International Electronic Journal of Pure and Applied Mathematics

Volume 7 No. 4 2014, 145-159
ISSN: 1314-0744
url: http://www.e.ijpam.eu
doi: http://dx.doi.org/10.12732/iejpam.v7i4.1

ANALYTICAL APPROXIMATE SOLUTION FOR NONLINEAR SPACE-TIME FRACTIONAL CAHN-HILLIARD EQUATION

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Abstract: The fractional derivatives in the sense of Caputo and the homotopy analysis method are used to construct an approximate solution for the nonlinear space-time fractional derivatives Cahn-Hilliard equation. The numerical results show that the approaches are easy to implement and accurate when applied to the nonlinear space-time fractional derivatives Cahn-Hilliard equation. This method introduces a promising tool for solving many space-time fractional partial differential equations. This method is efficient and powerful in solving wide classes of nonlinear evolution fractional order equat. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

AMS Subject Classification: 14F35, 26A33, 34A08
Key Words: homotopy analysis method, Caputo fractional derivative, space-time fractional derivatives, Cahn-Hilliard equation

Received: Ocotber 3, 2013

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International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 4 (2014)

1. Introduction

Fractional differential equations FDEs have found applications in many problems in physics and engineering [1,2,3,4]. Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Some of the recent analytical methods for solving nonlinear problems include the Adomian decomposition method ADM [5, 6], variational iteration method VIM [7], Homotopy-perturbation method HPM (see [8], [9]), and homotopy analysis method HAM [10-20]. The HAM, first proposed in 1992 by Liao [10], has been successfully applied to solve many problems in physics and science.

In this work, a new algorithm for solving space-time fractional derivatives Cahn-Hilliard equation is proposed based on HAM. Now consider the following space-time fractional Cahn-Hilliard equation [21]:

$$D_t^{\alpha} u - \gamma D_x^{\beta} u - 6u(D_x^{\beta} u)^2 - (3u^2 - 1)u_{xx} + u_{xxxx} = 0. \tag{1.1}$$

Here α and β are the parameters standing for the order of the fractional time and space derivatives, respectively and they satisfy $0 < \alpha \leq 1, 0 < \beta \leq 1$ and $x > 0$. We use the Caputo fractional derivative on the half axis R^+ (i.e. $t \in R^+$) ${}_t^C D_{0_+}^{\alpha}$ for time and the Caputo fractional derivative on the half axis R^+ (i.e. $x \in R$) ${}_x^C D_{0_+}^{\beta}$ for space.

The paper is organized as follows. In Sec**II**, some necessary details on the fractional calculus are provided. In Sec**III**, the Cahn-Hilliard equation with time-space fractional derivative is studied with the HAM. Finally, conclusions follow.

2. Preliminaries and Notations

In this section, we give some definitions and properties of the fractional calculus. Several definitions of fractional calculus have been proposed in the last two centuries. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order. There are many books [1-4] that develop fractional calculus and various definitions of fractional integration and differentiation, such as Grunwald-Letnikov's definition, Riemann-Liouville definition, Caputo's definition and generalized function approach.

Definition 2.1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m if and only if $f^{(m)} \in C_\mu$, $m \in \mathbb{N}$.Clearly $C_\mu \subset C_\gamma$ if $\gamma \leq \mu$.

Definition 2.2. The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geq 0$, of a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad (\alpha > 0), J^0 f(t) = f(t), \tag{2.1}$$

$\Gamma(\alpha)$ is the well- known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:

For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$ and $\gamma \geq -1$:

(1) $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,

(2) $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,

(3) $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$, where $\beta \geq 0$, and $\gamma \geq -1$.

Definition 2.3. The fractional derivative (D^α) of $f(t)$ in the Caputo's sense is defined as

$$D^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} f(\tau) d\tau, & n-1 < \alpha < n, \ n \in N, \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n. \end{cases} \tag{2.2}$$

According to the Caputo's derivative, we can easilly obtain the following expressions: $D^\alpha K = 0$; K is a constant

$$D^\alpha f(t) = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, & \beta > \alpha - 1, \\ 0, & \beta \preceq \alpha - 1. \end{cases} \tag{2.3}$$

Remark 1. In this paper, consider equation (1.1) (with time-and space fractional derivatives). When $\alpha \in IR^+$, we have:

$$D^\alpha f(t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \left\{ \begin{array}{ll} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,t)}{\partial t^n} d\tau, & n-1 < \alpha < n, n \in N, \\ \dfrac{d^n}{dt^n} u(x,t), & \alpha = n \in N \end{array} \right\}. \tag{2.4}$$

