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LABELING OF PALEY DIGRAPHS

R. Parameswari[1 §], R. Rajeswari[2]
[1,2]Sathyabama University
Chennai, Tamil Nadu 600119, INDIA

Abstract: In this paper we prove that some members of family of magic labeling such as vertex anti magic labeling, total bimagic labeling and total magic labeling for the small class of digraphs called Paley digraphs with q vertices and p set of edges where $q \equiv 3 \bmod 4$ and $p = \frac{q-1}{2}$.

Let G be a graph with vertex set as V and edge set as E. A graph $G(V, E)$ is called vertex anti magic if there is an injection $f : E \to \{1, 2, \dots |E|\}$ such that for each vertex v in V, sum of the edge labels incident to each vertex will all be different. A total bimagic labeling of a digraph with v vertices and e edges is a bijection $f : V(G) \cup E(G) \to \{1, 2, \dots, |V| + |E|$ such that for any vertex v_i the sum of the labels of outgoing edges of v_i together with the label of itself is equal to either of constants k_1 or k_2. A vertex magic total labeling of a graph $G = (V, E)$ is a bijection $f : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ such that for every vertex w, the sum of the labels of w and edge labels corresponding to all the edges incident with w is a constant.

AMS Subject Classification: 05C78
Key Words: Paley digraph, graph labeling, total magic labeling, vertex anti magic labeling, total bimagic labeling

1. Introduction

Paley graph first defined by Raymond Paley in 1933. Let F_q be a finite field with q elements such that either $q \equiv 1 \pmod 4$ or $q \equiv 3 \pmod 4$. Let ω be a primitive element in F_q and F^* the set of nonzero squares in F_q, so $F^* = \{\omega^2, \omega^4, \dots, \omega^{q-1} = 1\}$. The

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§Correspondence author

Paley graph denoted by $P(q)$ is the graph with vertex set F_q and edges all pairs $\{x, y\}$ such that $x - y \in F^*$ [2]. The class of Paley graphs is one of the two infinite families of self-complementary arc transitive graph, and also example of distance transitive graph, strongly regular graph and of conference graph [1]. Paley graphs of order q is $\left(\frac{q-1}{2}\right)$ regular, and every adjacent vertices have $\left(\frac{q-5}{4}\right)$ common neighbours, and every two nonadjacent vertices have $\left(\frac{q-1}{4}\right)$ common neighbours, which means that Paley graphs are strongly regular with parameters $\left[q, \left(\frac{q-1}{2}\right), \left(\frac{q-5}{4}\right), \left(\frac{q-1}{4}\right)\right]$.

Paley digraphs are directed analogs of Paley graphs that yield antisymmetric conference matrices. They were introduced by Graham & Spencer (1971) (independently of Sachs, Erdös, and Rényi) as a way of constructing tournaments with a property previously known to be held only by random tournaments in a Paley digraph, every small subset of vertices is dominated by some other vertex.

If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling. Most of the graph labeling problems have the following three common characteristics: a set of numbers for assignment of vertex labels, a rule that assigns a label to each edge and some condition(s) that these labels must satisfy.

An enormous body of literature has grown around the subject in the last 40 years. They introduce the family of labeling with attractive names such as graceful, harmonious, magic, anti magic, bimagic, cordial, square sum, square difference and prime etc. A useful survey to know about the numerous labeling methods is by J.A. Gallian [4]. Labeled graphs serve as useful models for a broad range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing and data base management.

