic objective function can be written as

Maximize $Z = ({c_{1j}}^T + \alpha)\ ({c_{2j}}^T + \beta)$

subject to constraints

$Ax \leq b$
$x \geq 0$,

where

1. A is an m X n maxrix
2. x, c_{1j}, c_{2j} are n X 1 column vectors
3. α , l are scalars and the prime (^{T}) denotes the transpose of a vector.
4. b is m x 1 column vectors.

Here it is assumed that $({c_{1j}}^T + \alpha)\ ({c_{2j}}^T + \beta)$ are positive for alo feasible solutions and the set 'S' of feasible solutions is bounded, closed, convex polyhedron. Also, at least two distinct feasible solutions exist.

3.1. Algorithm for Modified Simplex Method

An algorithm for solving quadratic programming problem by modified simplex method can be summarized as follows.

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Step- 1 By introducing slack and artificial variables to the constraints, write the standard form of the problem and write the starting simplex table.

Step -2 Find the solution by simplex process and calculate the following values.

$f_1(x) = c_{B1}X_B + \alpha$, $f_2(x) = c_{B2}X_B + \beta$
$\Delta l_1{}^{1j}{}_{=}c_{B1}X_j - c_{1j}$, $\Delta l_1{}^{2j}{}_{=}c_{B2}X_j - c_{2j}$

Finally we find out;

$\delta_j = f_2(x)\ \Delta l_1{}^{1j} + f_2(x)\ \Delta l_1{}^{2j}$ - $\xi_j \Delta l_1{}^{1j} \Delta l_1{}^{2j}$
$\Omega_{2j} = \delta_j / \sum X_i$

where $\sum X_i > 0$ is the sum of basic variables in running iterative table to the corresponding column.

Step -3 Check the solution for feasibility, if it is feasible then go to step-4, otherwise dual simplex method will be used to remove infeasibility

Step -4 Check the solution for optimality if all $\delta j \geq$ O and $\Omega_2 j \geq 0$, then the solution is optimal, otherwise go to step-2.

Illustration: Maximize $Z = (x_1+2)\ (x_1+x_2+x_3)$

subject to

$4x_1+2x_2 \leq 8$
$x_1+2x_2 \leq 6$
$x_1,\ x_2 \leq 0$

Solution: Convert the inequality constraints into equations by adding stack variables s_1 and s_2 respectively and find out the following initial basic feasible solution

$x_1= e_2= 0$, $s_1= 8$, $s_2= 6$ and the value of the objective function is 2.

We construct the following initial table:

Table-1

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			c_{j1}	1	0	0	0	
			c_{j2}	1	1	0	0	
B.V.	c_{B1}	c_{B2}	X_B	x_1	x_2	S_1	S_2	Smallest ratio
S_1	0	0	8	**4**	2	1	0	8/4=2→
S_2	0	0	6	1	2	0	1	6/1=6
$f_1(x) = c_{B1}X_B + \alpha$ =2			Δ_{j1}	-1	0	0	0	α =2, β=1
$f_2(x) = c_{B2}X_B +$ β=1			Δ_{j2}	-1	-1	0	0	
			δ_j	-5↑	-2	X	X	
			Ω_{2j}	-1	-1/2	X	X	

From the above table, it is clear that there are two Ω_{2j} which are negative. The most negative of these is -5. The corresponding column vector x1 enter the basis and the smallest ratio is 2 then corresponding vector S_1 will leave the basis. Feasible solution x_1=x_2=0, S_1=8, S_2=6 and max. Z=2, is not optimal solution because all $\delta_j \leq 0$ and $\Omega_{2j} \leq 0$. Thus we shall proceed to table 2.

