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PSEUDO SPLINE METHOD FOR SOLVING CERTAIN SECOND ORDER INITIAL VALUE PROBLEM OF ORDINARY DIFFERENTIAL EQUATION

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Abstract: In this paper, new set of low order schemes for solving certain second order initial value problem of differential equations directly are derived.

From the general method of deriving a spline function, schemes based on interpolation and collocation were derived. In the process of derivation we observed that it is possible to obtain the discrete scheme from the continuous scheme by collocating at a given point. Also from the process of derivation the Adams'-Stormer formula were recovered for the explicit scheme and the Cowell formula for the implicit scheme.

AMS Subject Classification: 65L05, 65L12
Key Words: initial value problems, ordinary differential equations, collocation, pseudo spline function

1. Introduction

Consider the second order initial value problem (IVP)

$$\begin{aligned} &y''(t) = f(t, y(t)), \\ &y(t_0) = y_0, \\ &y'(t_0) = y_1, \end{aligned} \tag{1.1}$$

where $a \leq t \leq b; a = t_0 < t_1 < t_2 < \ldots < t_{N-1} = b$, $N = \dfrac{(b-a)}{h}$, $N = 0, 1, \ldots, N-1$ and $h = t_{n+1} - t_n$ is called the step length. The conditions on the function $f(t, y(t))$ are such that existence and uniqueness of solution is guaranteed (see [3]).

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The numerical solution of the equation (1.1) had received lots of attention and it is still receiving such an attention due to the fact that many physical (Engineering) problems formulated into mathematical equation result in the above type.

There are two ways of treating the equation (1.1) viz: reduction of the equation to systems of two first order differential equation whose solution is generated in a step-by-step fashion by a formula which is regarded as discrete replacement of the equivalent system (see [1], [3], [4], [5]-[6]) and directly using the equation as it is with appropriate schemes to approximate the solution.

In the class of methods available in solving first order or systems of first order equations numerically, the most celebrated methods are the single-step and the multisteps methods. In a single step, an information at just one point is enough to advance the solution to the next point while for the multisteps (as the name suggests), information at more than one previous points will be required to advance the solution to the next point.

The Euler's method (the pioneering method), which is the oldest method, and the Runge-Kutta methods fall in the class of the single-step methods while the Adams methods are in the class of the multistep method.([1], [3], [4]-[6]).

The Adams method is divided into two namely the Adams-Bashforth (explicit) and Adams-Moulton (implicit). These two methods combined can be used as a predictor-corrector method. This class of method has been proved to be one of the most efficient method to solve certain class of IVP (non-stiff).

In the literature, the derivation of the Adams method had been extensively dealt with using the interpolatory polynomial for the discretized problem. For the derivation of linear multisteps method through interpolation and collocation (see [1], [4], [5] and [7]).

In this paper, we consider a new class of methods for solving (1.1) based on interpolation and collocation. Our method is based on a general method for deriving the spline functions.

The Adams-Stomers' formula and Cowells' formula were recovered for the explicit and implicit schemes respectively.

The paper is organized in the following order, §2 deals with description of piecewise interpolation functions, the derivation of our scheme features is §3. Brief analysis on the derived scheme shall be given in §4. The result from some numerical examples will be given to illustrate and validate our scheme in §5. The conclusion is given in the last section.

2. Piecewise-Interpolation

One of the methods of deriving the multisteps method is by polynomial interpolation for a set of discrete point, however, polynomial interpolation for a set of (N+1)points $\{t_k, y_k\}$ is frequently unsatisfactory because the interpolation error is related to higher derivatives of the interpolated function. To circumvent this, we discretized the interpolation domain and interpolate locally. The overall accuracy may be significantly improved even if the interpolation polynomial is of low order.

Interpolation functions obtained on this principle are piece-wise interpolation functions or splines. We define a spline function as follows:

Definition 2.1. A function $S(t)$ is called a spline of degree k if

(i). the domain of S is the interval $[a, b]$

(ii). $S, \; S', \; S'', \ldots, S^{(k-1)}$ are all continuous on $[a, b]$.