If all vertices in G have the same weight k, we call the labeling vertex-magic edge labeling or vertex-magic total labeling, respectively and we call k a magic constant. The original concept of total edge-magic graph is due to Kotzig and Rosa [3]. They called it a magic graph. Almost all the Gallian's survey focuses on undirected graphs. However in early 1980's Bloom and Hsu defined labeling for directed graphs. In 1990 Hartsfield and Ringer introduced the concept of an antimagic graph [5] In 1993 Bodendiek and Walther [7] introduced the concept of an (a, d)-arithmetic antimagic labeling. In particular, super vertex (a, d) antimagic labeling was introduced by Thirusangu et. al., 2009 [13]. A (p, q)-graph $G = (V, E)$ with p vertices and q edges is called total edge magic if there is a bijection $f : V \cup E \to \{1, 2, \ldots, |V| + |E|\}$ such that there exists a constant k for any edge uv in E, $f(u) + f(uv) + f(v) = k$. It becomes interesting when we arrive with magic type labeling summing to exactly two distinct constants say k_1 or k_2. Edge bimagic total labeling for undirected graph was introduced by J. Baskar Babujee [8] and studied in [9] as (1,1) edge bimagic labeling.

In [12] K. Thirusangu, Atulya K. Nagar, R. Rajeswari introduced Magic labeling

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

and super vertex (a, d) antimagic labeling for digraphs. Here we prove the same for a small class of digraph called Paley digraphs.

2. Preliminaries

Let $G = G(V, E)$ be a finite, strong regular and directed graph with p vertices and q edges. In this paper we deal with the labeling with domain either the set of all vertices or the set of all edges or the set of all vertices and edges. We call these labeling as the vertex labeling or the edge labeling or the total labeling respectively.

Definition 1. The vertex-weight of a vertex v in G under an edge labeling to be the sum of edge labels corresponding to all edges incident with v. Under a total labeling, vertex-weight of v is defined as the sum of the label of v and the edge labels corresponding to all the edges incident with v. If all vertices in G have the same weight k, we call the labeling vertex-magic edge labeling or vertex-magic total labeling, respectively and we call k a magic constant.

Definition 2. Let G be a graph with vertex set as V and edge set as E. If there exists a bijection $f : E \to \{1, 2, \dots, |E|\}$ such that for each vertex v in V sum of the edge labels of each vertex will all be different. This labeling is called (a, d) vertex anti magic labeling.

Definition 3. A total bimagic labeling of a digraph with v vertices and e edges is a bijection $f : V(G) \cup E(G) \to \{1, 2, \dots, |V| + |E|\}$ such that for any vertex v_i, the sum of the labels of outgoing edges of v_i together with the label of itself is equal to either of constants k_1 or k_2.

Definition 4. Let p be a prime number and n be a positive integer such that $p^n \equiv 1 \pmod{4}$. The graph $P = (V, E)$ with $V(P) = Fp^n$ and $E(P) = \{(x, y) : x, y \in Fp^n, x - y \in (F^*p^n)^2\}$ is called Paley graph of order p^n.

Definition 5. Let q be a prime number such that $q \equiv 3 \pmod{4}$. The graph $P = (V, E)$ with $V(P) = F_q$ and $E(P) = \{(x, y) : x, y \in F_q, x - y \in (F_q^*)^2\}$ is called Paley digraph of order q.

3. Main Results

3.1. (a, d) Vertex Anti Magic Labeling of Paley Digraph

Algorithm 3.1.1. Input: The Finite field elements of order q, where q is the prime number with $q \equiv 3 \pmod{4}$.

Step 1: Using Definition 5 construct the Paley digraph $P(q)$.

Step 2: Denote the vertex set of $P(q)$ as $V = \{v_1, v_2, v_3, \dots, v_q\}$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

Step 3: Denote the edge set of $P(q)$ as
$E(E_{r1}, E_{r2}, \ldots, E_{rp}) = \{e_1, e_2, e_3, \ldots, e_{pq}\}$
where $r_1, r_2, \ldots, r_p$ are the residues of q and $p = \frac{q-1}{2}$ and
$E_{r1} = \{e_1, e_2, e_3, \ldots, e_q\}$
$E_{r2} = \{e_{q+1}, e_{q+2}, e_{q+3}, \ldots, e_{2q},\}$
$E_{r3} = \{e_{2q+1}, e_{2q+2}, e_{2q+3}, \ldots, e_{3q},\}$
$\vdots$
$E_{rp} = \{e_{q+1}, e_{q+2}, e_{q+3}, \ldots, e_{pq},\}$
E_{r1} = Set of all out going arcs from v_i generated by r_1
E_{r2} = Set of all out going arcs from v_i generated by r_2
$\vdots$
E_{rp} = Set of all out going arcs from v_i generated by r_p.
Step 4: Define $f_{rk} : E \to \{q+1, q+2, \ldots, (p+1)q\}$
for $k = 1, 2, 3, \ldots, p$ as