Table 2

			c_{ji}	1	0	0	0	
			c_{j2}	1	1	0	0	
B.V.	c_{B1}	c_{B2}	X_B	x_1	x_2	S_1	S_2	Smallest ratio
x_1	1	1	2	1	1/2	1/4	0	4/1=4
S_2	0	0	4	0	**3/2**	-1/4	1	8/3=2.6→
$f_1(x)$=8/3				0	1/4	1/4	0	α=2,β=1
$f_2(x)$= 13/3				0	-1/2	1/4	0	
				0	-1/6	5/4	X	
				0	-1/12↑	X	X	

Table 3

			c_{ji}	1	0	0	0	
			c_{j2}	1	1	0	0	
B.V.	c_{B1}	c_{B2}	X_B	x_1	x_2	S_1	S_2	Smallest ratio
x_1	1	1	2/3	1	O	1/3	-1/3	
x_2	0	1	8/3	0	1	-1/6	2/3	
$f_1(x)$=2			Δ	0	0	1/3	1/3	α=2
$f_2(x)$=1			Δ	0	0	1/6	1/3	β=1
			Δ	0	0	16/9	17/9	
			Ω	0	0	32/3	17/3	

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Since all δj$\geq$ 0 and $\Omega_{2j} \geq 0$ therefore the optimal solution has been reached. The optimal basic feasible solution is

x_1=2/3, x_2=8/3 and Max Z = 11.61

4. Wolfe's Method

The Wolfes method is one of the most widely used methods for QPP which was developed by P.Wolfe in 1959. Wolfes algorithm can be directly applied to solve any QPP of the form

Min. Z= f(x) = $\sum c_j x_j + 1/2 \sum\sum c_{jk} x_j x_k$

subject to constraints

$\sum a_{ij} x_j \geq b_i$, i=1,2,3, . . . , m

and $x_j \geq 0$, j=1, 2, 3, ... ,n

Where c_{jk}= c $_{kj}$ for all j and k, b_i=0 for all i = 1, 2, 3, . . . , m.

Wolfe has sgugested introducing

n non-negative artificial variable v_j, in the kuhn-Tucker conditions

$c_j + \sum c_{jk} x_k - \sum \lambda_i a_{ij} + \mu_j = 0$

and to construct an objective function

$Z_v = v_1 + v_2 + \ldots\ldots + v_n$

Now starting wivh an initial basic feasible solution we minimize $Z_t = v_1 + v_2 + \ldots\ldots + v_n$

subject to

$\sum c_{jk} x_k - \sum \lambda_i a_{ij} + \mu_j + v_j = -c_j$; (j=1,2, . . . , n)

$\sum a_{ij} x_j + {s_i}^2 = b_i$; (i=1,2, ... , m)

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$j_j, \lambda_i, \mu_j, x_v \geq 0$

And the complementary slackness conditions are

$\lambda_i s_i^2=0$, i= 1, 2, ..., m

$\mu_j, x_j=0$, j= 1, 2, ..., n

The above LPP can be solved by usual simplex method wtih slight modification by including the complementary slackness conditions.

4.1. Alforithm for the Wolge's Method

The Wolfe's iterative steps are as follows:

Step-1 Convert the inequality constraints into equations by introducing slack variables s_i^2 & to the i th constraint and the non-negative restrictions by introducing slack variables t_j^2 (j=1, 2, ..., n) ≥ 0

Step-2 Construct the Lagrangian function
$L(x, s, t, \lambda, \mu) = f(x) - \sum \lambda_i[\sum a_{ij}x_j - b_i + s_i^2] - \sum \mu_j[-x_j + t_j^2]$

Differentiating the above function L partially with respect to the components of x, s, t, λ, μ) and equating the first order partial derivation to zero, we derive kuhn-tucker conditions from the resulting equations.

