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GOLDEN QUATERNIONIC STRUCTURES

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Abstract: In this paper, we introduce the concept of Golden quaternion and then show that we cannot have golden quaternions, golden dual quaternions and golden hyperbolic quaternions. But; we show that we have golden biquaternions and investigate its basic properties. Furthermore, we obtain the Golden split quaternions, Golden dual split quaternions and Golden hyperbolic split quaternions by using the Lorentz metric, respectively and investigate their basic properties.

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1. Introduction

The Golden ratio is an irrational number defined to be $\frac{1+\sqrt{5}}{2}$ which is the real positive root of the equation $x^2 - x - 1 = 0$. It has been of interest to mathematicians, physicists, philosophers, architects, artists and even musicians since antiquity. It has been called the Golden mean, the Golden section, the divine proportion, the Fibonacci number and the mean of Phidias. It has a value of 1.61803... and is denoted ϕ, from the first letter of the name of the mathematician Phidias. We say that a line-segment is divided in the ratio of the Golden section, or in the Golden

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ratio, if the larger subsegment is related to the smaller exactly as the whole segment is related to the larger segment [15].

The Golden ratio is a popular ratio since the time of Euclid and we encounter it in many fields; for example, in dimensions of the human body, in the structure of musical compositions, and in the ratios of harmonious sound frequencies [12]. Furthermore, it plays an important role in architecture and visual arts. The Golden proportion and the Golden rectangle (which is spanned by two sides in the Golden proportion) have been found in the harmonious proportions of temples, churches, statues, paintings, pictures and fractals [5, 7].

The golden ratio is also used in the analysis of financial markets, in strategies such as Fibonacci retracement. In recent years, it has played an important role in differential geometry. M. Crasmareanu and C.E. Hretcanu [2] defined the Golden structure with the structure polynomial $Q(x) = x^2-x-I$ on a differentiable manifold and hence introduced the Golden ratio in differential geometry, which gives rise to the Golden differential geometry. Furthermore, E. Suresh defined Golden biquaternions, which also gives rise to the quaternion theory [3]. Our work is also based on E. Suresh's study.

We can summarize this study as follows, briefly. First, we recall some basic concepts related to quaternions, biquaternions (complexified quaternions), split quaternions, dual split quaternions and hyperbolic split quaternions, respectively, in Section 2. In the next section, we introduce the concept of Golden quaternion, we investigate its basic properties, and present some examples. Moreover, we obtain split quaternions by using the Lorentz metric in quaternions. The split quaternion product is different from quaternion product since the metric is different. Hence, we separately investigate split quaternions as the golden quaternions.

2. Preliminaries

In this section, to make the paper self-contained, we list some basic concepts and notations related to quaternions, biquaternions (complex quaternions), split quaternions, dual split quaternions and hyperbolic split quaternions, respectively.

A quaternion algebra $\mathbb{H}$ is an associative, non-commutative division ring with four basic elements $\{1, e_1, e_2, e_3\}$ satisfying the equalities

$$\begin{aligned} e_1^2 &= e_2^2 = e_3^2 = -1, \\ e_1 e_2 &= -e_2 e_1 = e_3, \\ e_2 e_3 &= -e_3 e_2 = e_1, \end{aligned}$$

$$e_3e_1 = -e_1e_3 = e_2.$$

We write any quaternion in the form $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ or $q = S_q + \overrightarrow{V_q}$ where the symbols $S_q = q_0$ and $\overrightarrow{V_q} = q_1e_1 + q_2e_2 + q_3e_3$ denote the scalar and vector parts of q. If $S_q = 0$ then q is called a pure quaternion. The conjugate of the quaternion q is denoted by K_q, and defined as $K_q = S_q - \overrightarrow{V_q}$. The norm of a quaternion q is defined by $\sqrt{qK_q} = \sqrt{K_q q} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ and denoted by N_q and we say that $q_0 = \frac{q}{N_q}$ is a unit quaternion where $q \neq 0$. Every unit quaternion can be written in the form $q_0 = \cos\theta + \overrightarrow{S_0}\sin\theta$ where $\overrightarrow{S_0}$ is a unit vector satisfying the equality $\overrightarrow{S_0}^2 = \overrightarrow{S_0}\overrightarrow{S_0} = -1$ and is called axis of the quaternion [10] . The quaternion product of two quaternions $p = p_0 + p_1e_1 + p_2e_2 + p_3e_3$ and $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ is defined as