$$f_{rk}(e_{(k-1)q+i}) = \begin{cases} (k-1)q + i & 1 \le i \le q \quad k \text{ odd} \\ kq + 1 - i & 1 \le i \le q \quad k \text{ even} \end{cases}$$

Output: (a, d) Anti-magic labeling for Paley digraph $P(q)$

Theorem 6. *Every directed Paley graph admits (a, d) vertex anti-magic labeling.*

Proof. Consider any prime number q which is congruent to 3 mod 4. Construct a Paley directed graph with vertices $v_1, v_2, \ldots, v_q$ as the set of finite field elements and the edges as the ordered pairs (x, y) such that $x, y \in F$ and $x - y \in (F_q^*)^2$. To prove every directed Paley graphs admits (a, d) anti-magic labeling, we have to show that for any vertex v_i, the sum of the labels of its outgoing arcs are distinct and the set of all such distinct elements corresponds to V is equal to $\{a, a+d, a+2d, \ldots, a+(q-1)d\}$, where a and d are any two positive integers.

Consider an arbitrary vertex $v_i \in V$ of the directed Paley graph. By the construction of the Paley digraph $P(q)$, we have q vertices and each vertex has exactly $\left(\frac{q-1}{2}\right)$ outgoing arcs and $\left(\frac{q-1}{2}\right)$ incoming arcs. The edge sets are denoted as $E(E_{r1}, E_{r2}, E_{r3}, \ldots, E_{rp})$ and it is defined in the algorithm. Now define

$$f_{rk} : E \to \{q+1, q+2, \ldots, (p+1)q\} \quad \text{for} \quad k = 1, 2, 3, \ldots, p \quad \text{as}$$

$$f_{rk}(e_{(k-1)q+i}) = \begin{cases} (k-1)q + i & 1 \le i \le q \quad k \text{ odd} \\ kq + 1 - i & 1 \le i \le q \quad k \text{ even} \end{cases}$$

Hence for each vertex v_i, $1 \le i \le q$
Sum of the labels

$$S = i + 2q + 1 - i2q + i + 4q + 1 - i + \cdots + (p-1)q + 1 - i + (p-1)q + i$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

$$= i + \left(\frac{p-1}{2}\right)[1 + pq + q] \quad \text{for } 1 \leq i \leq q$$

Moreover for any two integers i, j such that $i \neq j$, $f(v_i) \neq f(v_j)$, and for any two vertices v_i, v_j the sum of the labels of its outgoing arcs are distinct. The initial value of the sum is $1 + \left(\frac{p-1}{2}\right)[1 + pq + q] = a$ (say) and for the consecutive vertices, the sum differs by $1 = d$ (say). This proves that the vertex sums form an arithmetic progression $\{a, a + d, a + 2d, \ldots, a + (q - 1)d\}$. Hence, the directed Paley graph admits (a, d) vertex anti-magic labeling. □

Example 7. Consider the Paley digraph $P(11)$ of order 11, where 11 is the prime number with $11 \equiv 3$ (mod4) The Paley digraph $P(11)$ and its (a, d) anti-magic labeling is shown in Figure 1.