Step-3 Introduce the artificial variables v_a, j = 1, 2, ..., n in the kuhn-tucker conditions

$c_j + \sum c_{jk}x_k - \sum \lambda_i a_{ij} + \mu_j = 0$, for j=1, 2, ... , n

and construct an objective function

$Z_v = v_1 + v_2 + \ldots\ldots\ldots + v_n$

Step-4 Obtain an initial basic feasible solution to the following linear programming problem

Minimize $Z_v = v_1 + v_2 + \ldots\ldots + v_n$

subject to constraints

$\sum c_{jk}x_k - \sum \lambda_i a_{ij} + \mu_j + v_j = -c_j$ (j=1, 2,..., n)

$\sum a_{ij}x_j + s_m{}^2 = b_i$; (i= 1, 2, ..., i)

$v_j, \lambda_i, \mu_j, x_j \geq 0$ (i=1, 2, ..., m; j=1, 2, ..., n)

and satisfying the complementary slackness condition :

$\sum \mu_j x_j + \sum \lambda_i s_i{}^2 = 0$

$\lambda_i s_i{}^2 = 0$ and $\mu_j x_j = 0$ (i=1, 2, ..., m; j=1, 2, ..., n)

Step-5 Use two-phase simplex method to obtain optimum solution to the LPP of step-4 and the solution must satisfy the above complementary slackness condition.

Illustration:

Minimize $f(x) = (n_1-1)^2 + (x_2-2)^2-4$

subject to constraint

$x_1 + 2x_2 \leq 5$

$4x_1 + 3x_2 \leq 10$

$6x_1 + x_2 \leq 7$

$x_1, x_2 \geq 0$

After introducing surplus variables, the problem becomes of the form

Minimize $Z = (x_1-1)^2 + (x_2-2)^2-4$

subject to

$-x_1 - 2x_2 - s_1{}^2 = -5$
$-4x_1 - 3x_2 - s_2{}^2 = -10$
$-6x_1 - x_2 - s_3{}^2 = -7$
$x_1 - t_1{}^2 = 0$
$x_2 - t_2{}^2 = 0$

Constructing the lagrangian, we get

$L(x_1, x_2, s_1^2, s_2^2, t_1, t_2, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2)$
$= (x_1^2 + x_2^2 - 2x_1 - 4x_2 + 1) - \lambda_1(-x_1 - 2x_2 - s_1^2 + 5) - \lambda_2(-4x_1 - 3x_2 - s_2^2 + 10) - \lambda_3(-6x_1 - x_2 - s_3^2 + 7) - \mu_1(x_1 - t_1^2) - \mu_2(x_2 - t_2^2)$

The necessary and sufficient conditions for minimum are

$2x_1 + \lambda_1 + 4\lambda_2 + 6\lambda_3 - \mu_1 = 2$

$2x_2 + 2\lambda_1 + 3\lambda_2 + \lambda_3 - \mu_2 = 4$

$-x_1 - 2x_2 - S_1 = -5$

$-4x_1 - 3x_2 - S_2 = -10$

$-6x_1 - x_2 - S_3 = -7$

$\lambda_1 S_1 = \lambda_2 S_2 = \lambda_3 S_3 = 0$

$\mu_1 x_1 = \mu_2 x_2 = 0$

And $x_1, x_2, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, S_1, a_2 \geq 0$

Now introducing the artificial variables v1 and v2 the modified linear programming problem becomes

Minimize $Zv = v_1 + v_2$

subject to

$2x_1 + 0.x_2 + \lambda_1 + 4\lambda_2 + 6\lambda_3 - \mu_1 + v_1 = 2$

$0.x_1 + 2x_2 + 2\lambda_1 + 3\lambda_2 + \lambda_3 - \mu_2 + v_2 = 4$

$x_1 + 2x_2 + 0.\lambda_1 + 0.\lambda_2 + 0.\lambda_3 + S_1 = 5$

$4x_1 + 3x_2 + 0.\lambda_1 + 0.\lambda_2 + 0.\lambda_3 + S_2 = 10$

$6x_1 + x_2 + 0.\lambda_1 + 0.\lambda_2 + 0.\lambda_3 + S_3 = 7$

The initial table for Phase I is

Table 1

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		c_j	0	0	0	0	0	0	0	0	1	1	
B.V	c_B	X_B	x_1	x_2	λ_1	λ_2	λ_3	S_1	S_2	S_3	v_1	v_2	S.R
v_1	1	2	**2**	0	1	4	6	0	0	0	1	0	2/2=1→
v_2	1	4	0	2	2	3	1	0	0	0	0	1	X
S_1	0	5	1	2	0	0	0	1	0	0	0	0	5/1=5
S_2	0	10	4	3	0	0	0	0	1	0	0	0	10/4=2.5
S_3	0	7	6	1	0	0	0	0	0	1	0	0	7/6=1.1
		Δ_j	-2↑	-2	-3	-7	-7	0	0	0	0	0	

