

SYMMETRIC AND GENERATING FUNCTIONS

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Abstract: In this paper, we calculate the generating functions by using the concepts of symmetric functions. Although the methods cited in previous works are in principle constructive, we are concerned here only with the question of manipulating combinatorial objects, known as symmetric operators. The proposed generalized symmetric functions can be used to find explicit formulas of the Fibonacci numbers, and of the Tchebychev polynomials of first and second kinds.

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1. Introduction

By studying the Fibonacci sequence ($F_{n+2} = F_{n+1} + F_n$ with $F_0 = F_1 = 1$), we note its close connection with the equation $x^2 = x + 1$, whose roots are the golden numbers Φ_1 and Φ_2 . It is also noticed that the eigenvalues of the symmetric matrix

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1.1)$$

represent the two golden numbers Φ_1 and Φ_2 of Fibonacci sequence [3]. Consequently, we obtain the following Vieta's formulas

$$\sigma_1 = \lambda_1 + \lambda_2 = 1 \text{ and } \sigma_2 = \lambda_1 \lambda_2 = -1, \quad (1.2)$$

where σ_1, σ_2 are called elementary symmetric functions of real roots λ_1, λ_2 , respec-

tively. So, the eigenvectors of matrix M are multiples of

$$v_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}. \quad (1.3)$$

If we assume that $|\lambda_1| > |\lambda_2|$, then for any positive integer n , we have [3]

$$M^n = \begin{pmatrix} S_n(\lambda_1 + \lambda_2) & \sigma_2 S_{n-1}(\lambda_1 + \lambda_2) \\ S_{n-1}(\lambda_1 + \lambda_2) & \sigma_2 S_{n-2}(\lambda_1 + \lambda_2) \end{pmatrix}, \quad (1.4)$$

where $S_n(\lambda_1 + \lambda_2) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}$.

In this paper, we are interested in the use of symmetric functions to generate the well-known Fibonacci numbers and Tchebychev polynomials of first and second kinds. In this framework, some necessary preliminaries and definitions are given in Section 2. In Section 3, we propose a new theorem which allows the determination of the generating functions. The proposed theorem is based on symmetric functions and a new proposition on the symmetric operators. In Section 4, some applications are given for the generating functions of Fibonacci numbers and Tchebychev polynomials.

2. Preliminaries

2.1. Definition of Symmetric Functions in Several Variables

Consider an equation of degree n of the form

$$(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n) = 0, \quad (2.1)$$

with $\lambda_1, \lambda_2, \dots, \lambda_n$ being real roots. If we expand the left hand side, we obtain

$$x^n - \sigma_1 x^{n-1} + \sigma_2 x^{n-2} - \sigma_3 x^{n-3} + \cdots + (-1)^n \sigma_n = 0, \quad (2.2)$$

where $\sigma_1, \sigma_2, \dots, \sigma_n$ are homogeneous and symmetrical polynomials in $\lambda_1, \lambda_2, \dots, \lambda_n$. To be more accurate, these polynomials can be denoted as $\sigma_i(\lambda_1, \lambda_2, \dots, \lambda_n)$ with $i = 1, 2, \dots, n$, or simply as $\sigma_i^{(n)}$.

The general formula of the polynomials $\sigma_i^{(n)}$ are given by [8]

$$\sigma_i^{(n)} = \sum_{m_1 + m_2 + \cdots + m_n = i} \lambda_1^{m_1} \lambda_2^{m_2} \cdots \lambda_n^{m_n}, \quad (2.3)$$

with $m_1, m_2, \dots, m_n = 0$ or 1 .

The polynomials $\sigma_i^{(n)}$ can be considered as the sum of all distinct products that can be formed by monomial polynomials C_n^i . It is noticed that $\sigma_i^{(n)} = 0$ for $i > n$.

2.2. Symmetric Functions

Let A and B be two alphabets, we denote by $S_n(A - B)$ the coefficients of the rational sequence of poles A and zeros B as follows [2]

$$\sum_{n=0}^{\infty} S_n(A - B)z^n = \frac{\prod_{b \in B} (1 - bz)}{\prod_{a \in A} (1 - az)}. \tag{2.4}$$

Equation (2.4) can be rewritten in the following form

$$\sum_{n=0}^{\infty} S_n(A - B)z^n = \left(\sum_{n=0}^{\infty} S_n(A)z^n \right) \times \left(\sum_{n=0}^{\infty} S_n(-B)z^n \right), \tag{2.5}$$

with

$$S_n(A - B) = \sum_{j=0}^n S_{n-j}(-B)S_j(A). \tag{2.6}$$

The polynomial whose roots are B is written as

$$S_n(x - B) = \sum_{j=0}^n S_{n-j}(-B)x^j, \text{ with } \text{card}(B) = n. \tag{2.7}$$

On the other hand, if A has cardinality equal to 1, i.e., $A = \{x\}$, then equality (2.4) can be rewritten as follows [1]

$$\sum_{n=0}^{\infty} S_n(x - B)z^n = \frac{\prod_{b \in B} (1 - bz)}{(1 - xz)} = 1 + \dots + S_{n-1}(x - B)z^{n-1} + \frac{S_n(x - B)}{(1 - xz)}z^n, \tag{2.8}$$

where $S_{n+k}(x - B) = x^k S_n(x - B)$ for all $k \geq 0$.