$$pq = S_pS_q - \left\langle \overrightarrow{V_p}, \overrightarrow{V_q} \right\rangle + S_q\overrightarrow{V_p} + S_p\overrightarrow{V_q} + \overrightarrow{V_p} \wedge \overrightarrow{V_q},$$

where $\langle , \rangle$ and $\wedge$ are Euclidean inner product and vector product, respectively [6].

2.1. Complexified Quaternions

Let $1, e_1, e_2, e_3$ be the quaternion basis elements and $z_0, z_1, z_2, z_3 \in \mathbb{C}$. Then

$$q = z_0 + z_1e_1 + z_2e_2 + z_3e_3$$

is said to be a biquaternion (complexified quaternion). As a consequence of this definition, a biquaternion q can also be written as

$$q = q^{'} + iq^{''}$$

where $q^{'} = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ and $q^{''} = a_0^* + a_1^*e_1 + a_2^*e_2 + a_3^*e_3$ are real and imaginary biquaternion components, respectively. Scalar and vector parts of biquaternion q are denoted by $S_q = z_0$ and $\overrightarrow{V_q} = z_1e_1 + z_2e_2 + z_3e_3$, respectively. The complex numbers z_0, z_1, z_2, z_3 commute with the quaternion basis elements $1, e_1, e_2, e_3$. The quaternion algebra $\mathbb{H}$ is a normed division algebra. The algebra of complexified quaternion $\mathbb{P} = \mathbb{H} \otimes \mathbb{C}$ is not a division algebra since the norm can be zero. The norm of biquaternion $q = z_0 + z_1e_1 + z_2e_2 + z_3e_3$ is defined as $N_q = \sqrt{z_0^2 + z_1^2 + z_2^2 + z_3^2}$ and we say that $q_0 = \frac{q}{\sqrt{N_q}}$ is a unit biquaternion where $N_q \neq 0$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

2.2. Split Quaternions

First we recall some basic notions from Lorentzian geometry. The Minkowski 3−space E_1^3 is the Euclidean space endowed with the metric

$$g(\overrightarrow{u}, \overrightarrow{v}) = -u_1v_1 + u_2v_2 + u_3v_3,$$

where $\overrightarrow{u} = (u_1, u_2, u_3)$, $\overrightarrow{v} = (v_1, v_2, v_3) \in E^3$. Since g is an indefinite metric, a vector $\overrightarrow{u}$ in E_1^3 may have one of the three casual characters, i.e., it is spacelike if $g(\overrightarrow{u}, \overrightarrow{u}) > 0$ or $\overrightarrow{u} = 0$, timelike if $g(\overrightarrow{u}, \overrightarrow{u}) < 0$ and null (light-like) if $g(\overrightarrow{u}, \overrightarrow{u}) = 0$ and $\overrightarrow{u} \neq 0$ [9]. In [13], the Lorentzian vector product $\overrightarrow{u} \wedge_L \overrightarrow{v}$ of $\overrightarrow{u}$ and $\overrightarrow{v}$ is defined as

$$\overrightarrow{u} \wedge_L \overrightarrow{v} = \begin{vmatrix} -e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

The semi-Euclidean 4−space with 2−index is denoted E_2^4. The inner product of this semi-Euclidean space is given by

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_{E_2^4} = -u_1v_1 - u_2v_2 + u_3v_3 + u_4v_4,$$

and we say that $\overrightarrow{u}$ is timelike, spacelike or lightlike if $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{E_2^4} < 0$, $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{E_2^4} > 0$ and $\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{E_2^4} = 0$, respectively for the vector $\overrightarrow{u}$ in E_2^4. Split quaternions $\widehat{\mathbb{H}}$ are identified with the semi-Euclidean space E_2^4. Besides, the subspace of $\widehat{\mathbb{H}}$ consisting of pure split quaternions $\widehat{\mathbb{H}_0}$ is defined with the Minkowski 3−space [8].