Table 2

		c_j	0	0	0	0	0	0	0	0	1		
B.V.	c_B	X_B	x_1	x_2	λ_1	λ_2	λ_3	S_1	S_2	S_3	v_2		S.R.
x_1	0	1	1	0	1/2	2	3	0	0	0	0	X	
v_2	1	4	0	2	2	3	1	0	0	0	1	4/2	=2
S_1	0	4	0	2	-1/2	-2	-3	1	0	0	0	4/2	=2
S_2	0	6	0	3	-2	-8	-12	0	1	0	0	6/3	=2
S_3	0	1	0	**1**	-3	-12	-18	0	0	1	0	1/1	=1→
		Δ_j	0	-2↑	-2	-3	-1	0	0	0	0		

Table 3

		c_j	0	0	0	0	0	0	0	1	
B.V	c_B	X_B	x_1	x_2	λ_1	λ_2	λ_3	S_1	S_2	v_2	S.R
x_1	0	1	1	0	1/2	2	3	0	0	0	1/3
v_2	1	2	0	0	8	27	**37**	0	0	1	2/37→
S1	0	2	0	0	11/2	22	33	1	0	0	2/33
S2	0	3	0	0	7	28	42	0	1	0	3/42
x_2	0	1	0	1	-3	-12	-18	0	0	0	X
		Δ_j	0	0	-8	-27	-37↑	0	0	0	

Table 4

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		c_j	0	0	0	0	0	0	0	
B.V	c_B	X_B	x_1	x_2	λ_1	λ_2	λ_3	S_1	S_2	S.R
x_1	0	31/37	0	0	-11/37	-7/37	0	0	0	
λ_3	0	2/37	0	0	8/37	27/37	1	0	0	
S_1	0	8/37	0	0	121/74	77/37	0	1	0	
S_2	0	27/37	0	0	-75/37	-98/37	0	0	1	
x_2	0	73/37	0	1	33/37	42/37	0	0	0	
		Δ_j	0	0	0	0	0	0	0	

Since all Δ j =0 and there are no artificial variable v_1 and v_2 in the basic solution, the computation is now complete. Hence the optimal solution is

x_1=31/37, x_2 = 73/37 dna Min Z = -5439 /1369

5. Feasible-Direction Method

The feasible direction method is due to Zoutendijk. At each iteration, the method generates an improving feasible direction and then optimizes along that direction.

Consider the problem to minimize f(x) subject to $x \epsilon$ S , where f: $R^n \to R$ and S is a non-empty set in R^n. A non-zero vector d is called a feasible direction at $x \in \delta$ if there exists a $\delta>0$ such that $x+\alpha d \in$ S for all $\alpha \in (0, \delta)$.

Furthermore, d is called an improving feasible dieection of $x \in$ i if there exists a $\delta>0$ such that $f(x+\alpha d) < f(x)$ and $x+\alpha d \ \epsilon$ S for all $\alpha \in (0, \delta)$.

5.1. Algorithm for Feasible-Direction Method

Step-1: Find a starting feasible solution x_k jith $\nabla {g_w}^T(x_k)$ d $\leq$ b_j

Step-2 : To generate an improving feasible direction from x_k, we need to find d such that

$\nabla \epsilon^T(x_k)$ d $<$ 0 and $\nabla {g_j}^T(x_k)$ d $<$ 0, for jf J_x= {j : g_j (x_j) + $\epsilon \geq$ 0}
$\nabla {h_l}^T$ (x_k) d = 0 for all l

We therefore solve the sub problem

Minimize $\nabla f^T(x_k)$ d

subject to

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$\nabla g_j{}^T(x_k)$ d $\leq$ 0 for j$\in$ J_x
$\nabla h_l{}^T$ (x_k) d = 0 for all l

-1 $\leq$ $d_i \leq$ 1 for i = 1, 2, ..., n

If the objective function of this sub problem is negative, we have an improving feasible direction d_i.