Here, also need two basic properties about them:

$$\begin{aligned} D^\alpha J^\alpha f(t) &= f(t), \\ (J^\alpha D^\alpha) f(t) &= f(t) - \sum_{k=0}^{\infty} f^{(k)}(0^+) \frac{t^k}{k!}, \ x > 0. \end{aligned} \tag{2.5}$$

3. Basic Idea of Homotopy Analysis Method (HAM)

To describe the basic ideas of the HAM, we consider the following differential equation:

$$N[D_t^\alpha u(x,t)] = 0, \quad t > 0, \tag{3.1}$$

where N is a nonlinear operator for this problem, D_t^α stand for the fractional derivative, x and t denotes independent variables, and $u(x,t)$ is an unknown function.

By means of the HAM, one first construct zero-order deformation equation

$$(1-q)\mathcal{L}(\phi(x,t;q) - u_0(x,t)) = qhH(t)N[D_t^\alpha \phi(x,t,q)], \tag{3.2}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, $\mathcal{L}$ is an auxiliary linear operator, $u_0(x,t)$ is an initial guess. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(x,t;0) = u_0(x,t), \quad \phi(x,t;1) = u(x,t). \tag{3.3}$$

Liao [10]-[11] expanded $\phi(x,t;q)$ in Taylor series with respect to the embedding parameter q, as follows:

$$\phi(x,t;q) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t)q^m, \tag{3.4}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;q)}{\partial q^m}\mid_{q=0}. \tag{3.5}$$

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter h and the auxiliary function $H(t)$ are selected such that the series (3.4) is convergent at $q = 1$, then we have from (3.4)

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{3.6}$$

Let us define the vector

$$u_n^{\rightarrow}(t) = \{u_0(x,t), u_1(x,t), u_2(x,t), \ldots, u_n(x,t)\}. \tag{3.7}$$

Differentiating (3.2) m times with respect to q, then setting $q = 0$ and dividing them by $m!$, we have the mth-order deformation equation

$$\mathcal{L}(u_m(x,t) - \varkappa_m u_{m-1}(x,t)) = hH(t)\mathcal{R}_m(u_{m-1}^{\rightarrow}), \tag{3.8}$$

where

$$\mathcal{R}_m(u_{m-1}^{\rightarrow}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\mathcal{D}_t^\alpha \phi(x,t;q)]}{\partial q^{m-1}}\mid_{q=0}, \tag{3.9}$$

and

$$\varkappa_m = \begin{cases} 0 & m \leq 1, \\ 1 & m > 1. \end{cases} \tag{3.10}$$

Applying the Riemann-Liouville integral operator J^αon both side of (3.8), we have

$$u_m(x,t) = \varkappa_m u_{m-1}(x,t) - \varkappa_m \sum_{i=0}^{n-1} u_{m-1}^i(0^+)\frac{t^i}{i!} + hH(t)J^\alpha \mathcal{R}_m(\vec{u_{m-1}}), \tag{3.11}$$

the mth-order deformation equation (3.8) is linear and thus can be easily solved, especially by means of symbolic computation software such as Mathematica, and Maple

Laio [11] proved that, as long as a series solution given by the HAM converges, it must be one exact solutions. So, it is important to ensure that the solution series is convergent. Note that the solution series is convergent. Note that the solution series contain the auxiliary parameter h,which we can choose properly by plotting the so-called h-curve to ensure solution series converge.

4. Application

In this section, to demonstrate the effectiveness of our approach, we will apply the HAM to construct approximate solutions for the Cahn-Hilliard equation with space-time frational derivatives equation (1.1)

Example 1. Consider the following form of the space-fractional equation for $(\gamma = 1)$

$$D_t^\alpha u - u_x - 6u(u_x)^2 - (3u^2 - 1)u_{xx} + u_{xxxx} = 0, \tag{4.1}$$

with the initial condition (see [21])

$$u(x,0) = f(x) = \tanh(\frac{\sqrt{2}}{2}x). \tag{4.2}$$

The exact solution of (4.1) for the special case $\alpha = \beta = 1$ is:

$$u(x,t) = \tanh(\frac{\sqrt{2}}{2}(x+t)). \tag{4.3}$$

For application of homotopy analysis method, in view of (4.1) and the initial condition given in (4.2), it in convenient to choose $u_0(x,t) = f(x) = \tanh(\frac{\sqrt{2}}{2}x)$ as the initial approximate.