A split quaternion algebra is an associative, non-commutative non-division ring with four basic elements $\{1, e_1, e_2, e_3\}$ satisfying the equalities

$$\begin{aligned} e_1^2 &= -1, \quad e_2^2 = e_3^2 = 1, \\ e_1e_2 &= -e_2e_1 = e_3, \\ e_2e_3 &= -e_3e_2 = -e_1, \\ e_3e_1 &= -e_1e_3 = e_2. \end{aligned}$$

The split quaternion algebra is the even subalgebra of the Clifford algebra of 3-dimensional Lorentzian space. Scalar and vector parts of split quaternion q are denoted by $S_q = q_0$ and $\overrightarrow{V_q} = q_1e_1 + q_2e_2 + q_3e_3$, respectively. The split quaternion product of two split quaternions $p = p_0 + p_1e_1 + p_2e_2 + p_3e_3$ and $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ is defined as

$$pq = S_pS_q + g(\overrightarrow{V_p}, \overrightarrow{V_q}) + S_q\overrightarrow{V_p} + S_p\overrightarrow{V_q} + \overrightarrow{V_p} \wedge_L \overrightarrow{V_q},$$

where $g(,)$ and $\wedge_L$ are Lorentzian inner product and vector product, respectively. If $S_q = 0$ then q is called pure split quaternion. Let $q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 = S_q + \overrightarrow{V_q}$ be a split quaternion. The conjugate of a split quaternion, denoted K_q, is defined as $K_q = S_q - \overrightarrow{V_q}$. Since the vector parts of q and K_q differ only in sign, we have $I_q \overset{def}{=} qK_q = K_q q$. We say that a split quaternion q is spacelike, timelike or lightlike if $I_q < 0$, $I_q > 0$, $I_q = 0$, respectively. Obviously, $-I_q = -q_0^2 - q_1^2 + q_2^2 + q_3^2$ is identified with $\langle q, q \rangle_{E_2^4}$ for the split quaternion q. The norm of $q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ is defined as $N_q = \sqrt{\left| q_0^2 + q_1^2 - q_2^2 - q_3^2 \right|}$. If $N_q = 1$ then q is called the unit split quaternion and $q_0 = \frac{q}{N_q}$ is a unit split quaternion for $N_q \neq 0$. Also, spacelike and timelike quaternions have multiplicative inverses and they satisfy the property $qq^{-1} = q^{-1}q = 1$. They are constructed as $q^{-1} = \frac{K_q}{I_q}$. Lightlike quaternions have no inverses [14]. Furthermore, we have the following.

Vector part of any spacelike quaternion is spacelike since $q_0^2 + q_1^2 - q_2^2 - q_3^2 < 0$ and $0 < q_0^2 < -q_1^2 + q_2^2 + q_3^2 = g(\overrightarrow{V_q}, \overrightarrow{V_q})$. But, vector part of any timelike quaternion can be spacelike, timelike or null in E_1^3. This is important especially for polar forms and rotations [14].

i) Every spacelike qauternion can be written in the form

$$q = N_q(\sinh\theta + \overrightarrow{S_0}\cosh\theta)$$

where $\sinh\theta = \frac{q_0}{N_q}$, $\cosh\theta = \frac{\sqrt{-q_1^2+q_2^2+q_3^2}}{N_q}$ and $\overrightarrow{S_0} = \frac{q_1e_1+q_2e_2+q_3e_3}{\sqrt{-q_1^2+q_2^2+q_3^2}}$ is a spacelike unit vector in E_1^3 and $\overrightarrow{S_0}^2 = 1$.

ii) Every timelike qauternion with spacelike vector part can be written in the form

$$q = N_q(\cosh\theta + \overrightarrow{S_0}\sinh\theta)$$

where $\cosh\theta = \frac{q_0}{N_q}$, $\sinh\theta = \frac{\sqrt{-q_1^2+q_2^2+q_3^2}}{N_q}$ and $\overrightarrow{S_0} = \frac{q_1e_1+q_2e_2+q_3e_3}{\sqrt{-q_1^2+q_2^2+q_3^2}}$ is a spacelike unit vector in E_1^3 and $\overrightarrow{S_0}^2 = 1$.