The summation is actually limited to a finite number of terms since $S_{-k}(\cdot) = 0$ for all $k > 0$. In particular, we have

$$\prod_{b \in B} (x - b) = S_n(x - B) = S_0(-B)x^n + S_1(-B)x^{n-1} + S_2(-B)x^{n-2} + \dots, \tag{2.9}$$

where $S_k(-B)$ are the coefficients of the polynomials $S_n(x - B)$ for $0 \leq k \leq n$. This coefficients are zero for $k > n$.

For example, if all $b \in B$ are equal, i.e., $B = nb$, then we have

$$S_n(x - nb) = (x - b)^n. \tag{2.10}$$

By choosing $b = 1$, i.e., $B = \left\{ \underbrace{1, 1, \dots, 1}_n \right\}$, we obtain

$$S_k(-n) = (-1)^k \binom{n}{k} \text{ and } S_k(n) = \binom{n+k-1}{k}. \tag{2.11}$$

By combining (2.6) and (2.11), we obtain the following expression

$$S_n(A-nx) = S_n(A) - \binom{n}{1} S_{n-1}(A)x + \binom{n}{2} S_{n-2}(A)x^2 - \dots + (-1)^n \binom{n}{n} x^n. \tag{2.12}$$

For any pair (x, y) we can associate the divided difference ∂_{xy} defined by [8]

$$\partial_{xy}(f) = \frac{f(x, y, z, \dots) - f(y, x, z, \dots)}{x - y}. \tag{2.13}$$

3. Results

In this section, we provide some definitions and a new proposition which will be useful for the next theorem.

Definition 1. The inverse of the sequence $\sum_{n=0}^{\infty} S_n(A)z^n$ is the sequence $\sum_{n=0}^{\infty} S_n(-A)z^n$, that is

$$\sum_{n=0}^{\infty} S_n(A)z^n = \frac{1}{\sum_{n=0}^{\infty} S_n(-A)z^n}. \tag{3.1}$$

Definition 2. The symmetric operator π_{xy}^n is defined by [5]

$$\pi_{xy}^n f(x) = \frac{x^n f(x) - y^n f(y)}{x - y}. \tag{3.2}$$

Proposition 3. Given an alphabet $E_2 = \{e_1, e_2\}$, then for any positive integer k , the operator $\pi_{e_1 e_2}^k$ satisfied the following formula

$$\pi_{e_1 e_2}^k f(e_1) = f(e_1) S_{k-1}(e_1 + e_2) + e_2^k \partial_{e_1 e_2} f(e_1). \tag{3.3}$$

Proof. From (3.2) we have

$$\begin{aligned} \pi_{e_1 e_2}^k f(e_1) &= \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \\ &= \frac{e_1^k f(e_1) - e_2^k f(e_1) + e_2^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2} \end{aligned}$$

$$= \frac{f(e_1) [e_1^k - e_2^k] + e_2^k [f(e_1) - f(e_2)]}{e_1 - e_2}$$

Using the formulas (1.4) and (2.13) we obtain

$$\pi_{e_1 e_2}^k f(e_1) = f(e_1) S_{k-1}(e_1 + e_2) + e_2^k \partial_{e_1 e_2} f(e_1).$$

This completes the proof of proposition 3. □

Theorem 4. Given two alphabets $E_2 = \{e_1, e_2\}$ and $A_2 = \{a_1, a_2\}$, then

$$\sum_{n=0}^{\infty} S_n(A_2) S_n(e_1 + e_2) z^n = \frac{1 - a_1 a_2 e_1 e_2 z^2}{\left(\sum_{n=0}^{\infty} S_n(-A_2) e_1^n z^n \right) \left(\sum_{n=0}^{\infty} S_n(-A_2) e_2^n z^n \right)} \quad (3.4)$$