By means of HAM, we choose the linear operator

$$\mathcal{L}[\phi(x,t;q)] = \frac{\partial \phi(x,t;q)}{\partial t} = D_t^\alpha, \tag{4.4}$$

with property $\mathcal{L}[c] = 0$, where c is a constant. We define a nonlinear operator as

$$N[\phi(x,t;q)] = D_t^\alpha \phi(x,t;q) - \phi_x(x,t;q) - 6\phi(x,t;q)(\phi_x(x,t;q))^2 - (3(\phi(x,t;q))^2 - 1)\phi_{xx}(x,t;q) + \phi_{xxxx}(x,t;q). \quad (4.5)$$

We construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}_(\phi(x,t;q) - u_0(x,t)) = qhH(t)N[\phi(x,t,q)].$$

For $q = 0$ and $q = 1$, we can write

$$\begin{aligned} \phi(x,t;0) =& u_0(x,t) = u(x,0), \\ \phi(x,t;1) =& u(x,t). \end{aligned} \quad (4.6)$$

Thus, we obtain the m^{th}-order deformation equations

$$\mathcal{L}(u_m(x,t) - \varkappa_m u_{m-1}(x,t)) = hH(t)\mathcal{R}_m(u_{m-1}^{\rightarrow}),$$

where

$$\mathcal{R}_m = D_t^\alpha u_{m-1} + (u_{m-1})_{xxxx} + (u_{m-1})_{xx} - 6\sum_{i=0}^{m-1}\sum_{j=0}^{i}(u_j)_x(u_{i-j})_x u_{m-1-i} - 3\sum_{i=0}^{m-1}\sum_{j=0}^{i} u_j u_{i-j}(u_{m-1-i})_{xx} - (u_{m-1})_x. \quad (4.7)$$

In order to obey both the rule of solution expression and the rule of the coefficient ergodicity [11], the auxiliary function can be determined uniquely $H(t) = 1$.

Now the solution of the m^{th} -order deformation equations (4.7) for $m \geq 1$ become

$$u_m(x,t) = (\varkappa_m + h)u_{m-1}(x,t) + (\varkappa_m - 1)hu_{m-1}(x,t) + h\ j^\alpha((u_{m-1})_{xxxx} - (u_{m-1})_x + (u_{m-1})_{xx} - 6\sum_{i=0}^{m-1}\sum_{j=0}^{i}(u_j)_x(u_{i-j})_x u_{m-1-i} - 3\sum_{i=0}^{m-1}\sum_{j=0}^{i} u_j u_{i-j}(u_{m-1-i})), \quad (4.8)$$

The approximate solutions of equation (4.1) take the following form

$$u_1(x,t) = -h\ f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad (4.9)$$

$$u_2(x,t) = (1+h)u_1(x,t) + h^2 f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (4.10)$$

$$u_3(x,t) = (1+h)u_2(x,t) - h^3 f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + h^2(1+h)f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \quad (4.11)$$

and so on. After some calculation, we get:

$$\begin{aligned}
f(x) &= \tanh(\frac{\sqrt{2}}{2}x),\\
f_1(x) &= f_x + 6ff_x^2 + 3f^2 f_{xx} - f_{xx} - f_{xxxx},\\
f_2(x) &= -(f_1)_{xxxx} - (f_1)_{xx} + (f_1)_x + 6f_x^2 f_1 + 12ff_x(f_1)_x + 3f^2(f_1)_{xx} + 6ff_1f_{xx},\\
f_3(x) &= (f_2)_x - (f_2)_{xxxx} - (f_2)_{xx} + 6f_2f_x^2 + 12f_xf_1(f_1)_x\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\\
&\quad + 6(f_1^2)_xf_2\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + 12ff_x(f_2)_x + 3f^2(f_2)_{xx} + 6ff_1(f_1)_{xx}\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)}\\
&\quad + 3f_xf_1^2\frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} + 6ff_2f_{xx},\\
&\quad \vdots
\end{aligned}$$