iii) Every timelike qauternion with timelike vector part can be written in the form

$$q = N_q(\cos\theta + \overrightarrow{S_0}\sin\theta)$$

where $\cos\theta \frac{q_0}{N_q}$, $\sin\theta = \frac{\sqrt{q_1^2-q_2^2-q_3^2}}{N_q}$ and $\overrightarrow{S_0} = \frac{q_1e_1+q_2e_2+q_3e_3}{\sqrt{q_1^2-q_2^2-q_3^2}}$ is a timelike unit vector in E_1^3 and $\overrightarrow{S_0}^2 = -1$.

2.3. Dual Numbers and Dual Split Quaternions

After we give a brief summary of dual numbers, we recall dual split quaternions.

Each element of the set

$$\mathbb{D} = \left\{ A = a + \varepsilon a^* | a, a^* \in \mathbb{R} \text{ and } \varepsilon \neq 0, \varepsilon^2 = 0 \right\}$$

is called a dual number. A dual number $A = a + \varepsilon a^*$ can be expressed in the form $A = ReA + \varepsilon DuA$, where $ReA = a$ and $DuA = a^*$. Addition and multiplication of two dual numbers $A = a + \varepsilon a^*$ and $B = b + \varepsilon b^*$ are defined as

$$\begin{aligned} A + B &= (a + b) + \varepsilon(a^* + b^*), \\ AB &= ab + \varepsilon(ab^* + a^* b) \end{aligned}$$

respectively and the conjugate of $A = a + \varepsilon a^*$ is defined as $\overline{A} = a - \varepsilon a^*$ [6].

A dual split quaternion Q is defined by

$$Q = A_0 + A_1 e_1 + A_2 e_2 + A_3 e_3$$

where $A_0, A_1, A_2, A_3 \in \mathbb{D}$. As a consequence of this definition, a dual split quaternion Q can also be written as

$$Q = q + \varepsilon q^*$$

where $q = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3$ and $q^* = a_0^* + a_1^* e_1 + a_2^* e_2 + a_3^* e_3$ are real and dual split quaternion components, respectively. Scalar and vector parts of dual split quaternion Q are denoted by $S_Q = A_0$ and $\overrightarrow{V_Q} = A_1 e_1 + A_2 e_2 + A_3 e_3$, respectively. The Hamilton conjugate of Q, denoted K_Q, is defined as $K_Q = S_Q - \overrightarrow{V_Q} = \overline{q} + \varepsilon \overline{q^*}$. The norm of a dual split quaternion Q is defined by $\sqrt{QK_Q} = \sqrt{K_Q Q} = \sqrt{A_0^2 + A_1^2 - A_2^2 - A_3^2}$ and denoted by N_Q. For dual split quaternions with non-zero norm, the inverse is defined as $Q^{-1} = \frac{K_Q}{N_Q^2}$ [11].

2.4. Hyperbolic Numbers and Hyperbolic Split Quaternions

In this last section, we give a brief summary of hyperbolic numbers and hyperbolic split quaternions.

Consider numbers of the form

$$w = x + jy,$$

where x and y are real numbers, and j is a commuting element satisfying the relation

$$j^2 = 1.$$

Such a number system is known as the hyperbolic number system. Addition and multiplication of two hyperbolic numbers $z = x_1 + jy_1$ and $w = x_2 + jy_2$ are, defined as

$$\begin{aligned} z + w &= (x_1 + x_2) + j(y_1 + y_2), \\ zw &= (x_1x_2 + y_1y_2) + j(x_1y_2 + x_2y_1). \end{aligned}$$

Moreover, given any hyperbolic number $w = x + jy$, we define the conjugate of w, denoted $\overline{w}$, to be

$$\overline{w} = x - jy,$$

(See [1]).

