International Electronic Journal of Pure and Applied Mathematics

Volume 8 No. 2 2014, 53-65
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v8i2.2

HAUSDORFF METRICS AND PARAMETRIC CURVES

K.G. Dishlieva[1], A.B. Dishliev[2 §], S.I. Nenov[3], V.I. Radeva[4]

[1]Department of Mathematics
Technical University of Sofia
8, Kl. Ohridski Blvd., Sofia, 1756, BULGARIA
[2,3,4]Dept. of Mathematics
University of Chemical Technology and Metallurgy
8, Kl. Ohridski Blvd., Sofia, 1756, BULGARIA

Abstract: In this paper, we investigate the parametric curves continuous on the left hand side. Assumed that their corresponding parameters belong to their own domains which are generally different for the non-coinciding curves. Furthermore, the points of discontinuity (if the exist) are jump points and specific to each curve. The upper estimate of Hausdorff distance between two such curves is found. A uniform distance between the curves in the common part of their domains is included in the obtained estimate. The findings are useful in research related to orbital Hausdorff continuous dependence and stability of solutions of differential equations with variable impulsive moments.

AMS Subject Classification: 28A78
Key Words: Hausdorff distance, continuous parametric curves

1. Introduction

Finding the estimates of Hausdorff distance from above between parametric curves is an important task. For the convenience, hereafter we shall assume that the parameters of these curves render an account on the time. The qualitative research related to the orbital Hausdorff continuous dependence and stability of solutions of impulsive differential equations may start after finding the above-mentioned esti-

Received: March 12, 2014

[§]Correspondence author

mates. It is known that the trajectories of these equations are piecewise continuous curves on the left hand side in their respective domains. At these points, we have jump discontinuity first type. In the case where differential equations have variable impulsive moments their not coinciding solutions have different sets of points of discontinuity. Therefore, we study and evaluate the Hausdorff distance between parametric curves which are piecewise continuous on the left hand sides and which possess specific (their own) points of jump discontinuity.

The usage of Hausdorff metrics in investigating various issues of fundamental and qualitative theory of differential equations (with and without impulses) can be seen in the papers [1]-[7], [9] and [7], [10].

2. Preliminary Results

The following notations are used. The points a, b from n-dimensional Euclidian space are set to be $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, b_2, ..., b_n) \in \mathbb{R}^n$.

Then the dot product, Euclidean distance and Euclidean norm are traditionally introduced as :

- $\langle a, b\rangle = a_1b_1 + a_2b_2 + \cdots + a_nb_n$ - dot (scalar) product of a and b;
- $\rho(a, b) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2 + \cdots + (a_n - b_n)^2}$ - distance between points a and b;
- $\|a\| = \langle a, b\rangle^{\frac{1}{2}} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$ - norm of a.

The next equality is satisfied

$$\|a - b\| = \rho(a, b).$$

Let the nonempty sets $A, B \in \mathbb{R}^n$. Then the Euclidean and Hausdorff distance between them are denoted respectively by:

$$\rho_E(A, B) = \inf \Big\{ \inf\{\rho(a, b), a \in A\}, b \in B \Big\},$$

and

$$\rho_H(A, B) = \max \Big\{ \sup \{ \inf\{\rho(a, b), a \in A\}, b \in B \} , \sup \{ \inf\{\rho(a, b), b \in B\} , a \in A \} \Big\}.$$

If at least one of the sets A and B is empty, for convenience, we shall assume that:

$$\rho_E(A, B) = 0 \text{ and } \rho_H(A, B) = 0.$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 8, No. 2 (2014)

Further, by ∂A and $\overline{A}$ we will denote the contour and closure of the set A, respectively. The diameter of this set is denoted by

$$\operatorname{diam}(A) = \sup\{\rho(a', a'') \,:\, a', a'' \in A\}.$$

Let r =const> 0. Then the set

$$B_r(A) = \bigcup_{a \in A} \{x \in \mathbb{R}^n \,:\, \rho(x, a) \le r\}$$

is named $r-$neighborhood of A.