In this case the approximate solution of time-fractional equation (4.1) according to the HAM, we can conclude that

$$\begin{aligned}
u_{app} &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...\\
u_{app} &= f(x) - h\ f_1\frac{t^\alpha}{\Gamma(\alpha+1)} - h(1+h)f_1\frac{t^\alpha}{\Gamma(\alpha+1)}\\
&\quad + h^2f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + (1+h)(-h(1+h)f_1\frac{t^\alpha}{\Gamma(\alpha+1)}\\
&\quad + h^2f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}) - h^3f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\\
&\quad + h^2(1+h)f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots
\end{aligned} \tag{4.13}$$

And so on setting $h = -1$,we get an accurate approximation solution in the following form:

$$\begin{aligned}
u_{app}\,|_{\text{ADM}} &= u_{app}\,|_{\text{HPM}} = u_{app}\,|_{\text{HAM}}\\
&= f(x) + f_1(x)\frac{t^\alpha}{\Gamma(\alpha+1)} + f_2(x)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + f_3(x)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + ...
\end{aligned} \tag{4.14}$$

This solution (4.14) is exactly the same solution obtained in [21].

Figures 1 and 2 show the exact and HAM solutions of time-fractional with $h = -1, n = 4$, also Figure 3 show $h-$curve.

Example 2. Consider the following form of the time-fractional equation for $(\gamma = 1)$

$$u_t - D_x^\beta u - 6u(D_x^\beta u)^2 - (3u^2-1)u_{xx} + u_{xxxx} = 0, \tag{4.15}$$

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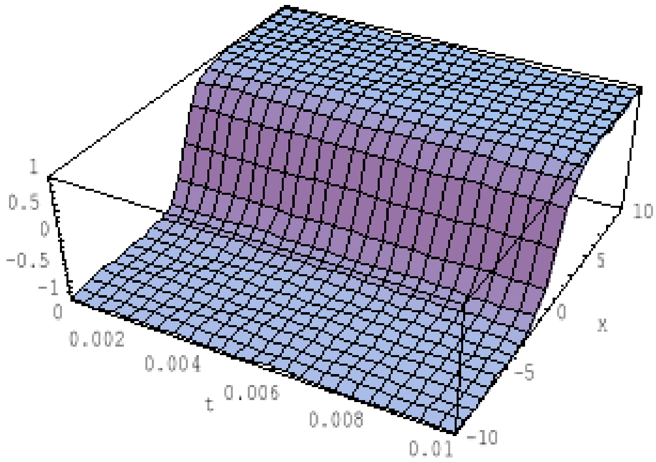

Figure 1(a): Exact solution of equation (4.1)

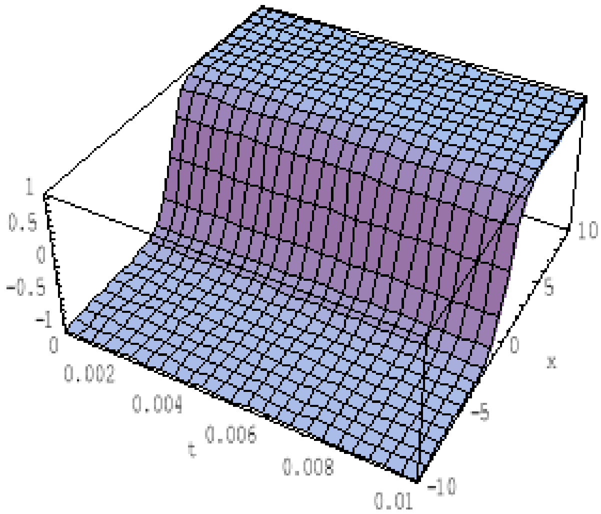

Figure 1(b): HAM solution with $\alpha = 1$

with the initial condition, see [21],

$$u(x, 0) = f(x) = x^2, \tag{4.16}$$

For application of homotopy analysis method, in view of (4.15) and the initial condition given in (4.16), it in convenient to choose $u_0(x, t) = f(x) = x^2$ as the initial approximate.

