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ON PARTIAL ORBITS AND STABILIZERS

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Abstract: Within the framework of partial actions of groups, we introduce here the partial stabilizer and prove that it coincides with the global stabilizer. In addition, we define the partial orbits and show that they are completely determined by the global orbits. Finally, we use this result to prove that the partial action (X, α) is $n - transitive$ if and only if its enveloping action (T, β) is $n - transitive$.

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1. Introduction

Partial actions of groups constitute a natural generalization of global actions. They appeared in the theory of operator algebras (see [3]). The formal definition of this concept was given by Exel in 1998 ([4]). Later in 2003, Abadie ([1]) introduced the notion of enveloping action and found that any partial action possesses an enveloping action. Many studies have shown that partial actions are a powerful tool to generalize many well-known results of global actions (see [2], [5] and the literature quoted therein).

In 2009, Choi and Lim ([2]) introduced the concept of *transitive* partial action. Among other results, they proved that each transitive partial group action is the restriction of a group coset action. Although in that paper they use implicitly the concepts of partial stabilizer and partial orbits, they do not study their properties and their relationship to global notions given in the enveloping action of the partial action base.

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The main goal of this paper is to study some classical concepts of global actions in the context of partial actions. In Section 1 we present some basic results of global and partial actions. In Section 2 we define the partial stabilizer and prove that this set coincides with the global stabilizer. In addition, we define the partial orbits and show that they are completely determined by the global orbits. Finally, we use this result to prove that the partial action (X, α) is n-transitive if and only if its enveloping action (T, β) is n-transitive.

2. Global and Partial Actions

In this section we define the global and partial actions and present some properties of them. Other details can be found in [6] and [2].

Definition 1. Let Y be a nonempty set and G a group with identity element 1. A global action of G on Y is a function from $G \times Y$ into Y satisfying:

1. $1 \cdot y = y$ for all $y \in Y$.
2. $(gh) \cdot y = g \cdot (h \cdot y)$ for all $y \in Y$ and all $g, h \in G$.

An action of G on Y is completely determined by the set of bijections $\gamma = \{\gamma_g \mid g \in G\}$ where $\gamma_g : Y \to Y$ is defined by $\gamma_g(y) = g \cdot y$ for each $y \in Y$ and each $g \in G$. We denote this action by γ or (Y, γ).

Given a global action (Y, γ) several important sets can be defined. The *stabilizer* of $y \in Y$ is the set $G_y = \{g \in G : \gamma(y) = g \cdot y = y\}$ and it can be proved that G_y is a subgroup of G. The *orbit* of $y \in Y$ is the set $O_y = \{g \cdot y : g \in G\}$. The collection $\{O_y : y \in G\}$ determines a partition of Y, which is denoted by Y/G. A global action is called *transitive* if it determines a single orbit.

Partial actions of groups appeared independently in various areas of mathematics, in particular, in the study of operator algebras. The formal definition of this concept was given by Exel in 1998 ([4]). We use the definition of partial action given in [3].

Definition 2. A partial action α of the group G on the set X is a collection of subsets S_g, $g \in G$, of X and bijections $\alpha_g : S_{g^{-1}} \to S_g$ such that for all $g, h \in G$ the following statements hold:

1. $S_1 = X$ and α_1 is the identity of X.
2. $S_{(gh)^{-1}} \supseteq \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.
3. $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for all $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.

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The partial action α will be denoted by (X, α) or $\alpha = \{S_g, \alpha_g\}_{g \in G}$. Examples of partial actions can be obtained by restricting a global action to a subset. More exactly, suppose that G acts on Y by bijections $\gamma_g : Y \to Y$ and let X be a subset of Y. Set $S_g = X \cap \gamma_g(X)$ and let α_g be the restriction of γ_g to $S_{g^{-1}}$, for each $g \in G$. Then it is easy to see that $\alpha = \{S_g, \alpha_g\}_{g \in G}$ is a partial action of G on X. In this case, α is called the *restriction* of γ to X. In fact, it can be proved that any partial action (X, α) possesses a minimal global action (T, β) (enveloping action of (X, α)), such that α is the restriction of β to X ([1], Theorem 1.1).

3. Results

In this section we consider a nonempty set X and a partial action $\alpha = \{S_g, \alpha_g\}_{g \in G}$ of the group G on X. By Theorem 1.1 of [1] there exists an enveloping action (T, β) for (X, α). That is, there exist a set T and a global action $\beta = \{\beta_g \mid g \in G\}$ of G on T, where each β_g is a bijection of T, such that the partial action α is given by restriction. Thus, we can assume that $X \subseteq T$, T is the orbit of X, $S_g = X \cap \beta_g(X)$ for each $g \in G$ and $\alpha_g(x) = \beta_g(x)$ for all $g \in G$ and all $x \in S_{g^{-1}}$.

Definition 3. Let (X, α) be a partial action with enveloping action (T, β). The partial orbit of $x \in X$ is defined by

$$O_x^\alpha = \{y \in X : \exists g \in G,\ x \in S_{g^{-1}} \text{ and } \alpha_g(x) = y\}.$$

The next proposition shows that the restriction of global orbits are partial orbits. Moreover, we give a method for going up partial orbits to global orbits.

Proposition 4. *Let (X, α) be a partial action with enveloping action (T, β). The following statements hold:*

1. $O_x = \bigcup_{g \in G} \beta_g(O_x^\alpha)$ *for each* $x \in X$.
2. $O_x^\alpha = O_x \cap X$ *for each* $x \in X$.

Proof. 1. Let $x \in X$. The inclusion $O_x \subseteq \bigcup_{g \in G} \beta_g(O_x^\alpha)$ is evident since $x \in O_x^\alpha$. On the other hand, $z \in \bigcup_{g \in G} \beta_g(O_x^\alpha)$ implies that $z = \beta_{gh}(x)$ for some $g, h \in G$ and the result follows.