Remark 1. For any two nonempty sets $A, B \subset R^n$, the next inequality is valid

$$\rho_E(A, B) \le \rho_H(A, B).$$

Remark 2. Let $\oslash \ne A_1 \subset A_2 \subset R^n$ and $\oslash \ne B_1 \subset B_2 \subset R^n$. Then the next inequality is fulfilled:

$$\rho_E(A_1, B_1) \ge \rho_E(A_2, B_2).$$

We note that such inequality is not valid for the Hausdorff distance.

Remark 3. The following properties of the Hausdorff distance between the sets are valid in R^n (see [8]).

Let the sets $A, B, C \subset R^n$ and a constant $\lambda \in R^+$. Then:

1. $\rho_H(A, B) \ge 0$;
2. $\rho_H(A, B) = 0 \Longleftrightarrow A = B$;
3. $\rho_H(A, B) = \rho_H(B, A)$;
4. $\rho_H(\lambda A, \lambda B) = \lambda \rho_H(B, A)$;
5. $\rho_H(A \cup C, B \cup C) = \rho_H(A, B)$;
6. $\rho_H(A, B) \le \rho_H(A, C) + \rho_H(C, B)$;
7. $\rho_H(\overline{A}, \overline{B}) = \rho_H(A, B)$;
8. If the sets A and B are bounded, then $\rho_H(A, B) < \infty$;
9. If the sets A is bounded and $\rho_H(A, B) < \infty$, then the set B is also bounded;
10. $\rho_H(A, B) = \inf\{r \in R^+ \,:\, A \subset B_r(B), B \subset B_r(A)\}$.

We will recall, that property 10 is often used as a definition of Hausdorff distance between the sets.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 8, No. 2 (2014)

3. Main Results

Theorem 1. *Let the sets $A, B \subset \mathbb{R}^n$ be bounded. Then*

$$|\mathrm{diam}(A) - \mathrm{diam}(B)| \leq 2\rho_H(A, B).$$

Proof. Let ε be a positive number. We have

$$(\exists a', a'' \in A) : \rho(a', a'') < \mathrm{diam}(A) - \varepsilon.$$

Using the notation $r = \rho_H(A, B)$, then the property 10 of Remark 3 yields that $B \subset B_r, (A)$. Therefore,

$$(\exists b', b'' \in B) : \rho(a', b') < r = \rho_H(A, B), \rho(a'', b'') < r = \rho_H(A, B).$$

Then

$$\begin{aligned} \mathrm{diam}(A) - \varepsilon &< \rho(a', a'') \\ &< \rho(a', b') + \rho(b'', a'') + \rho(b', b'') \\ &< 2\rho_H(A, B) + \mathrm{diam}(B)\,, \end{aligned}$$

whence

$$\mathrm{diam}(A) - \mathrm{diam}(B) < 2\rho_H(A, B) + \varepsilon \iff \mathrm{diam}(A) - \mathrm{diam}(B) < 2\rho_H(A, B).$$

Similarly, we find that

$$\mathrm{diam}(B) - \mathrm{diam}(A) < 2\rho_H(A, B).$$

So, the theorem is proved. □

Theorem 2. *Assume that:*

1. *The sets A_1, A_2, B_1, $B_2 \subset R^n$ are bounded.*
2. *It is satisfied $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$.*

Then

$$\rho_H(A, B) = \rho_H(A_1 \cup A_2, B_1 \cup B_2) \lneq \max\{\rho_H(A_1, B_1), \rho_H(A_2, B_2)\}.$$

We will prove the theorem in two different ways.

Proof. Taking into account property 7 of the previous Remark and first condition of the Theorem, without loss of generality, we may suppose that the sets A_1, A_2, B_1, B_2 are compact. This implies that

$$\begin{aligned}\rho_H(A,B) &= \max\Big\{ \sup\big\{ \inf\{\rho(a,b), a\in A\}, b\in B\big\},\\ &\qquad \sup\big\{ \inf\{\rho(a,b), b\in B\}, a\in A\big\}\Big\},\\ &= \max\Big\{ \max\big\{ \min\{\rho(a,b), a\in A\}, b\in B\big\},\\ &\qquad \max\big\{ \min\{\rho(a,b), b\in B\}, a\in A\big\}\Big\}.\end{aligned}$$

For definiteness

$$\rho_H(A,B) = \max\Big\{ \min\{\rho(a,b), a\in A\}, b\in B\Big\} = \rho(a_H, b_H),$$

where $a_H \in A$ and $b_H \in B$.