By means of HAM, we choose the linear operator

$$\mathcal{L}[\phi(x, t; q)] = \frac{\partial \phi(x, t; q)}{\partial t}, \tag{4.17}$$

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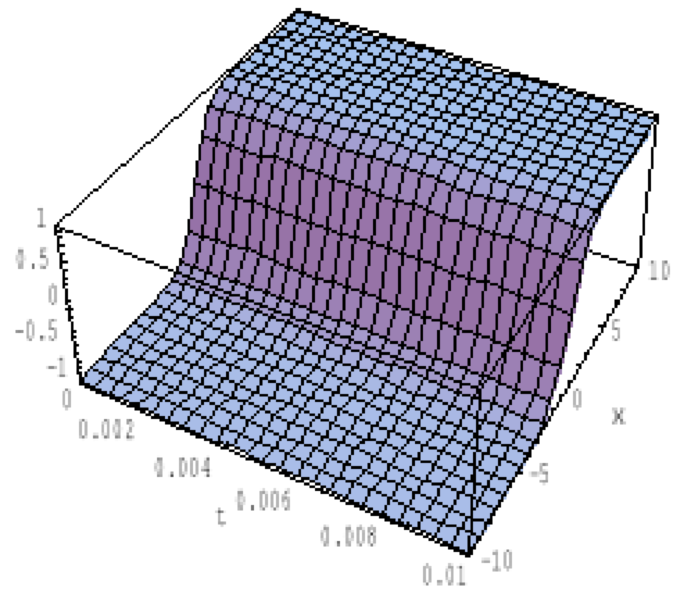

Figure 2(a): HAM solution with $\alpha = 0.5$

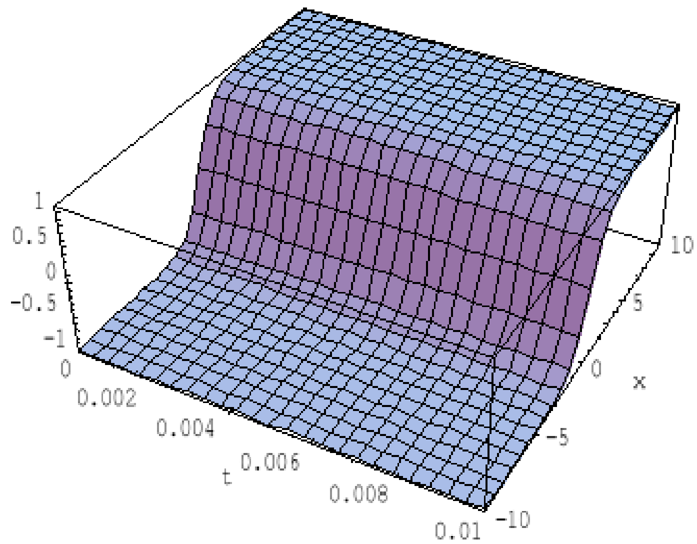

Figure 2(b): HAM solution with $\alpha = 0.75$

with property $\mathcal{L}[c] = 0$, where c is a constant. We define a nonlinear operator as

$$N[\phi(x,t;q)] = \phi_t(x,t;q) - D_x^\beta \phi_x(x,t;q) - 6\phi(x,t;q)(D_x^\beta \phi(x,t;q))^2 \\ - (3(\phi(x,t;q))^2 - 1)\phi_{xx}(x,t;q) + \phi_{xxxx}(x,t;q). \quad (4.18)$$

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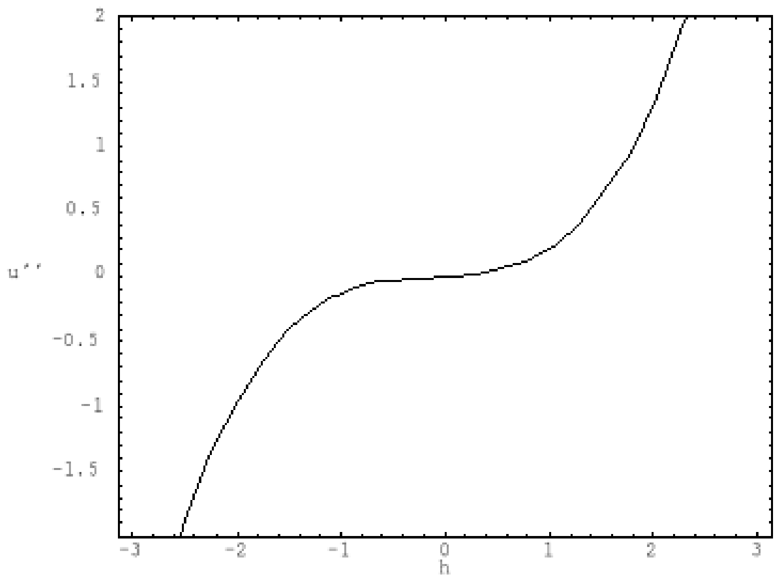