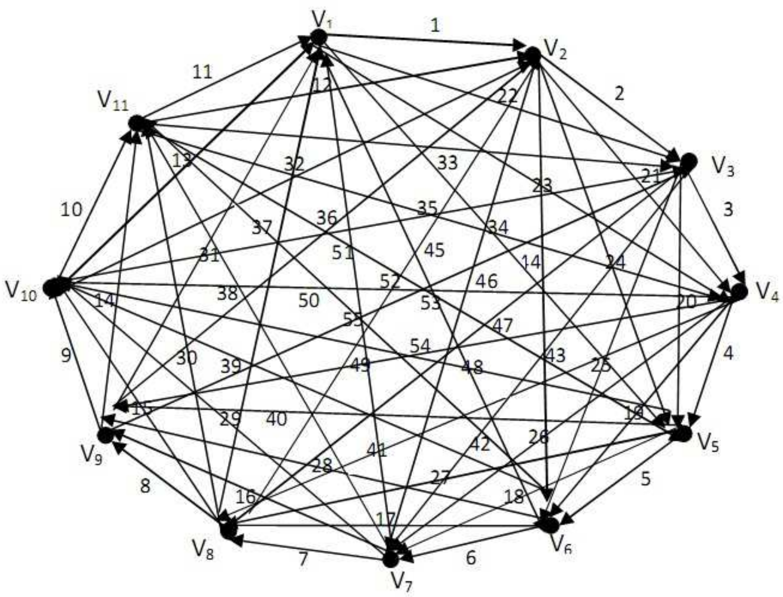

Figure 1: (a, d) Vertex anti magic total labeling of $P(11)$

3.2. Total Bimagic Labeling of Paley Digraph

Algorithm 3.2.1. Input: The Finite field of order q, where q is the prime number with $q \equiv 3$ (mod4)

Step 1: Using Definition 5 construct the Paley digraph $P(q)$.

Step 2 : Denote the vertex set of $P(q)$ as $V = \{v_1, v_2, v_3, \ldots, v_q\}$.

Step 3: Denote the edge set of $P(q)$ as

$$E(E_{r1}, E_{r2}, E_{r3}, \ldots, E_{rp}) = \{e_1, e_2, e_3, \ldots, e_{pq}\}$$

where $r_1, r_2, \ldots, r_p$ are the residues of q and $p = \left(\frac{q-1}{2}\right)$
E_{r1} = Set of all out going arcs from v_i generated by r_1
E_{r2} = Set of all out going arcs from v_i generated by r_2
E_{r3} = Set of all out going arcs from v_i generated by r_3
$\vdots$
E_{rp} = Set of all out going arcs from v_i generated by r_p
and each E_{ri}, $i = 1, 2, \ldots, p$ contains q edges.

Step 4: Define $f : V \cup E \to \{1, 2, \ldots |V \cup E|\}$ as follows

$f(v_i) = i \quad 1 \le i \le q$

$$f_{rk}(e_i) = \begin{cases} \left[p + \left(\frac{k+1}{2}\right)\right] q + 1 - (k-1)q - i, & 1 \le i \le p+1 \\ \left(\frac{k+3}{2}\right) q + 1 - i, & p+2 \le i \le q, k \text{ odd} \end{cases}$$

$$f_{rk}(e_i) = \begin{cases} p + \frac{k}{2} q + i, & 1 \le i \le p+1 \\ \left(p - \frac{k}{2}\right) q + p + i, & p+2 \le i \le q, k \text{ even} \end{cases}.$$

Output: Total Bimagic labeling for Paley digraph $P(q)$

Theorem 8. *Every directed Paley graph admits Total bimagic labeling.*

Proof. Consider any prime number q which is congruent to 3 mod 4. Construct a Paley directed graph with vertices $v_1, v_2, \ldots, v_q$ as the set of elements of finite field and the edges as the ordered pairs (x, y) such that $x, y \in F$ and $x - y \in (F_q^*)^2$. To prove every directed Paley graph admits Total Bimagic labeling, we have to show that for any vertex v_i, the sum of the labels of outgoing edges from v_i and itself is a constant k_1 or k_2. Consider any arbitrary vertex $v_i \in V$ of the Paley digraph $P(q)$. By the construction of the Paley digraph, we have each vertex of $P(q)$ has exactly $\left(\frac{q-1}{2}\right)$ outgoing edges and $\left(\frac{q-1}{2}\right)$ incoming edges. The edge sets are denoted as $E(E_{r1}, E_{r2}, E_{r3}, \ldots, E_{rp}) = \{e_1, e_2, e_3, \ldots, e_{pq})$ and it is defined as in the algorithm. Now define the map f and f_{rk} as defined in step 4 of the above algorithm.