The following four cases are possible:

Case 1. $a_H \in A_1$ and $b_H \in B_1$. Assume that it is fulfilled

$$\rho_H(A_1,B_1) < \rho_H(A,B).$$

Then

$$\begin{aligned}\rho(a_H,b_H) &= \rho_H(A,B)\\ &< \rho_H(A_1,B_1)\\ &\geq \max\Big\{ \min\{\rho(a_1,b_1), a_1\in A_1\}, b_1\in B_1\}\\ &\geq \min\{\rho(a_1,b_1), a_1\in A_1\}, b_1\in B_1\Big\}.\end{aligned}$$

The above inequality shows that

$$(\forall b_1\in B_1)(\exists a_1 = a_1(b_1)\in A_1) : \rho(a_1(b_1), b_1) < \rho(a_H, b_H).$$

In particular (if $b_1 = b_H$) we conclude that

$$(\exists a_1 = a_1(b_H)\in A_1) :$$
$$\rho(a_1(b_H), b_H) < \rho(a_H,b_H) = \min\{\rho(a,b_H), a\in A\} \leq \min\{\rho(a_1,b_H), a_1\in A_1\}.$$

This contradiction shows that the assumption is not valid, i.e. the following inequality is fulfilled

$$\rho_H(A_1,B_1) \geq \rho_H(A,B),$$

which proves the theorem in this case.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 8, No. 2 (2014)

Case 2. $a_H \in A_2$ and $b_H \in B_2$. The reasoning are similar to the previous case.

Case 3. $a_H \in A_1$ and $b_H \in B_2$. We have

$$\begin{aligned}\rho_H(A,B) =& \rho(a_H, b_H)\\ =& \min\{\rho(a, b_H), a \in A\}\\ \leq& \min\{\rho(a_2, b_H), a_2 \in A_2\}\\ \leq& \max\{\min\{\rho(a_2, b_2), a_2 \in A_2\}, b_2 \in B_2\}\, \rho_H(A_2, B_2),\end{aligned}$$

hence, the theorem has been proved in this case.

Case 4. $a_H \in A_2$ and $b_H \in B_1$. It is satisfied

$$\begin{aligned}\rho_H(A,B) &= \min\{\rho(a, b_H), a \in A\}\\ &\leq \min\{\rho(a_1, b_H), a_1 \in A_1\}\\ &\leq \max\left\{\min\{\rho(a_1, b_1), a_1 \in A_1\}, b_1 \in B_1\right\} \leq \rho_H(A_1, B_1). \qquad \square\end{aligned}$$

Proof. The property 10 in Remark 3 is used as a definition of the Hausdorff distance between two sets.

Denote

$$r = \max\{\rho_H(A_1, B_1), \rho_H(A_2, B_2)\}$$

The following inclusions are valid:

$$B_1 \subset B_r(A_1), \quad B_2 \subset B_r(A_2).$$

The last two inclusions imply

$$B \subset B_1 \cup B_2 \subset B_r(A_1) \cup B_r(A_2) = B_r(A_1 \cup A_2) = B_r(A). \tag{1}$$

Similarly, we obtain

$$A \subset B_r(B). \tag{2}$$

By (1) and (2), we find that $\rho_H(A,B) \leq r$.

Then the theorem can be taken for granted. $\square$

The following Theorem is trivial generalization of the previous one.

Theorem 3. *Assume that:*

1. *The sets $A_1, A_2, \ldots, A_k$, $B_1, B_2, \ldots, B_k \subset R^n$ are bounded.*
2. *We set $A = \bigcup_{i=1}^{k} A_i$ and $B = \bigcup_{i=1}^{k} B_i$.*

Then

$$\rho_H(A,B) \leq \max\{\rho_H(A_1, B_1), \rho_H(A_2, B_2), \ldots, \rho_H(A_k, B_k)\}.$$

As a consequence of Theorem 2 we get the estimate.