(iii). there are points t_i (called knots) such that $a = t_1 < t_2 < \ldots < t_n = b$ and such that S is a polynomial of degree k on each sub-interval $[t_i, t_{i+1}]$, $i = 1, \ldots, n-1$. subject to the interpolating conditions.

(iv). $S(t_i) = y(t_i) \qquad \forall t \in [t_i, t_{i+1}] \;\; i = 1, \ldots n-1.$

(v). $S_r^{(j)}(t_i) = S_{r+1}^{(j)}(t_i); \;\; j = 1, \ldots, k-1, \;\; r = 1, \ldots, n-1, \;\; i = 2, \ldots, n-1.$

Condition (iv) is the collocation while (v) is the continuity condition, only on interior knots.

We shall now use the piece-wise cubic and quartic interpolation spline functions to derive our methods.

3. Derivation of the Scheme

3.1. Pseudo Cubic Spline Function

Let $S(t)$ be the desired function, since we are considering a piecewise cubic spline, its second derivative is piecewise linear on $[t_{n-1}, t_n]$, then the linear Lagrange interpolation formula gives the representation for $S''(t)$ at the given points t_{n-1} and t_n as,

$$\frac{S''(t) - S''(t_{n-1})}{(t - t_{n-1})} = \frac{S''(t_n) - S''(t)}{(t_n - t)}. \tag{3.1}$$

Simplifying (3.1) gives

$$S''(t) = \frac{1}{(t_n - t_{n-1})} \left\{ (t - t_{n-1}) S''(t_n) + (t_n - t) S''(t_{n-1}) \right\} \tag{3.2}$$

Integrating equation (3.2) twice we have,

$$S(t) = \frac{1}{(t_n - t_{n-1})} \left\{ \frac{S''(t_n)}{6} (t - t_{n-1})^3 + \frac{S''(t_{n-1})}{6} (t_n - t)^3 \right\} + A(t_n - t) + B(t - t_{n-1}) \tag{3.3}$$

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where A and B are constants. To determine these constants, (3.3) is collocated at two points say $t = t_{n-1}$ and $t = t_n$, this yield

$$S(t_{n-1}) = \frac{S''(t_{n-1})}{6}(t_n - t_{n-1})^2 + A(t_n - t_{n-1}) \tag{3.4}$$

and

$$S(t_n) = \frac{S''(t_n)}{6}(t_n - t_{n-1})^2 + B(t_n - t_{n-1}). \tag{3.5}$$

From (3.4) and (3.5) we have that

$$A = \frac{1}{(t_n - t_{n-1})}\left\{S(t_{n-1}) - \frac{S''(t_{n-1})}{6}(t_n - t_{n-1})^2\right\} \tag{3.6}$$

and

$$B = \frac{1}{(t_n - t_{n-1})}\left\{S(t_n) - \frac{S''(t_n)}{6}(t_n - t_{n-1})^2\right\} \tag{3.7}$$

Substitute for A and B in (3.3), we have

$$\begin{aligned} S(t) &= \frac{1}{(t_n - t_{n-1})}\left\{\frac{S''(t_n)}{6}(t - t_{n-1})^3 + \frac{S''(t_{n-1})}{6}(t_n - t)^3\right\} + \\ &\frac{1}{(t_n - t_{n-1})}\left\{S(t_{n-1}) - \frac{S''(t_{n-1})}{6}(t_n - t_{n-1})^2\right\}(t_n - t) \\ &+ \frac{1}{(t_n - t_{n-1})}\left\{S(t_n) - \frac{S''(t_n)}{6}(t_n - t_{n-1})^2\right\}(t - t_{n-1}) \end{aligned} \tag{3.8}$$

Collocating (3.8) at $t = t_{n+1}$ yields

$$S(t_{n+1}) = 2S(t_n) - S(t_{n-1}) + h^2 S''(t_n) \tag{3.9}$$

By collocation property, we have

$$y_{n+1} = 2y_n - y_{n-1} + h^2 y''_n$$

which becomes

$$y_{n+1} = 2y_n - y_{n-1} + h^2 f_n \tag{3.10}$$

when the equation (1.1) is used.