Hence for each vertex v_i, $1 \le i \le p+1$.

Sum of the labels

$$\begin{aligned} S &= i + (p+1)q + 1 - i + p + q + i + \cdots + \\ &\quad \left[p + \left(\frac{p-1}{2}\right) q + i + \left[\left(p + \frac{p+1}{2}\right) q + 1 - (p-1)q - i\right]\right] \\ &= \left(\frac{p+1}{2}\right) pq + \left[\frac{p+1}{2}\frac{p+3}{2}\right] q + \frac{p+1}{2} + \frac{p^2-p}{2} \\ &= \frac{q}{8}[4p^2 + 8p + 4] + \frac{p^2+1}{2} \end{aligned}$$

which is a constant k_1 (say) for all p and q.
For each vertex v_i, $p+2 \le i \le q$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

Sum of the labels

$$\begin{aligned} S &= i + 2q + 1 - i + (p-1)q + p + i + 3q + 1 - i + \dots \\ &\quad + \left(p - \frac{p-1}{2}\right) q + p + i + \left(\frac{p+3}{2}\right) q + 1 - i \\ &= \left(\frac{p+1}{2}\right) 1 + \left(\frac{p-1}{2}\right) p + \left[(p-1) + (p-2) + \dots + \left(p - \frac{p-1}{2}\right)\right] q \\ &= \frac{q}{8}[4p^2 + 4p + 8] + \frac{p^2+1}{2} \end{aligned}$$

Which is a constant k_2 (say) for all p and q, obviously $k_1 \neq k_2$
Hence, the Paley digraph admits Total Bimagic labeling. □

Example 9. Consider the Paley digraph $P(7)$ of order 7, where 7 is the prime number with $7 \equiv 3$ (mod4) The Paley digraph $P(7)$ admits Total Bimagic labeling with two common constants $k_1 = 61$ and $k_2 = 54$ is shown in Figure 2.

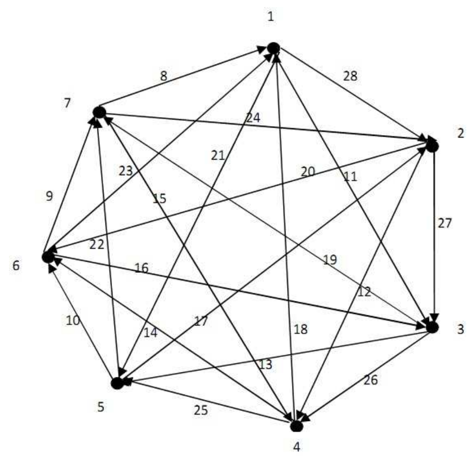

Figure 2: Total BiMagic labeling of Paley digraph $P(7)$

3.3. Magic Labeling of Paley Digraph

In this section we show the existence of vertex-magic labeling for Paley digraph and present an algorithm to get a magic labeling for a Paley digraph of order q.

Algorithm 3.3.1. Input: The Finite field elements of order q, where q is the prime number with $q \equiv 3$ (mod4).

Step 1 : Using Definition 5 construct the Paley digraph $P(q)$.

Step 2: Denote the vertex set of $P(q)$ as $V = \{v_1, v_2, v_3, \dots, v_q\}$.