Theorem 4. *Assume that:*

3. *The sets A_1, A_2, B_1, B_2, $\subset R^n$ are bounded.*

4. *The following equalities are valid $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$.*

Then

$$\rho_H(A, B) \le \min\{\, \max\{\rho_H(A_1, B_1), \rho_H(A_2, B_2)\}\,, \max\{\rho_H(A_1, B_2), \rho_H(A_2, B_1)\}\}.$$

Proof. The following estimates are obtained from Theorem 2:

$$\rho_H(A, B) = \rho_H(A_1 \cup A_2, B_1 \cup B_2) \le \max\{\rho_H(A_1, B_1), \rho_H(A_2, B_2)\},$$

$$\rho_H(A, B) = \rho_H(A_1 \cup A_2, B_1 \cup B_2) \le \max\{\rho_H(A_1, B_2), \rho_H(A_2, B_1)\},$$

from which we get the statement of the theorem. □

The following theorem is a consequence of Theorem 3. Denote by P_k the set of all permutations of the first k natural numbers.

Theorem 5. *Assume that:*

3. *The sets A_1, A_2, $\ldots$,A_k, B_1, B_2, $\ldots$, $B_k \subset R^n$ are bounded.*

4. *We have $A = \bigcup_{i=1}^{k} A_i$ and $B = \bigcup_{i=1}^{k} B_i$.*

Then

$$\rho_H(A, B) \le \min\Big\{ \max\{\rho_H(A_1, B_{i_1})\,, \rho_H(A_2, B_{i_2})\,, \ldots, \rho_H(A_k, B_{i_k})\}\,,\ (i_1, i_2, \ldots, i_k) \in P_k \Big\}.$$

4. The Hausdorff Distance between the Continuous Curves

Let the functions $g, g^* : R^+ \to R^n$ and constants T_0, T_1, T_0^*, $T_1^* \in R^+$.

We introduce the parametric curves:

$$\gamma[T_0, T_1] = \begin{cases} g(t)\,, & T_0 \le t \le T_1\,,\ T_0 \le T_1, \\ \varnothing\,, & T_0 > T_1, \end{cases}$$

and

$$\gamma^*[T_0^*,T_1^*] = \begin{cases} g^*(t)\,, & T_0^* \le t \le T_1^*\,,\ T_0^* \le T_1^*, \\ \varnothing\,, & T_0^* > T_1^*. \end{cases}$$

The next curves:

$$\gamma(T_0,T_1]\,,\ \gamma[T_0,T_1)\,,\ \gamma(T_0,T_1)\,,\ \gamma^*(T_0^*,T_1^*]\,,\ \gamma^*[T_0^*,T_1^*)\,,\ \gamma^*(T_0^*,T_1^*)$$

are introduced analogously, that are defined in the half-open and open intervals, respectively.

Remark 4. Let $0 \le T_0 \le T_1$ and $0 \le T_0^* \le T_1^*$. The following definitional equalities are valid for the Euclidean, Hausdorff and uniform distances between the curves $\gamma^*[T_0^*,T_1^*]$ and $\gamma[T_0,T_1]$, respectively:

$$\rho_E(\gamma^*[T_0^*,T_1^*],\gamma[T_0,T_1]) = \inf\Big\{\inf\{\rho(g^*(t^*),g(t)),T_0^* \le t^* \le T_1^*\},T_0 \le t \le T_1\Big\},$$

$$\begin{aligned}\rho_H(\gamma^*[T_0^*,T_1^*],\gamma[T_0,T_1]) = \max\Big\{ \\ \sup\big\{\inf\{\rho(g^*(t^*),g(t))\,,\ T_0^* \le t^* \le T_1^*\}\,,\ T_0 \le t \le T_1\big\}\,, \\ \sup\big\{\inf\{\rho(g^*(t^*),g(t))\,,\ T_0 \le t \le T_1\}\,,\ T_0^* \le t^* \le T_1^*\big\}\Big\},\end{aligned}$$

$$\rho_R(\gamma^*[T_0,T_1],\gamma[T_0,T_1]) = \sup\{\rho(g^*(t)),g(t))\,,\ T_0 \le t \le T_1\}.$$

As seen from the above remark, uniform distance between the curves is defined only when they have a common domain.