Let $1, e_1, e_2, e_3$ be the split quaternion basis elements and $Q_0, Q_1,\ Q_2, Q_3 \in H$. Then the expression

$$q = Q_0 + Q_1e_1 + Q_2e_2 + Q_3e_3$$

is a hyperbolic split quaternion. As a consequence of this definition, a hyperbolic split quaternion Q can also be written as

$$Q = Q^{'} + jQ^{''}$$

where $Q^{'} = Q_0^{'} + Q_1^{'}e_1 + Q_2^{'}e_2 + Q_3^{'}e_3$ and $Q^{''} = Q_0^{''} + Q_1^{''}e_1 + Q_2^{''}e_2 + Q_3^{''}e_3$ are real and hyperbolic split quaternion components, respectively. Scalar and vector parts of hyperbolic split quaternion Q are denoted by $S_Q = Q_0$ and $\overrightarrow{V_Q} = Q_1e_1 + Q_2e_2 + Q_3e_3$, respectively. The conjugate of Q, denoted K_Q, is defined as $K_Q = S_Q - \overrightarrow{V_Q}$. The norm of a hyperbolic split quaternion Q is defined by $\sqrt{QK_Q} = \sqrt{K_QQ} = \sqrt{Q_0^2 + Q_1^2 - Q_2^2 - Q_3^2}$ and denoted by N_Q. For hyperbolic split quaternions with non-zero norm, the inverse is defined as $Q^{-1} = \frac{K_Q}{N_Q^2}$.

3. Golden Quaternions

In this section, first we introduce the concept of the golden quaternion and then show that we cannot have golden, golden dual and golden hyperbolic quaternions. Next, we investigate golden biquaternions and present some examples. Finally, we introduce the concept of the golden split quaternion, golden dual split quaternions and golden hyperbolic split quaternion, respectively. Next, we investigate these quaternions and present some examples.

Definition 3.1. Let Q be a quaternion if Q satisfy the equation

$$Q^2 = Q + 1,$$

then we say that Q is a golden quaternion.

Now, let $Q = S_Q + \overrightarrow{V_Q}$, then we have

$$Q^2 = S_Q^2 - \left\langle \overrightarrow{V_Q}, \overrightarrow{V_Q} \right\rangle + 2S_Q\overrightarrow{V_Q}.$$

In this case, the equation $Q^2 - Q = 1$ is equivalent to

$$S_Q^2 - \left\langle \overrightarrow{V_Q}, \overrightarrow{V_Q} \right\rangle + 2S_Q\overrightarrow{V_Q} - S_Q - \overrightarrow{V_Q},$$

and we get

$$\begin{cases} S_Q^2 - \left\langle \overrightarrow{V_Q}, \overrightarrow{V_Q} \right\rangle - S_Q = 1 \\ 2S_Q\overrightarrow{V_Q} - \overrightarrow{V_Q} = 0. \end{cases} \tag{1}$$

Hence we get

$$S_Q = \frac{1}{2} \quad \text{or} \quad \overrightarrow{V_Q} = 0.$$

Then there are two cases for golden quaternions:

$$\begin{aligned} &i)\ \overrightarrow{V_Q} &= &\ 0 \Rightarrow Q = \frac{1 \pm \sqrt{5}}{2}, \\ &ii)\ \overrightarrow{V_Q} &\neq &\ 0 \Rightarrow S_Q = \frac{1}{2} \text{ and } \left\langle \overrightarrow{V_Q}, \overrightarrow{V_Q} \right\rangle = -\frac{5}{4}. \end{aligned} \tag{2}$$

Hence, we can say that there aren't golden quaternions. Furthermore, we cannot have golden dual or golden hyperbolic quaternions.

3.1. Golden Biquaternions

But, we can have the golden biquaternions. Now, we try to explain the golden biquaternion. According to (2) we have

$$S_Q = \frac{1}{2} \text{ and } \left\langle \overrightarrow{V_Q}, \overrightarrow{V_Q} \right\rangle = -\frac{5}{4}.$$

Now, let Q be a golden biquaternion, namely, $Q = z_0 + z_1e_1 + z_2e_2 + z_3e_3$, where $z_i \in \mathbb{C}, i = 0, 1, 2, 3$. Then we have

$$z_0 = \frac{1}{2} \text{ and } z_1^2 + z_2^2 + z_3^2 = -\frac{5}{4}. \tag{3}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