Proof. Let $f(e_1) = \sum_{n=0}^{\infty} e_1^n S_n(A_2) z^n$, then the left hand side of formula (3.4) can be written as

$$\begin{aligned} \pi_{e_1 e_2} f(e_1) &= \pi_{e_1 e_2} \left(\sum_{n=0}^{\infty} S_n(A_2) e_1^n z^n \right) \\ &= \frac{e_1 \sum_{n=0}^{\infty} S_n(A_2) e_1^n z^n - e_2 \sum_{n=0}^{\infty} S_n(A_2) e_1^n z^n}{e_1 - e_2} \\ &= \sum_{n=0}^{\infty} S_n(A_2) \left(\frac{e_1^{n+1} - e_2^{n+1}}{e_1 - e_2} \right) z^n \\ &= \sum_{n=0}^{\infty} S_n(A_2) S_n(E_2) z^n, \end{aligned}$$

and the right hand side of this formula can be written as

$$\begin{aligned} &\pi_{e_1 e_2} \left[\frac{1}{\left(\sum_{n=0}^{\infty} S_n(-A_2) e_1^n z^n \right) \left(\sum_{n=0}^{\infty} S_n(-A_2) e_2^n z^n \right)} \right] \\ &= \frac{e_1 \left(\sum_{n=0}^{\infty} S_n(-A_2) e_2^n z^n \right) - e_2 \left(\sum_{n=0}^{\infty} S_n(-A_2) e_1^n z^n \right)}{(e_1 - e_2) \left(\sum_{n=0}^{\infty} S_n(-A_2) e_1^n z^n \right) \left(\sum_{n=0}^{\infty} S_n(-A_2) e_2^n z^n \right)} \\ &= \frac{e_1(1 - a_1 e_2 z)(1 - a_2 e_2 z) - e_2(1 - a_1 e_1 z)(1 - a_1 e_1 z)}{(e_1 - e_2) \left(\sum_{n=0}^{\infty} S_n(-A_2) e_1^n z^n \right) \left(\sum_{n=0}^{\infty} S_n(-A_2) e_2^n z^n \right)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{e_1 (1 - e_2(a_1 + a_2)z + a_1a_2e_2^2z^2) - e_2 (1 - e_1(a_1 + a_2)z + a_1a_2e_1^2z^2)}{(e_1 - e_2) \left(\sum_{n=0}^{\infty} S_n(-A_2)e_1^n z^n \right) \left(\sum_{n=0}^{\infty} S_n(-A_2)e_2^n z^n \right)} \\
 &= \frac{1 - a_1a_2e_1e_2z^2}{\left(\sum_{n=0}^{\infty} S_n(-A_2)e_1^n z^n \right) \left(\sum_{n=0}^{\infty} S_n(-A_2)e_2^n z^n \right)}.
 \end{aligned}$$

This completes the proof of Theorem 4. □

4. Applications to the Generating Functions

In this section, we attempt to give results for some well-known generating functions. In fact, we will use Theorem 3.4 to derive Fibonacci numbers and Tchebychev polynomials of second kind. Moreover, the generating functions for some special cases of Fibonacci numbers and Tchebychev polynomials are given.

Case 1: For $a_1 = 1$ and $a_2 = 0$, one can apply Theorem 4 to arrive at [3]

$$\sum_{n=0}^{\infty} S_n(e_1 + [-e_2])z^n = \frac{1}{(1 - e_1z)(1 - e_2z)}. \tag{4.0}$$

In (4.0) replace e_2 by $(-e_2)$, and choose e_1, e_2 such that: $e_1 - e_2 = 1, e_1e_2 = 1$ to obtain

$$\sum_{n=0}^{\infty} S_n(e_1 + [-e_2])z^n = \frac{1}{1 - z - z^2}, \text{ with } F_n = S_n(e_1 + [-e_2]), \tag{4.1}$$

where F_n are Fibonacci numbers.

Also, if we replace e_1 by $(2e_1), e_2$ by $(-2e_2)$ with the condition $4e_1e_2 = -1$, then there follows that

$$\begin{aligned}
 \sum_{n=0}^{\infty} S_n(2e_1 + [-2e_2])z^n &= \frac{1}{1 - 2(e_1 - e_2)z + z^2}, \text{ with } U_n(e_1 - e_2) \\
 &= S_n(2e_1 + [-2e_2]). \tag{4.2}
 \end{aligned}$$

where U_n are the Tchebychev polynomials of second kind.