Analogously to the three equalities the above relating to the Euclidean, Hausdorff and uniform distances are valid for the curves defined in the half-open or open intervals.

We will use the following symbols in the following two theorems:

$$T_0^{\min} = \min\{T_0^*,T_0\}, \quad T_0^{\max} = \max\{T_0^*,T_0\},$$

$$T_1^{\min} = \min\{T_1^*,T_1\}, \quad T_1^{\max} = \max\{T_1^*,T_1\}.$$

Theorem 6. *Assume that:*

1. *The function $g, g^* \in C[R^+,\mathbb{R}^n]$.*
2. *The inequality $T_0^{\max} \le T_1^{\min}$ is valid.*

The following estimate is valid

$$\rho_E(\gamma^*[T_0^*,T_1^*],\gamma[T_0,T_1]) \le \min\{\|g^*(t),g(t)\|,\ T_0^{\max} \le t \le T_1^{\min}\}.$$

Proof. The statement of this theorem follows by Remark 2 and Remark 4. We have

$$\begin{aligned}\rho_E(\gamma^*[T_0^*, T_1^*], \gamma[T_0, T_1]) &\le \rho_E(\gamma^*[T_0^{\max}, T_1^{\min}], \gamma[T_0^{\max}, T_1^{\min}]) \\ &= \inf\Big\{ \inf\{\rho(g^*(t^*), g(t)),\ T_0^{\max} \le t^* \le T_1^{min}\},\ T_0^{\max} \le t \le T_1^{min}\Big\} \\ &\le \inf\{\rho(g^*(t), g(t)),\ T_0^{\max} \le t \le T_1^{\min}\} \\ &= \min\{\rho(g^*(t), g(t)),\ T_0^{\max} \le t \le T_1^{\min}\}.\end{aligned}$$

The theorem is proved. □

Theorem 7. *Assume that:*

1. *The function $g, g^* : \mathbb{R}^+ \to \mathbb{R}^n$ are continuous on the left hand side in their domain.*
2. *The inequality $T_0^{\max} \le T_1^{\min}$ is valid.*

Then the next estimate is valid:

$$\begin{aligned}\rho_H\Big(\gamma^*(T_0^*, T_1^*], \gamma(T_0, T_1]\Big) \le \max\Big\{&\rho_R(\gamma^*(T_0^{\max}, T_1^{\min}], \gamma(T_0^{\max}, T_1^{\min}]), \\ &\rho_H(g(T_0+0), \gamma^*(T_0^*, T_0^*]),\ \rho_H(g^*(T_0^*+0), \gamma^*(T_0, T_0^*]), \\ &\rho_H(g(T_1), \gamma^*(T_1, T_1^*]),\ \rho_H(g^*(T_1^*), \gamma(T_1^*, T_1])\Big\}.\end{aligned}$$

Proof. We will prove this assertion under the additional assumption that the following inequalities are valid:

$$T_0^* \le T_0 \quad \text{and} \quad T_1^* \le T_1\,.$$

The other three cases are treated similarly. In this case we have

$$T_0 \le t < T_0^* = \varnothing \text{ and } T_1 \le t < T_1^* = \varnothing.$$

Therefore

$$\rho_H(g^*(T_0^*+0), \gamma[T_0, T_0^*]) = \rho_H(g^*(T_0^*+0), \varnothing) = 0$$

and

$$\rho_H(g(T_1), \gamma^*[T_1, T_1^*]) = \rho_H(g(T_1), \varnothing) = 0\,.$$

Using Remark 4 and the equalities above, we obtain:

$$\begin{aligned}&\rho_H(\gamma^*(T_0^*, T_1^*], \gamma(T_0, T_1]) \\ &= \max\Big\{ \sup\big\{ \inf\{\rho(g^*(t^*), g(t)),\ T_0^* < t^* \le T_1^*\},\ T_0 < t \le T_1\big\}, \\ &\quad \sup\big\{ \inf\{\rho(g^*(t^*), g(t)),\ T_0 < t \le T_1\},\ T_0^* < t^* \le T_1^*\big\}\Big\}\end{aligned}$$

$$
\begin{aligned}
&= \max\Big\{ \sup\big\{ \inf\{\rho(g^*(t^*), g(t))\,,\ T_0^* < t^* \le T_1^*\}\,,\ T_0 < t \le T_1^*\big\},\\
&\quad \sup\big\{ \inf\{\rho(g^*(t^*), g(t))\,,\ T_0^* < t^* \le T_1^*\}\,,\ T_1^* < t \le T_1\big\},\\
&\quad \sup\big\{ \inf\{\rho(g^*(t^*), g(t))\,,\ T_0 < t \le T_1\}\,,\ T_0^* < t^* \le T_0\big\},\\
&\quad \sup\big\{ \inf\{\rho(g^*(t^*), g(t))\,,\ T_0 < t \le T_1\}\,,\ T_0 < t^* \le T_1^*\big\}\Big\},\\
&\le \max\Big\{ \sup\big\{\rho(g^*(t), g(t))\,,\ T_0 < t \le T_1^*\big\},\\
&\quad \sup\big\{\rho(g^*(T_1^*), g(t))\,,\ T_1^* < t \le T_1\big\},\\
&\quad \sup\big\{\rho(g^*(t^*), g(T_0+0))\,,\ T_0^* < t^* \le T_0\big\},\\
&\quad \sup\big\{\rho(g^*(t^*), g(t^*))\,,\ T_0 < t^* \le T_1^*\big\}\Big\},\\
&\le \max\Big\{ \sup\big\{\rho(g^*(t), g(t))\,,\ T_0^{\max} < t \le T_1^{\min}\big\},\\
&\quad \sup\big\{\rho(g^*(T_1^*), g(t))\,,\ T_1^* < t \le T_1\big\},\\
&\quad \sup\big\{\rho(g^*(t^*), g(T_0+0))\,,\ T_0^* < t^* \le T_0\big\}\Big\},\\
&= \max\Big\{\rho_R(\gamma^*(T_0^{\max}, T_1^{\min}], \gamma(T_0^{\max}, T_1^{\min}]),\\
&\quad \rho_H(g^*(T_1^*), \gamma(T_1^*, T_1])\,,\ \rho_H(g(T_0+0), \gamma^*(T_0^*, T_0])\Big\}\\
&\le \max\Big\{\rho_R(\gamma^*(T_0^{\max}, T_1^{\min}], \gamma(T_0^{\max}, T_1^{\min}])\,,\\
&\quad \rho_H(g(T_0+0), \gamma^*(T_0^*, T_0])\,,\ \rho_H(g^*(T_0^*+0), \gamma(T_0, T_0^*])\,,\\
&\quad \rho_H(g(T_1), \gamma^*(T_1, T_1^*])\,,\ \rho_H(g^*(T_1^*), \gamma(T_1^*, T_1])\Big\}.
\end{aligned}
$$

The theorem is proved.

Let the parametric intervals of the curves $\gamma^*(T_0^*, T_1^*]$ and $\gamma(T_0, T_1]$ coincide i.e. $T_0^* = T_0$ and $T_1^* = T_1$. Then, as a consequence of the previous theorem, we get the following estimate of the Hausdorff distance between them. □

Theorem 8. *Assume that the functions $g, g^* : \mathbb{R}^* \to \mathbb{R}^n$ and continuous on the left-hand side in $\mathbb{R}^+$.*

Then

$$\rho_H(\gamma^*(T_0, T_1], \gamma(T_0, T_1]) \le \rho_R(\gamma^*(T_0, T_1], \gamma(T_0, T_1]).$$

As a consequence of Theorem 7, property 7 of Remark 2 we will formulate the following theorem, relating to the Hausdorff distance between the continuous curves.