Figure 3: The h-curve of the 4^{th}-order of approximation for $x = t = 2$

We construct the zeroth-order deformation equation

$$(1-q)\mathcal{L}_(\phi(x,t;q) - u_0(x,t)) = qhH(t)N[\phi(x,t,q)].$$

For $q = 0$ and $q = 1$, we can write

$$\begin{aligned}\phi(x,t;0) =& u_0(x,t) = u(x,0),\\ \phi(x,t;1) =& u(x,t).\end{aligned} \tag{4.19}$$

Thus, we obtain the m^{th}-order deformation equations

$$\mathcal{L}(u_m(x,t) - \varkappa_m u_{m-1}(x,t)) = hH(t)\mathcal{R}_m(u^{\rightarrow}_{m-1}),$$

where

$$\begin{aligned}\mathcal{R}_m(u^{\rightarrow}_{m-1}) =& (u_{m-1})_t + (u_{m-1})_{xxxx} - D^\beta_x u_{m-1} + (u_{m-1})_{xx}\\ &- 6\sum_{i=0}^{m-1}\sum_{j=0}^{i} D^\beta_x u_j D^\beta_x u_{i-j} u_{m-1-i} - 3\sum_{i=0}^{m-1}\sum_{j=0}^{i} u_j u_{i-j}(u_{m-1-i})_{xx}.\end{aligned} \tag{4.20}$$

In order to obey both the rule of solution expression and the rule of the coefficient ergodicity [11], the auxiliary function can be determined uniquely $H(t) = 1$.

Now the solution of the m^{th} -order deformation equations (4.20) for $m \geq 1$ become

$$u_m(x,t) = \varkappa_m u_{m-1}(x,t) + h\int_0^t \mathcal{R}_m(u^{\rightarrow}_{m-1})dt, \tag{4.21}$$

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and so on, we substitute the initial condition (4.16) into the system (4.21) with the aid of Maple; the approximate solutions of equation (4.15) take the following form

$$u_1(x,t) = -h\ f_1(x)t\ , \tag{4.22}$$

$$\begin{aligned} u_2(x,t) =& (1+h)u_1(x,t) + \frac{h^2t^2}{2}(f_2(x)+6f_3(x)+f_4(x)-f_5(x)), \\ & ... \end{aligned} \tag{4.23}$$

and so on. After some calculation, we get:

$$\begin{aligned} f(x) =& x^2, \\ f_1(x) =& 6x^4 + \frac{\Gamma(3)}{\Gamma(3-\beta)}x^{2-\beta} + 6x^{6-2\beta}(\frac{\Gamma^2(3)}{\Gamma^2(3-\beta)}) - 2, \\ f_2(x) =& \frac{\Gamma(3)}{\Gamma(3-2\beta)}x^{2-2\beta} + 6\frac{\Gamma^2(3)}{\Gamma^2(3-\beta)}\frac{\Gamma(7-2\beta)}{\Gamma(7-3\beta)}x^{6-3\beta} + 6\frac{\Gamma(5)}{\Gamma(5-\beta)}x^{4-\beta}, \\ f_3(x) =& \frac{2\Gamma^2(3)}{\Gamma(3-\beta)\Gamma(3-2\beta)}x^{6-3\beta} \\ &+ \frac{12\Gamma^2(3)}{\Gamma^2(3-\beta)}\frac{\Gamma(3)\Gamma(7-2\beta)}{\Gamma(3-\beta)\Gamma(7-3\beta)}x^{8-3\beta} + 12\frac{\Gamma(3)\Gamma(5)}{\Gamma(5-\beta)\Gamma(3-\beta)}x^{6-\beta} \\ &+ \frac{\Gamma^2(3)}{\Gamma^2(3-\beta)}(\frac{\Gamma(3)}{\Gamma(3-\beta)}x^{6-3\beta} + \frac{6\Gamma^2(3)}{\Gamma^2(3-\beta)}x^{10-4\beta} + x^{8-2\beta} - 2x^{4-2\beta}, \\ f_4(x) =& 12\frac{\Gamma(3)}{\Gamma(3-\beta)}x^{4-\beta} + 72\frac{\Gamma^2(3)}{\Gamma^2(3-\beta)}x^{8-2\beta} + 72x^6 - 24x^2 \\ &+ 3(2-\beta)(1-\beta)\frac{\Gamma(3)}{\Gamma(3-\beta)}x^{4-\beta} \\ &+ 18(6-2\beta)(5-2\beta)\frac{\Gamma^2(3)}{\Gamma^2(3-\beta)}x^{8-2\beta} \\ &+ 3.72x^6 - (2-\beta)(1-\beta)\frac{\Gamma(3)}{\Gamma(3-\beta)}x^{-\beta} \\ &- 6(6-2\beta)(5-2\beta)\frac{\Gamma^2(3)}{\Gamma^2(3-\beta)}x^{4-2\beta} - 72x^2, \\ & \vdots \end{aligned} \tag{4.24}$$

