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A NOTE ON THE PARTIAL SUM $\sum_{n=1}^{x} n \ln n$ BY USING STIRLING'S APPROXIMATION

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Abstract: The Diverging partial summation formula involving natural numbers is of great interest in mathematics. In this note we have found an almost complete form of partial sum n-log-n.

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1. Introduction

Evaluation of exact value of a divergent series for large upper limit value is a time consuming process. It would linearly depend on the upper limit. In this case having an almost closed form of partial sum is of great help. Here the goal is to express result of the sum in terms of the upper limit. For example, we have

$$\sum_{n=1}^{x} n = \frac{x(x+1)}{2}$$

Instead of having to add x natural numbers , we can use the RHS of the above equation and evaluate the sum easily.

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2. The Expansion

The almost closed form expansion of the partial sum $\sum_{n=1}^{x} n\ln n$ is given

$$\sum_{n=1}^{x} n\ln n = \frac{x^2\ln x}{2} + \frac{x\ln x}{2} + \frac{\ln x}{12} - \frac{x^2}{4} + \frac{\ln(2\pi)}{8} + \frac{1}{720x^2} - \frac{1}{5040x^4} - \frac{1}{15120x^6} + \frac{7}{38016x^8} + 0.0190198444761.$$

3. The Proof

Consider the series expansion of RHS of above equation

$$\begin{aligned}
\sum_{n=1}^{x} n\ln n &= \ln 1 + 2\ln 2 + 3\ln 3 + \cdots + x\ln x \\
&= \ln(1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 3\cdots\cdots x\cdot x\cdot x\cdot x\cdot x) \\
&= \ln((1\cdot 2\cdot 3\cdots x)(2\cdot 3\cdots x)(3\cdots x)\cdots(\overline{x-1}\cdot x)(x)) \\
&= \ln(\frac{x!}{0!}\cdot\frac{x!}{1!}\cdot\frac{x!}{2!}\cdots\frac{x!}{(x-1)!}) \\
&= \ln\left(\frac{(x!)^x}{\prod_{n=0}^{x-1} n!}\right) \\
&= x\ln(x!) - \ln(\prod_{n=0}^{x-1}(n!)) \\
&= x\ln(x!) - \sum_{n=0}^{x-1}\ln(n!) \\
&= (x+1)\ln(x!) - \sum_{n=1}^{x}\ln(n!)
\end{aligned} \tag{1}$$

The Stirling's approximation formula [1] for factorial is

$$n! = \sqrt{2\pi n}\ \cdot n^n\cdot e^{-n}\cdot\exp\left(\sum_{K=2}^{N}\frac{b_k}{k(k-1)n^{k-1}}\right) \tag{2}$$

where b_k is the kth Bernoulli number.[1] Expanding (2), we get

$$n! = n^n\sqrt{2\pi n}\cdot\exp\left(-n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680x^7} + \frac{1}{1188n^9}\right) \tag{3}$$

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Taking logarithm on both sides of equation (3),we get

$$\ln(n!) = \left(n\ln n + \frac{\ln(2\pi n)}{2} - n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680x^7} + \frac{1}{1188n^9}\right) \quad (4)$$

using (4) in (1), we get

$$\begin{aligned}\sum_{n=1}^{x} n\ln(n) =&(x+1)\left(x\ln x + \frac{\ln(2\pi x)}{2} - x + \frac{1}{12x} - \frac{1}{360x^3} + \frac{1}{1260x^5}\right.\\ &\left.-\frac{1}{1680x^7} + \frac{1}{1188x^9}\right)\\ &-\sum_{n=1}^{x}\left(n\ln n + \frac{\ln(2\pi n)}{2} - n + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5}\right.\\ &\left.-\frac{1}{1680n^7} + \frac{1}{1188n^9}\right). \qquad (5)\end{aligned}$$

Now, we have the following results.

$$\begin{aligned}\sum_{n=1}^{x}\frac{\ln(2\pi n)}{2} &= \frac{x\ln(2x)}{2} + \frac{1}{2}\sum_{n=1}^{x}\ln(n)\\ &= \frac{x\ln(2\pi)}{2} + \frac{\ln(x!)}{2}\end{aligned}$$

Also,

$$\sum_{n=1}^{x} n = \frac{x(x+1)}{2}$$

We also have the results of the following partial sums[6].

$$\sum_{n-1}^{x}\frac{1}{n} = \ln(x) + \gamma + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4}$$

$$\sum_{n-1}^{x}\frac{1}{n^3} = \zeta(3) - \frac{1}{2x^2} + \frac{1}{2x^3} - \frac{1}{4x^4} + \frac{1}{12x^6}$$

$$\sum_{n-1}^{x}\frac{1}{n^5} = \zeta(5) - \frac{1}{4x^4} + \frac{1}{2x^5} - \frac{5}{12x^6} + \frac{7}{24x^6}$$

$$\sum_{n-1}^{x}\frac{1}{n^7} = \zeta(7) - \frac{1}{6x^6} + \frac{1}{2x^7} - \frac{7}{128x^8} + \frac{7}{10x^{10}}$$

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$$\sum_{n-1}^{x}\frac{1}{n^9}=\zeta(9)-\frac{1}{8x^8}+\frac{1}{2x^9}-\frac{3}{4x^{10}}+\frac{11}{8x^{12}}$$

where $\zeta(k)$ is Riemann-Zeta function [2]evaluated at k .And γ is Euler–Mascheroni constant. Using the above results in equation (5)and simplifying,we get

$$\sum_{n=1}^{x}n\ln n=\frac{x^2\ln x}{2}+\frac{x\ln x}{2}+\frac{\ln x}{12}-\frac{x^2}{4}+\frac{\ln(2\pi)}{8}+\frac{1}{720x^2}$$
$$-\frac{1}{5040x^4}-\frac{1}{15120x^6}+\frac{7}{38016x^8}+0.0190198444761$$

This completes the proof.

4. Comparison with the Exact Results

The following table gives the value $\sum_{n=1}^{x} n\ln n$ exactly, one computed by our formula and relative error[3].

Number (x)	Exact Value	By using Formula	Relative Error
1	0	$6.2949398239781 \cdot 10^{-5}$	$O(10^{-5})$
2	1.3862943611199	1.3861191806913	$O(10^{-4})$
5	18.2744982337743	18.2744537802143	$O(10^{-6})$
10	102.0828305519349	102.082818052516	$O(10^{-7})$
10^2	20756.741958037946	20756.741957901191	$O(10^{12})$
10^3	3207332.341531321	3207332.3415313107	$O(10^{-16})$
10^4	435563071.31695175	435563071.31695193	$O(10^{-16})$
10^5	55065202972.33287	55065202972.332565	$O(10^{-12})$
10^6	$6.6577621867387 \cdot 10^{12}$	$6.6577621867388 \cdot 10^{12}$	$O(10^{-14})$
10^7	$7.8090486313828 \cdot 10^{14}$	$7.809048631384 \cdot 10^{14}$	$O(10^{-13})$
10^8	$8.9603404640793 \cdot 10^{16}$	$8.9603404640796 \cdot 10^{16}$	$O(10^{-14})$
10^9	$1.0111632928834 \cdot 10^{19}$	$1.0111634948835 \cdot 10^{19}$	$O(10^{-14})$

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