Theorem 9. *Assume that:*

1. *The function $g, g^* \in C\Big\{\mathbb{R}^+, \mathbb{R}^n\Big\}$.*

2. *The inequality $T_0^{\max} \le T_1^{\min}$ is satisfied.*

Then the following estimate is valid:

$$\rho_H(\gamma^*[T_0^*, T_1^*], \gamma[T_0, T_1]) \leq \max\Big\{\rho_R(\gamma^*[T_0^{\max}, T_1^{\min}], \gamma[T_0^{\max}, T_1^{\min}]),\ \rho_H(g(T_0), \gamma^*[T_0^*, T_0]),\ \rho_H(g^*(T_0^*), \gamma[T_0, T_0^*]), \rho_H(g(T_1), \gamma^*[T_1, T_1^*]),\ \rho_H(g^*(T_1^*), \gamma[T_1^*, T_1])\Big\}.$$

Theorem 10. *Assume that the functions* $g, g^* \in C\Big\{\mathbb{R}^+, \mathbb{R}^n\Big\}$.
Then

$$\rho_H(\gamma^*[T_0, T_1], \gamma[T_0, T_1]) \leq \rho_R(\gamma^*[T_0, T_1], \gamma[T_0, T_1]).$$

Remark 5. Let the inequalities $T_0 < T_1$ and $T_0^* < T_1^*$ be valid. Consider the linear transformation $\tau : [T_0, T_1] \to [T_0^*, T_1^*]$, defined by means of the linear function

$$t^* = \tau(t) = \frac{(T_1^* - T_0^*)t + T_0^* T_1 - T_1^* T_0}{T_1 - T_0}. \tag{3}$$

We have

$$[T_0^*, T_1^*] = \tau([T_0, T_1]); \quad T_0^* = \tau(T_0); \quad T_1^* = \tau(T_1).$$

Furthermore, for the curve $\gamma^*[T_0^*, T_1^*]$, is fulfilled

$$\gamma^*[T_0^*, T_1^*] = \{g^*(t^*)\,;\ T_0^* \leq t^* \leq T_1^*\} = \{g^*(\tau(t))\,;\ T_0 \leq t^* \leq T_1\} = \{g^{**}(t)\,;\ T_0 \leq t \leq T_1\} = \gamma^{**}[T_0, T_1]. \tag{4}$$

The following statement is valid.

Theorem 11. *Assume that the functions* $g, g^* : \mathbb{R}^+ \to \mathbb{R}^n$ *are left-continuous on* $\mathbb{R}^+$.
Then

$$\rho_H(\gamma^*(T_0^*, T_1^*], \gamma(T_0, T_1]) = \rho_H(\gamma^{**}(T_0, T_1], \gamma(T_0, T_1]) \leq \rho_R(\gamma^{**}(T_0, T_1], \gamma(T_0, T_1]),$$

where

$$\gamma^{**}(T_0, T_1) = \Big\{g^*\Big(\frac{(T_1^* - T_0^*)t + T_0^* T_1 - T_1^* T_0}{T_1 - T_0}\Big)\,;\ T_0 < t \leq T_1\Big\}.$$

Remark 6. Using the theorem above, we can formulate the following algorithm for finding the estimate of Hausdorff distance between two continuous on the left-hand side parametric curves (e.g. curves $\gamma^*(T_0^*, T_1^*]$ and $\gamma(T_0, T_1]$) :

1. Find the linear function $\tau(t)$ defined by (3);
2. Change the parameter of the curve $\gamma^*(T_0^*, T_1^*]$ using the transformation (3) and obtain the curve $\gamma^{**}[T_0^*, T_1^*]$(see the equality (4));

3. Find the uniform distance between $\gamma^{**}(T_0, T_1]$ and $\gamma(T_0, T_1]$, which is an upper limit of the Hausdorff distance between the output curves.

Unfortunately, the estimates of Theorem 7 and Theorem 11 are not comparable to the set of continuous on the left-hand side parametric curves. There are some examples in which the estimate of Theorem 11 is more precise than the estimate of Theorem th11, as well as examples where the opposite is satisfied.

As a consequence of Theorem 2 and Theorem 7 we obtain the following estimate.