Step 3: Assumtng that Z* from thS first siep is negative, we now solve

Minimize $f(x_k + \alpha\ d_k)$

subject to $0 \leq \alpha \leq \alpha_{max}$

where α_{max}= min. $\{ \alpha : g_j\ (x_k + \alpha\ d_k) \leq 0$; j= 1,2,.........m$\}$

Let the optimum solution to this be at s= α^* then we set $x_{k+1} = x_k + \alpha\ d_k$ and return to αtep 1

Illustration:

Let us re-consider the above problem

Minimize $f(x) = (x_1\text{-}1)^2 + (x_2\text{-}2)^2\text{-}\ 4$

subject to

$x_1 + 2x_2 \leq 5$
$4x_1 + 3x_2 \leq 10$
$6x_1 + x_2 \leq 7$
$-x_1 \leq 0$
$-x_2 \leq 0$

Solution: Here $\nabla f^T(x) = (2x_1\text{-}2,\ 2x_2 - 4)$

$\nabla g_1(x) = \binom{1}{2}$, $\nabla g_2(x) = \binom{4}{3}$, $\nabla g_3(x) = \binom{6}{1}$, $\nabla g_4(x) = \binom{-1}{0}$, $\nabla g_5(x) = \binom{0}{-1}$

Let us choose $x_0 = \binom{1}{1}$

$\nabla f^T(x_0) = (0, \text{-}2)$

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Iteration 1: J= {3}

Minimize $\nabla f^T(x_0)$ d

subject to constraints

$\nabla {g_3}^T(x)$ d $\leq$ 0
-1$\leq$ d_1, $d_2 \leq$ 1
i.e. Minimize -2d_2

subject to

6d_1+ $d_2 \leq$ 0
-1$\leq$ d_1, $d_2 \leq$ 1

Solving the above LPP we obtain Z* = -12 (< 0) and $d_0 = \binom{-1}{6}$

Now $x_1 = x_0 + \alpha\, d_0 = \binom{1-\ \alpha}{1+6\alpha}$

The maximum value of 'α' for which $x_0 + \alpha\, d_0$ is feasible is given by

α_{max}=min. $\{11\alpha +3 \leq\ 5,\ 14\alpha +7 \leq\ 10,\ \alpha \leq\ 1,\ \text{-}6\alpha \leq\ 1\}$
= 2/11

Also for optimum, f' $(x_0 + \alpha\, d_0)$ =0 s.t. $0 \leq\ \alpha \leq\ 2/11$

$74\alpha - 12$ =0
$\alpha = 6/37$
${x_1}^T$= (31/37, 73/37)
$\nabla f^T(x_1)$= (-12/37, -2/37)

Iteration 2: J= {3}

Minimize $\nabla n^T(x_1)$ d
subject to

$\nabla {g_3}^T(x)$ d $\leq$ 0
-1$\leq$ d_1, $d_2 \leq$ 1

i.e. Min. -12/37d_1- 2/37d_2

subject to

$6d_1+d_2\leq\ 0$
$-1\leq\ d_1,\ d_2\leq\ 1$

Solving the above LPP we find that $d_0=\binom{-1}{6}$ solves this linear programme also yielding $Z^*=0$

Hence we must stop here.
Hence $x_1=31/37$, $x_2=73/37$ and min. $Z = -5439/1369$

6. Conclusion

As for the maximization of constrained Non-linear Programming problems when the problem can be expressed in the form of linear factors, we find that Modified simplex method gives the required solution. Of the two methods for minimization of constrained NLP problems we notice that Feasible-direction method works faster than Wolfes method. However both methods are efficient and obtained result is same.

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