In a similar form, collocating (3.8) at $t = t_{n+2}$ and using the property of collocation we have

$$y_{n+2} = 3y_n - 2y_{n-1} + h^2 (4f_n - f_{n-1}). \tag{3.11}$$

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3.2. Pseudo Quartic Spline Function

Since we are considering a piecewise quartic spline, it can be assumed that its third derivative is piecewise linear on $[t_n, t_{n+1}]$. The linear Lagrange interpolation formula gives the following representation for $S'(t)$ at the given points t_{n-1} and t_n, for all $t \in [t_{n-1}, t_n]$, as

$$\frac{S'''(t) - S'''(t_{n-1})}{(t - t_{n-1})} = \frac{S'''(t_n) - S'''(t)}{(t_n - t)} = \frac{S'''(t_n) - S'''(t_{n-1})}{(t_n - t_{n-1})} \tag{3.12}$$

From (3.12) we have

$$\frac{S'''(t) - S'''(t_{n-1})}{(t - t_{n-1})} = \frac{S'''(t_n) - S'''(t)}{(t_n - t)} \tag{3.13}$$

$$\frac{S'''(t) - S'''(t_{n-1})}{(t - t_{n-1})} = \frac{S'''(t_n) - S'''(t_{n-1})}{(t_n - t_{n-1})} \tag{3.14}$$

$$\frac{S'''(t_n) - S'''(t)}{(t_n - t)} = \frac{S'''(t_n) - S'''(t_{n-1})}{(t_n - t_{n-1})} \tag{3.15}$$

Simplifying equations (3.13)-(3.15), we have

$$S'''(t) = \frac{1}{(t_n - t_{n-1})} \left\{ (t - t_{n-1}) S'''(t_n) + (t_n - t) S'''(t_{n-1}) \right\} \tag{3.16}$$

Integrating (3.16),

$$S''(t) = \frac{1}{(t_n - t_{n-1})} \left\{ \frac{1}{2}(t - t_{n-1})^2 S'''(t_n) - \frac{1}{2}(t_n - t)^2 S'''(t_{n-1}) \right\} + A \tag{3.17}$$

Integrating (3.17) we have

$$S'(t) = \frac{1}{(t_n - t_{n-1})} \left\{ \frac{1}{6}(t - t_{n-1})^3 S'''(t_n) - \frac{1}{6}(t_n - t)^3 S'''(t_{n-1}) \right\} + A(t_n - t) + B. \tag{3.18}$$

On further integration (3.18) yields

$$S(t) = \frac{1}{(t_n - t_{n-1})} \left\{ \frac{1}{24}(t - t_{n-1})^4 S'''(t_n) - \frac{1}{24}(t_n - t)^4 S'''(t_{n-1}) \right\} + A(t_n - t)^2 + B(t - t_n) + C, \tag{3.19}$$

where A, B, and C are constants of integrations to be determined.

Collocate the equations (3.17)-(3.19) at $t = t_n$ to determine the unknown constants, we have

$$A = S''(t_n) - \frac{h}{2}S'''(t_n) \tag{3.20}$$

$$B = S'(t_n) - \frac{1}{6}(t_n - t_{n-1})^2 S'''(t_n) \tag{3.21}$$

and

$$C = S(t_n) - \frac{1}{24}(t_n - t_{n-1})^3 S'''(t_n) \tag{3.22}$$

Since $S(t)$ interpolates the function f at $t = t_n$, it implies that $S''(t_n) = f(t_n, y(t_n))$.