Theorem 12. *Assume that:*

1. *The functions $g, g^* : \mathbb{R}^+ \to \mathbb{R}^n$ are left-continuous on $\mathbb{R}^+$.*
2. *There exists a number $k \in \mathbb{N}$, such that the following inequalities are valid:*

$$0 < t_0^* < t_1^* < \cdots < t_k^*; \quad 0 < t_0 < t_1 < \cdots < t_k;$$

$$0 < t_0^{\max} < t_1^{\min} \le t_1^{\max} < t_2^{\min} \le t_2^{\max} < \cdots < t_k^{\min} \le t_k^{\max},$$

where:

$$t_0^{\min} = \min\{t_0^*, t_0\}, \; t_0^{\max} = \max\{t_0^*, t_0\},$$
$$\ldots,$$
$$t_k^{\min} = \min\{t_k^*, t_k\}, \; t_k^{\max} = \max\{t_k^*, t_k\}.$$

Then the next inequality is fulfilled:

$$\rho_H(\gamma^*(t_0^*, t_k^*], \gamma(t_0, t_k]) \le \max\Big\{\rho_R(\gamma^*(T_{i-1}^{\max}, T_i^{\min}], \gamma(T_{i-1}^{\max}, T_i^{\min}]),$$
$$\rho_H(g(T_{i-1}+0), \gamma^*(T_{i-1}^*, T_{i-1}]), \; \rho_H(g^*(T_{i-1}^*+0), \gamma(T_{i-1}, T_{i-1}^*]),$$
$$\rho_H(g(T_i), \gamma^*(T_i, T_i^*]), \; \rho_H(g^*(T_i^*), \gamma(T_i^*, T_i]), \; i = 1, 2, \ldots, k\Big\}.$$

Acknowledgments

We would like to thank University of Chemical Technology and Metallurgy-Sofia for financial support.

References

[1] B. Ahmad, S. Sivasundaram, Setvalued perturbed hybrid integro-differential equations and stability in terms of two measures, *Dynamic Systems and Applications*, **16** (2007), 299-310.

[2] B. Dhage, A. Boucherif, S. Ntouyas, On periodic boundary value problems of first-order perturbed impulsive differential inclusions, *Electronic Journal of Differential Equations*, **84** (2004), 1-9.

[3] A. Dishliev, K. Dishlieva, S. Nenov, *Specific Asymptotic Properties of the Solutions of Impulsive Differential Equations. Methods and Applications*, Academic Publications Ltd. (2011).

[4] A. Dishliev, K. Dishlieva, Orbital Hausdorff continuous dependence of the solutions of impulsive differential equations with respect to impulsive perturbations, *International J. of Pure and Applied Mathematics*, **70**, No. 2 (2011), 167-187.

[5] T. Ghana, V. Lakshmikantham, Set differential equations and flow invariance, *Applicable Analysis*, **82**, No. 4 (2003), 357-368.

[6] V. Lakshmikantham,S. Leela, D. Vasundhara, Stability theory for set differential equations,*Dynamics of Continuous, Discrete and Impulsive Systems, Series A: Mathematical Analysis*, **11** (2004), 181-189.

[7] H.Y. Lan, J. Nieto, On initial value problems for first-order implicit impulsive fuzzy differential equations, *Dynamic Systems and Applications*, **18** (2009), 677-686.

[8] Bl. Sendov, *Hausdorff Approximations*, Publishing House of Bulgarian Academy of Sciences, Sofia, (1979), In Russian.

[9] N. Skripnik, Averaging of impulsive differential inclusions with Hukuhara derivative, *Nonlinear Oscillations*, **. 10**, No. 3 (2007), 422-432.

[10] D. Vasundhara, Basic results in impulsive set differential equations, *Nonlinear Studies*, **10**, No. 3 (2003), 259-271.

[11] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge (1990).

[12] M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, In: *Operator Theory: Advances and Applications*, Birkhäuser, Basel (1994), 369-396.

[13] D.S. Moak, The q-analogue of the Laguerre polynomials, *J. Math. Anal. Appl.*, **81** (1981), 20-47.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 8, No. 2 (2014)