By using the previous formula (4.2), we can deduce that

$$\sum_{n=0}^{\infty} [S_n(2e_1 + [-2e_2]) - (e_1 - e_2)S_{n-1}(2e_1 + [-2e_2])] z^n = \frac{1 - (e_1 - e_2)z}{1 - 2(e_1 - e_2)z + z^2}. \quad (4.3)$$

Then the Tchebychev polynomials of first kind can be derived directly as follows [3]

$$T_n(e_1 - e_2) = [S_n(2e_1 + [-2e_2]) - (e_1 - e_2)S_{n-1}(2e_1 + [-2e_2])]. \quad (4.4)$$

Case 2: For $a_1 = 1$, $a_2 = x$, and $e_1 = 1$, $e_2 = y$, in an application of Theorem 3.4 yields the following result [4]

$$\sum_{n=0}^{\infty} [1 + x + \dots + x^n] [1 + y + \dots + y^n] z^n = \frac{1 - xyz^2}{[(1 - z)(1 - xz)(1 - yz)(1 - xyz)]}. \quad (4.5)$$

Case 3: By replacing e_2 by $(-e_2)$ and a_2 by $(-a_2)$, we obtain

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{1 - e_1e_2a_1a_2z^2}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)}. \quad (4.6)$$

This case consists of three related parts.

Firstly, by making the following restrictions: $a_1 - a_2 = 1$, $a_1a_2 = 1$, and $e_1 - e_2 = 1$, $e_1e_2 = 1$ in (4.6) we gives

$$\sum_{n=0}^{\infty} S_n(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n = \frac{1 - z^2}{1 - z - 4z^2 - z^3 + z^4} = \sum_{n=0}^{\infty} F_n^2 z^n. \quad (4.7)$$

This corresponds to the square of Fibonacci numbers [7] given by

$$F_n^2 = S_n(a_1 + [-a_2])S_n(e_1 + [-e_2]). \quad (4.8)$$

Secondly, by making the following restrictions: $e_1 - e_2 = 1$, $e_1e_2 = 1$, $a_1a_2 = -1$, and by replacing $(a_1 - a_2)$ by $2(a_1 - a_2)$ in (4.6), we get the identity of Foata [7], involving the product of Fibonacci numbers with Tchebychev polynomial of second kind as follows

$$\frac{1 + z^2}{1 - 2(a_1 - a_2)z + (3 - 4(a_1 - a_2)^2)z^2 + 2(a_1 - a_2)z^3 + z^4} = \sum_{n=0}^{\infty} F_n U_n(a_1 - a_2)z^n. \quad (4.9)$$

In the last case, choose a_i and e_i such that $e_1 e_2 = -1$, $a_1 a_2 = -1$, and by replace $(a_1 - a_2)$ by $2(a_1 - a_2)$, and $(e_1 - e_2)$ by $2(e_1 - e_2)$ in (4.6), to obtain the identity of Foata [7], involving the square of Tchebychev polynomials of second kind given by

$$\sum_{n=0}^{\infty} U_n(e_1 - e_2)U_n(a_1 - a_2)z^n = (1 - z^2) \left(1 - 4(e_1 - e_2)(a_1 - a_2)z + (4(a_1 - a_2)^2 + 4(e_1 - e_2)^2 - 2)z^2 - 4(e_1 - e_2)(a_1 - a_2)z^3 + z^4 \right)^{-1}. \quad (4.10)$$

Notice that, under the same restrictions and by using (4.2) and (4.4), and the fact that

$$S_{n-1}(2a_1 + [-2a_2]) = \frac{(2a_1)^n - (-2a_2)^n}{2a_1 + 2a_2}. \quad (4.11)$$

we obtain the identity of Foata [6], involving the product of Tchebychev polynomials of second kind with Tchebychev polynomials of first kind:

$$\sum_{n=0}^{\infty} U_n(e_1 - e_2)T_n(a_1 - a_2)z^n = \left(1 - 2(e_1 - e_2)(a_1 - a_2)z + (2(a_1 - a_2)^2 - 1)z^2 \right) \left(1 - 4(e_1 - e_2)(a_1 - a_2)z + (4(a_1 - a_2)^2 + 4(e_1 - e_2)^2 - 2)z^2 - 4(e_1 - e_2)(a_1 - a_2)z^3 + z^4 \right)^{-1}, \quad (4.12)$$

and also the identity of Foata [7], involving the square of Tchebychev polynomials of first kind:

$$\sum_{n=0}^{\infty} T_n(e_1 - e_2)T_n(a_1 - a_2)z^n = \left(1 - 3(e_1 - e_2)(a_1 - a_2)z + (2(a_1 - a_2)^2 + 2(e_1 - e_2)^2 - 1)z^2 - (e_1 - e_2)(a_1 - a_2)z^3 \right) \left(1 - 4(e_1 - e_2)(a_1 - a_2)z + (4(a_1 - a_2)^2 + 4(e_1 - e_2)^2 - 2)z^2 - 4(e_1 - e_2)(a_1 - a_2)z^3 + z^4 \right)^{-1}. \quad (4.13)$$

5. Conclusion

In this paper, a new theorem has been proposed in order to determine the generating functions. The proposed theorem is based on the symmetric functions. The obtained results agree with the results obtained in some previous works.

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