Thus we have
$$N_Q = z_0^2 + z_1^2 + z_2^2 + z_3^2 = -1. \tag{4}$$
We know that a unitary biquaternion is expressed as
$$Q_0 = \frac{Q}{\sqrt{N_Q}} = \frac{z_0 + z_1e_1 + z_2e_2 + z_3e_3}{\sqrt{z_0^2+z_1^2+z_2^2+z_3^2}}, \quad N_Q \neq 0.$$
If we consider
$$\cos\theta = \frac{z_0}{\sqrt{N_Q}}, \ \sin\theta = \frac{\sqrt{z_1^2+z_2^2+z_3^2}}{\sqrt{N_Q}} \ \text{ and } \overrightarrow{S_0} = \frac{z_1e_1 + z_2e_2 + z_3e_3}{\sqrt{z_1^2+z_2^2+z_3^2}}, \tag{5}$$
then we can write a unitary biquaternion as
$$Q_0 = \cos\theta + \overrightarrow{S_0}\sin\theta, \quad \theta \in \mathbb{C}. \tag{6}$$
Then, according to the equalities (3), (4), (5) and (6), a unitary golden biquaternion is in the form as follows:
$$Q_0 = \frac{1}{2i} + \frac{\sqrt{5}}{2}\overrightarrow{S_0}, \ \left\langle \overrightarrow{S_0}, \overrightarrow{S_0} \right\rangle = 1, \ \overrightarrow{S_0}^2 = -1, \tag{7}$$
and since $Q = Q_0\sqrt{N_Q}$, a golden biquaternion is also in the form as
$$Q = \frac{1}{2} + \frac{\sqrt{5}i}{2}\overrightarrow{S_0}, \ \left\langle \overrightarrow{S_0}, \overrightarrow{S_0} \right\rangle = 1, \ \overrightarrow{S_0}^2 = -1. \tag{8}$$
Now, let us see the matrix form of the golden biquaternion. Let Q be a biquaternion, i.e., $Q = z_0 + z_1e_1 + z_2e_2 + z_3e_3$, where $z_j \in \mathbb{C}$, $j = 0, 1, 2, 3$. The matrix form of Q is given by
$$\begin{pmatrix} z_0 + iz_1 & z_2 + iz_3 \\ -z_2 + iz_3 & z_0 - iz_1 \end{pmatrix}.$$
Hence, the matrix form of the golden biquaternion $Q = \frac{1}{2} + \frac{\sqrt{5}i}{2}\overrightarrow{S_0}$, where $\overrightarrow{S_0} = (s_1, s_2, s_3)$, is also given by
$$\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{5}}{2}s_1 & \frac{\sqrt{5}i}{2}s_2 - \frac{\sqrt{5}}{2}s_3 \\ -\frac{\sqrt{5}i}{2}s_2 - \frac{\sqrt{5}}{2}s_3 & \frac{1}{2} + \frac{\sqrt{5}}{2}s_1 \end{pmatrix}.$$
Now we consider the golden analogues of the examples of biquaternions given in [4].

Let's transform the variables using the transformation
$$T(z_1, z_2, z_3) = (iyz, izx, ixy)$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

where $i = \sqrt{-1}$ is the imaginary number of complex algebra and $x, y, z \in \mathbb{R}$. Under this transformation, we get

$$z_1^2 + z_2^2 + z_3^2 = -y^2 z^2 - z^2 x^2 - x^2 y^2 = -\frac{5}{4}. \tag{9}$$

There is a surface known as *Steiner's Roman Surface* which has the intrinsic equation

$$y^2 z^2 + z^2 x^2 + x^2 y^2 - r^2 xyz = 0. \tag{10}$$

Then, Q is a golden biquaternion if

$$Q = \frac{1}{2} + (iyz)e_1 + (izx)e_2 + (ixy)e_3$$

has the property that the vector (yz, zx, xy) spans a Steiner's Roman Surface.

Using (9), we can write (10) as

$$r^2 xyz = \frac{5}{4}.$$

Furthermore, if we used the transform

$$T(z_1, z_2, z_3) = (ix, iy, iz),$$

we have obtain the equation

$$x^2 + y^2 + z^2 = \frac{5}{4},$$

which corresponds to a sphere of radius $r = \frac{\sqrt{5}}{2}$.