Substitute (3.18)-(3.20) into (3.19) and simplifying we have,

$$\begin{aligned} S(t) = {} & S(t_n) + (t - t_n)S'(t_n) + (t - t_n)^2 S''(t_n) - \\ & \frac{1}{6}(t - t_n)(t_n - t_{n-1})^2 S'''(t_n) - \frac{1}{2}(t - t_n)^3 S'''(t_n) - \frac{1}{24}(t_n - t_{n-1})^3 S'''(t_n) + \\ & \frac{1}{24(t_n - t_{n-1})}\left\{(t - t_{n-1})^4 S'''(t_n) - (t_n - t)^4 S'''(t_{n-1})\right\} \end{aligned} \tag{3.23}$$

Collocating (3.23) at $t = t_{n+1}$ and using the property that $S(t) \approx y(t)$ and that $h = t_n - t_{n-1}$ we have the

$$y_{n+1} = y_n + hy'_n + h^2 y''_n - \frac{h^3}{24}\left\{y'''_n + y'''_{n-1}\right\} \tag{3.24}$$

If we also collocate (3.23) at $t = t_{n+2}$ and simplifying we have

$$y_{n+2} = y_{n+1} + y_n - y_{n-1} + 4h^2 y''_n - h^3\left\{\frac{2}{3}y'''_{n-1} + y'''_n\right\} \tag{3.25}$$

From the equation (1.1) and the properties

$$y'_n = \frac{y_n - y_{n-1}}{h}, \tag{3.26}$$

$$y'_n = \frac{y_{n+1} - y_{n-1}}{2h}, \tag{3.27}$$

$$y'''_n = f'_n = \frac{f_n - f_{n-1}}{h}, \tag{3.28}$$

and

$$y'''_n = f'_n = \frac{f_{n+1} - f_{n-1}}{2h}, \tag{3.29}$$

using (3.26) and (3.28) in the equation (3.24) we have

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h^2}{24}\left\{f_{n-2} + 23f_n\right\} \tag{3.30}$$

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When (3.26) and (3.29) are applied in the equation (3.24) we also have

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h^2}{48}\{f_{n-2} + f_{n-1} + 47f_n - f_{n+1}\} \tag{3.31}$$

Using (3.28) in the equation (3.25) we obtained

$$y_{n+2} = y_{n+1} + y_n - y_{n-1} + \frac{h^2}{3}\{9f_n + f_{n-1} + 2f_{n-2}\} \tag{3.32}$$

Also by applying (3.29) in the equation (3.25) we have

$$y_{n+2} = y_{n+1} + y_n - y_{n-1} + \frac{h^2}{6}\{2f_{n-2} + 3f_{n-1} + 22f_n - 3f_{n+1}\} \tag{3.33}$$

Various multisteps schemes can be derived from the equations (3.23) at different collocation points (say $t = t_{n+3}, t_{n+4}, \ldots$).

The schemes derived in this work are

$$y_{n+1} = 2y_n - y_{n-1} + h^2 f_n \tag{A}$$

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h^2}{24}\{f_{n-2} + 23f_n\} \tag{B}$$

$$y_{n+2} = 3y_n - 2y_{n-1} + h^2(4f_n - f_{n-1}) \tag{C}$$

$$y_{n+2} = y_{n+1} + y_n - y_{n-1} + \frac{h^2}{3}\{9f_n + f_{n-1} + 2f_{n-2}\} \tag{D}$$

$$y_{n+2} = y_{n+1} + y_n - y_{n-1} + \frac{h^2}{6}\{2f_{n-2} + 3f_{n-1} + 22f_n - 3f_{n+1}\} \tag{E}$$

$$y_{n+1} = 2y_n - y_{n-1} + \frac{h^2}{48}\{f_{n-2} + f_{n-1} + 47f_n - f_{n+1}\} \tag{F}$$

Schemes (A) - (E) are explicit while (F) is implicit.

4. Analysis of the Methods

4.1. The Local Truncation Error

Assume that $y \in C^4[a, b]$ for all x in $a \le x \le b$. Due to a standard approach by [5] we have been able to show that the local truncation error associated with the numerical algorithms derived in this work are given below:

$$
\begin{aligned}
e_A &= \frac{1}{12}h^4 y^{(iv)}(\zeta), \quad \zeta \in (x_{n-1}, x_{n+1})\\
e_B &= \frac{1}{12}h^3 y'''(\zeta), \quad \zeta \in (x_{n-1}, x_{n+1})\\
e_C &= \frac{5}{4}h^4 y^{(iv)}(\zeta), \quad \zeta \in (x_{n-1}, x_{n+2})\\
e_D &= -2h^2 y''(\zeta), \quad \zeta \in (x_{n-2}, x_{n+2})\\
e_E &= -2h^2 y'''(\zeta), \quad \zeta \in (x_{n-2}, x_{n+2})\\
e_F &= \frac{1}{12}h^3 y'''(\zeta), \quad \zeta \in (x_{n-2}, x_{n+1})
\end{aligned}
$$