2. Let $x \in X$. The inclusion $O_x^\alpha \subseteq O_x \cap X$ is clear. If $z \in O_x \cap X$ then $x = \beta_{g^{-1}}(z)$ for some $g \in G$. So, $x \in S_{g^{-1}}$ and $\alpha_g(x) = z$ which implies that $O_x \cap X \subseteq O_x^\alpha$ and the result follows. □

Recall that the orbits of the global action (T, β) induce a partition of T. So, by Proposition 4 the partial orbits of (X, α) induce a partition of X. We denote by T/G and X/α the set of all orbits of T and the set of all partial orbits of X respectively.

Theorem 5. *Let (X, α) be a partial action with enveloping action (T, β). Then, there exists a one-to-one correspondence between T/G and X/α.*

Proof. If $y \in T$ there exist $x \in X$ and $g \in G$ such that $\beta_g(x) = y$ and thus $O_x = O_y$. Let $x \in X$. For $O_x \in T/G$ we define $\phi(O_x) = O_x \cap X$. Then, by Proposition 4, the mapping ϕ is well defined and it is injective. In a similar way, the mapping φ defined by $\varphi(O_x^\alpha) = \bigcup_{g\in G} \beta_g(O_x^\alpha)$ for each $O_x^\alpha \in X/\alpha$ is a well defined function and it is injective (Proposition 4). Thus the result follows. □

Definition 6. A partial action (X, α) is said to be *transitive* if for any $x, y \in X$ there exists $g \in G$ such that $x \in S_{g^{-1}}$ and $\alpha_g(x) = y$.

A global action (Y, γ) is called n-transitive if the action on Y^n is transitive. In a similar way the n-transitive partial actions are defined. In [2] was proved that (X, α) is transitive if and only if (T, β) is transitive (Proposition 2.4). This fact is a direct consequence of Theorem 5. Moreover, this result can easily be extended to n-transitive actions. We denote by $|A|$ the cardinality of the set A.

Corollary 7. *Under the above assumptions, the following statements hold:*

1. $|T/G| = |X/\alpha|$.
2. (X, α) *is n-transitive if and only if* (T, β) *is n-transitive.*
3. (X, α) *is transitive if and only if* (T, β) *is transitive.*

In [2] the notion of a partial stabilizer was defined analogously to the global case and it was proved that this set is a subgroup of G (Proposition 2.5). However, in our case we follow a different path using enveloping actions.

Definition 8. Let α be a partial action of the group G on X. The $\alpha-$stabilizer of $x \in X$, is the set $G_x^\alpha = \{g \in G : x \in S_{g^{-1}}, \alpha_g(x) = x\}$.

If (T, β) is the enveloping action of the partial action (X, α) and $x \in X$, it is natural to ask what is the relationship between the sets G_x^α and G_x.

Proposition 9. *Let (X, α) be a partial action with enveloping action (T, β). Then, $G_x^\alpha = G_x$ for every $x \in X$.*

Proof. Let $x \in X$. If $g \in G_x^\alpha$, then $x \in S_{g^{-1}}$ and $\alpha_g(x) = x$. Since α is the restriction of the global action, we have that $x = \alpha_g(x) = \beta_g(x) = g \cdot x$. Hence, $g \in G_x$.

Conversely, if $g \in G_x$ then $g^{-1} \cdot x = x$. Thus, $x \in X \cap \beta_{g^{-1}}(X) = S_{g^{-1}}$ and it is clear that $\alpha_g(x) = x$. So, $g \in G_x^\alpha$ and we conclude that $G_x^\alpha = G_x$ for every $x \in X$. □

A direct consequence of the last proposition is the following result ([2], Proposition 2.5 and Theorem 2.6).

Corollary 10. *Let (X, α) be a partial action with enveloping action (T, β). The following statements hold:*

1. *The set G_x^α is a subgroup of G for each $x \in X$. In particular, if (X, α) is transitive then G_x and G_y are conjugate for all $x, y \in X$.*

2. *If (X, α) is transitive, then the action (T, β) is equivalent to the left coset action $(G, G/G_x^\alpha)$ for all $x \in X$.*

4. Discussion

Since the emergence of partial actions, many researchers have attempted to generalize classical results on global actions into the new context (see [3], [5] and the literature quoted therein). As each partial action on a set has an enveloping action ([1]), it is natural to consider two methods for the study of this theory. The first is to get the results within the same theory of partial actions. The second is to relate the concepts with global actions, that is, to pass through the enveloping action. In this way, the derived properties are transferred to the initial partial action. In the particular case of the stabilizer and the partial orbit, Choi and Lim ([2]) have implicitly used these concepts to obtain results on partial actions. They have used the first method, which makes the proofs longer, more difficult and tedious. On the other hand, the second method provides results that are more transparent, straightforward and elegant (see [5] and the literature quoted therein). The reason of this might be that, in this case, all the known results of global actions can be used. Although Choi and Lim introduced implicitly partial orbits and partial stabilizers, they did not study these concepts and investigate their relationship with the global concepts.

5. Conclusion

Very few authors have studied notions of partial orbits and stabilizers. In this paper we introduce and study these concepts explicitly. In particular we relate the partial orbits with the analogous concept in the enveloping action. We prove that any partial orbit is the intersection of the base set X with a global orbit of its enveloping action. We also consider transitive partial actions and extend a previous result by showing that a partial action is n-transitive if and only if its enveloping action is n-transitive. Finally, a partial stabilizer is shown to coincide with the corresponding global stabilizer. As a consequence of this result Proposition 2.5 and Theorem 2.6 of [2] are shown to be evident.

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