Step 3: Denote the edge set of $P(q)$ as

$E(E_{r1}, E_{r2}, \ldots, E_{rp}) = \{e_1, e_2, e_3, \ldots, e_{pq}\}$
where $r_1, r_2, \ldots, r_p$ are the residues of q and $p = \frac{q-1}{2}$
E_{r1} = Set of all out going arcs from v_i generated by r_1
E_{r2} = Set of all out going arcs from v_i generated by r_2
E_{r3} = Set of all out going arcs from v_i generated by r_3
$\vdots$
E_{rp} = Set of all out going arcs from v_i generated by r_p.

Step 4: Define $f : V \to \{1, 2, 3, \ldots, q\}$
as $f(v_i) = i$, for $1 \leq i \leq q$.

Step 5: Define $g_{rk} : E \to \{q+1, q+2, \ldots, (p+1)q\}$
for $k = 1, 2, 3, \ldots, p$ as

$$g_{rk}(e_i) = \begin{cases} 2kq + 1 - i, & (k-1)q + 1 \leq i \leq kq, k \text{ odd} \\ q + i, & (k-1)q + 1 \leq i \leq kq, k \text{ even} \end{cases}.$$

Output: Magic labeling for Paley digraph $P(q)$

Theorem 10. *Every directed Paley graph admits vertex magic total labeling.*

Proof. Consider any prime number q which is congruent to 3 mod 4. Construct a Paley directed graph with vertices $v_1, v_2, \ldots, v_q$ as the set of finite field elements and the edges as the ordered pairs (x, y) such that $x, y \in F$ and $x - y \in (F_q^*)^2$. To prove every directed Paley graphs admits vertex magic labeling, we have to show that for any vertex v_i, the sum of the labels of v_i and its outgoing arcs is a constant. Consider an arbitrary vertex $v_i \in V$ of the directed Paley graph. By construction of the Paley digraph, we have each vertex has exactly $\left(\frac{q-1}{2}\right)$, outgoing arcs and $\left(\frac{q-1}{2}\right)$, incoming arcs. The edge sets are denoted as $E(E_{r1}, E_{r2}, E_{r3}, \ldots, E_{rp})$ and it is defined in the algorithm. Now define the map f and g_{rk} as defined in step 4 and 5 of the above algorithm.

Hence for each vertex v_i, $1 \leq i \leq q$.

Sum of the labels $S = i + 2q + 1 - i + 2q + \cdots + (p-1)q + i + ((p+1)q + 1 - i. = \left(\frac{p+1}{2}\right)[1 + pq + q],$.

This is a constant for all $p$and q.

Hence, the directed Paley graph admits vertex magic total labeling. □

Example 11. Consider the Paley digraph $P(7)$ of order 7, where 7 is the prime number with $7 \equiv 3$ (mod4) The Paley digraph $P(7)$ and its magic total labeling is shown in Figure 3.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

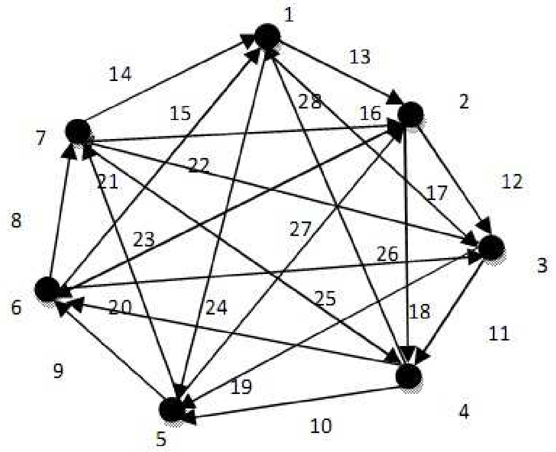

Figure 3: Vertex magic total labeling of $P(7)$

4. Conclusions

It is concluded that we proved that all Paley digraph admits some members of family of magic labeling such as (a, d) vertex anti-magic labeling, total bimagic labeling and total magic labeling. In future some more labeling can be investigated for the same digraph.

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