Order of the Methods: The schemes A, B, C, D, E and F are of order 3, 2, 2, 1, 1 and 2 respectively with the local truncation errors(**LTE**) given as $\frac{1}{12}h^4, \frac{1}{12}h^3, \frac{5}{4}h^4, -2h^2, -2h^2$ and $\frac{1}{12}h^3$

4.2. Consistency

For consistency, let us consider the general linear multistep method

$$\sum_{i=0}^{k} \alpha_i y_{n-i} = h \sum_{i=0}^{k} \beta_i f_{n-i}.$$

Define the polynomials

$$\rho(\zeta) = \sum_{i=0}^{k} \alpha_i \zeta^{k-i}, \quad \sigma(\zeta) = \sum_{i=0}^{k} \beta_i \zeta^{k-i}.$$

For consistency of numerical scheme for solving second order IVP directly, it is required that the following conditions are satisfied: $\rho(1) = 0, \rho'(1) = 0$ and $\rho''(1) = 2\sigma(1)$. Schemes A, B, C and F can be seen to be consistent while D and E are not.

4.3. Zero Stability

For a second order IVP, the stability criteria requires that the modulus of no roots of the Polynomial $\rho(\zeta)$ exceed 1, and that the multiplicity of the roots of modulus 1 be at most 2. By the analysis it is seen that schemes A, B and F satisfied the criteria and hence considered stable while C, D and E failed to meet the condition and hence unstable. The methods described by scheme A and scheme F are respectively the Adams - Stormers and Cowell schemes which can be seen to be zero stable.

4.4. Convergence of the Scheme

For any numerical scheme to be convergent consistency and stability is required hence schemes A, B and F are convergent even though the order is very low.

Our numerical results below will validate this assertion.

5. Numerical Examples

We shall consider the following problems;

1.

$$y'' = \lambda y, \quad y(0) = 1, \quad y'(0) = 0; \quad t \in [0, 1]$$

Exact Solution:

$$(a) \quad \lambda = 1; \quad y(t) = \frac{1}{2}(e^t + e^{-t}) = \cosh(t)$$

$$(b) \quad \lambda = -1; \quad y(t) = \cos(t)$$

2.

$$y'' = (t + y), \quad y(0) = 1; \quad y'(0) = 0; \quad t \in [0, 1]$$

Exact Solution: $y(t) = e^{(-t)} - t$

3.

$$y'' = \frac{8y^2}{1 + 2t}; \quad y(0) = 1; \quad y'(0) = -2, t \in [0, 1]$$

Exact Solution: $y = \dfrac{1}{1 + 2t}$

6. Discussion and Conclusion

In this paper, using the general method of deriving a spline function, we derived low order numerical schemes based on interpolation and collocation to approximate directly the solution of certain second order initial value problem (IVP). Six schemes A-F were derived, five of these schems are explicit while one is implicit. From the process of derivation the Adams'-Stormer formular (schemes A & B) were recovered for the explcit scheme and the Cowell formular (scheme F) for the implicit scheme. Two convergent explicit schemes A and B were applied to the problems 1, 2 and 3. The errors produced were given in the tables 1 and 2 respectively. It could be noticed that there is a better approximation with scheme A just because of the order of the scheme which is one higher than the order of the scheme B. Table 3 is presented just to show that scheme C though consistent, will not converge to the desired solution because it fails the zero stability criteria. Tables 4 displayed the errors produced for each problems when the schemes B& F are combined as predictor-corrector method while table 5 gives the maximum error produced by each scheme.