In this case the approximate solution of time-fractional equation (4.1) according to the HAM, we can conclude that

$$\begin{aligned} u_{app} =& u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ... \\ u_{app} =& f(x) - h\ f_1(x)t + (1+h)u_1(x,t) + \frac{h^2t^2}{2}(f_2(x)+6f_3(x)+f_4(x)-f_5(x)). \end{aligned}$$

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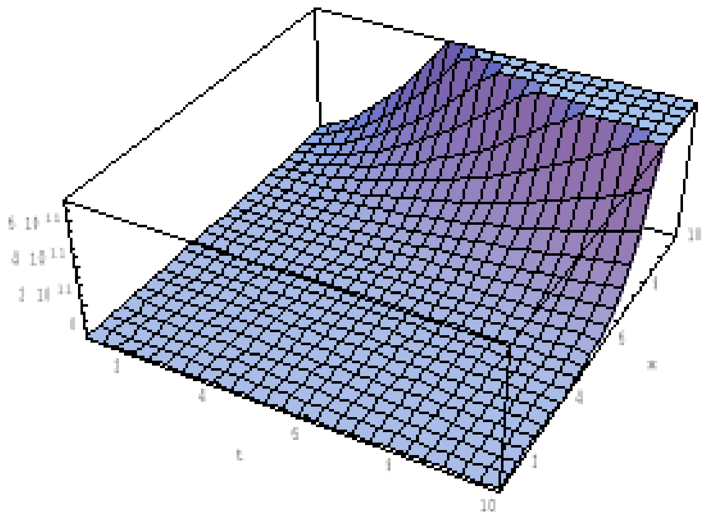

Figure 4(a): Solution of equation (4.15) obtained by HAM for $\beta = 0.5$ and 1. HAM solution with $\beta = 0.5$

And so on setting $h = -1$,we get an accurate approximation solution in the following form:

$$u_{app}\,|_{\mathrm{ADM}} = u_{app}\,|_{\mathrm{HPM}} = u_{app}\,|_{\mathrm{HAM}}$$
$$= f(x) + f_1(x)t + \frac{t^2}{2}(f_2(x) + 6f_3(x) + f_4(x) - f_5(x)) + \ldots \quad (4.25)$$

This solution (4.25) is exactly the same solution obtained in [21].

5. Conclusion

In this paper, the homotopy analysis method has been successfully applied to obtain the numerical solutions of the nonlinear space-time fractional derivative Cahn-Hilliard equation. The reliability of this method and the reduction in computation give this method a wider applicability. The HAM contains a certain auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region and the rate of convergence of the series solution. It is also demonstrated that the Adomian decomposition method and the homotopy perturbation method are special cases of the HAM. The HAM is clearly a very efficient and powerful technique for finding the numerical solutions of the proposed equation. It therefore provides more realistic series solutions that generally converge very rapidly for real physical problems.

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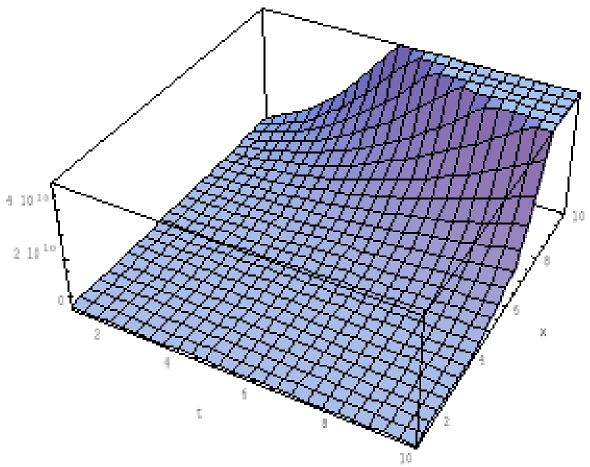

Figure 4(b): Solution of equation (4.15) obtained by HAM for $\beta = 0.5$ and 1. HAM solution with $\beta = 1$

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