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Table 1: Error of y(t) for the problems using **(**A) (h = 0.1)

t	Problem**1a**	Problem**1b**	Problem **2**	Problem **3**
0.3	8.3778269e-006	8.2889380e-006	7.5428257e-006	1.3227513e-003
0.4	2.5342822e-005	2.4659486e-005	2.1986111e-005	3.3607682e-003
0.5	5.1275305e-005	4.8741924e-005	4.2824800e-005	5.9182881e-003
0.6	8.6732480e-005	8.0009892e-005	6.9679600e-005	8.9387980e-003
0.7	1.3245700e-004	1.1778851e-004	1.0228730e-004	1.2413006e-003
0.8	1.8938825e-004	1.6126475e-004	1.4049284e-004	1.6355386e-002
0.9	2.5867662e-004	2.0949991e-004	1.8424289e-004	2.0783634e-002
1.0	3.4170076e-004	2.6144401e-004	2.3358102e-004	2.5717605e-002

Table 2: Error of y(t) for the problems using **(**B) (h = 0.1)

t	Problem**1a**	Problem**1b**	Problem **2**	Problem **3**
0.3	1.6864539e-005	1.6470092e-005	1.0243290e-004	1.5111759e-003
0.4	5.9410732e-005	5.7269028e-005	3.1905331e-004	3.9181648e-003
0.5	1.3695722e-004	1.2977775e-004	6.6384314e-004	6.9829798e-003
0.6	2.5950970e-004	2.4074412e-004	1.1533074e-003	1.0617410e-002
0.7	4.3795310e-004	3.9614058e-004	1.8067691e-003	1.4796126e-002
0.8	6.8425677e-004	6.0103333e-004	2.6467056e-003	1.9522503e-002
0.9	1.0116957e-003	8.5946521e-004	3.6991317e-003	2.4814036e-002
1.0	1.4350912e-003	1.1743539e-003	4.9940353e-003	3.0695757e-002

Table 3: Error of y(t) for the problems using **(**C) (h = 0.1)

t	Problem**1a**	Problem**1b**	Problem **2**	Problem **3**
0.3	0.0198742	0.0201242	0.0198608	0.1401852
0.4	0.0199723	0.0200223	0.0219496	0.0835493
0.5	0.0806877	0.0792879	0.0846348	0.5132403
0.6	0.0419373	0.0380379	0.0540130	0.0342884
0.7	0.2270257	0.2129777	0.2432390	1.5272631
0.8	0.0106756	0.0291224	0.0285210	0.8687707
0.9	0.6302546	0.5705597	0.6681494	5.0095166
1.0	0.4592996	0.4989043	0.3573469	5.5188468

Table 4: Error of y(t) for the problems using (B & F) (h = 0.1)

t	Problem**1a**	Problem**1b**	Problem **2**	Problem **3**
0.3	2.1083440e-005	2.0584826e-005	1.0726263e-004	1.0075301e-003
0.4	7.2185729e-005	6.9497144e-005	3.3407417e-004	2.6429889e-003
0.5	1.6283089e-004	1.5392045e-004	6.9507254e-004	4.7524868e-003
0.6	3.0331937e-004	2.8032773e-004	1.2075368e-003	7.2747616e-003
0.7	5.0492478e-004	4.5434338e-004	1.8916955e-003	1.0189526e-003
0.8	7.8010494e-004	6.8061771e-004	2.7710791e-003	1.3496069e-002
0.9	1.1427306e-003	9.6271622e-004	3.8729210e-003	1.7203644e-002
1.0	1.6083346e-003	1.3030252e-003	5.2286127e-003	2.1326912e-002

Table 5: Maximum Error produced for $y(t)$ for the problems with the convergent Schemes (h = 0.1)

Problem	Scheme **A**	Scheme **B**	Scheme **B & F**
1a	3.4170076e-004	1.4350912e-003	1.6083346e-003
1b	2.6144401e-004	1.1743539e-003	1.3030252e-003
2	2.3358102e-004	4.9940353e-003	5.2286127e-003
3	2.5717605e-002	3.0695757e-002	2.1326912e-002

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