If we use the transforms

$$\begin{aligned}
T(z_1, z_2, z_3) &= (ix, y, z),\\
T(z_1, z_2, z_3) &= (ix, iy, z),\\
T(z_1, z_2, z_3) &= (ix, y, iz),\\
T(z_1, z_2, z_3) &= (x, iy, z),\\
T(z_1, z_2, z_3) &= (x, iy, iz),\\
T(z_1, z_2, z_3) &= (x, y, iz),
\end{aligned}$$

then the solution surfaces will be hyperboloids.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

4. Golden Split Quaternions

Now, we investigate the golden split quaternion, i.e., we try to obtain the split quaternions Q that satisfy $Q^2 = Q + 1$. Let $Q = S_Q + \overrightarrow{V_Q}$. Then we have

$$Q^2 = QQ = S_Q^2 + g(\overrightarrow{V_Q}, \overrightarrow{V_Q}) + 2S_Q\overrightarrow{V_Q}.$$

Thus, the equation $Q^2 - Q = 1$ is equivalent to

$$S_Q^2 + g(\overrightarrow{V_Q}, \overrightarrow{V_Q}) + 2S_Q\overrightarrow{V_Q} - S_Q - \overrightarrow{V_Q} = 1,$$

and we get

$$\begin{cases} S_Q^2 + g(\overrightarrow{V_Q}, \overrightarrow{V_Q}) - S_Q = 1 \\ 2S_Q\overrightarrow{V_Q} - \overrightarrow{V_Q} = 0. \end{cases} \tag{11}$$

Thus, we have

$$S_Q = \frac{1}{2} \quad \text{or} \quad \overrightarrow{V_Q} = 0.$$

Therefore, there are exactly two cases for the golden split quaternions:

$$\begin{array}{rcll} i)\ \overrightarrow{V_Q} & = & 0 \Rightarrow Q = \dfrac{1 \pm \sqrt{5}}{2}, \\ ii)\ \overrightarrow{V_Q} & \neq & 0 \Rightarrow S_Q = \dfrac{1}{2} \text{ and } g(\overrightarrow{V_Q}, \overrightarrow{V_Q}) = \dfrac{5}{4}. \end{array} \tag{12}$$

Hence, we can have the golden split, golden dual split or golden hyperbolic split quaternions. Now, let us try to explain these golden split quaternions, respectively. If we have a golden split quaternion with $\overrightarrow{V_Q} \neq 0$, then according to (12) we get

$$S_Q = \frac{1}{2} \text{ and } g(\overrightarrow{V_Q}, \overrightarrow{V_Q}) = \frac{5}{4}.$$

Now, let Q be a golden split quaternion, namely, $Q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$,where $q_j \in \mathbb{R}$, $j = 0, 1, 2, 3$. Hence we get

$$q_0 = \frac{1}{2} \text{ and } -q_1^2 + q_2^2 + q_3^2 = \frac{5}{4}. \tag{13}$$

Thus we have

$$I_Q = q_0^2 + q_1^2 - q_2^2 - q_3^2 = -1, \tag{14}$$

which shows that the golden split quaternion Q is a spacelike quaternion. From [14], we know that every spacelike quaternion can be written in the form

$$Q = N_Q(\sinh\theta + \overrightarrow{\mathcal{S}_0}\cosh\theta), \tag{15}$$

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)

where $\sinh\theta = \frac{q_1}{N_Q}$, $\cosh\theta = \frac{\sqrt{-q_1^2+q_2^2+q_3^2}}{N_Q}$ and $\overrightarrow{\mathcal{S}_0} = \frac{q_1e_1+q_2e_2+q_3e_3}{\sqrt{-q_1^2+q_2^2+q_3^2}}$ is a spacelike unit vector in E_1^3. Then, according to (13), (14) and (15), a golden split quaternion is in the form

$$Q = \frac{1}{2} + \overrightarrow{\mathcal{S}_0}\frac{\sqrt{5}}{2},\ g(\overrightarrow{\mathcal{S}_0}, \overrightarrow{\mathcal{S}_0}) = 1,\ \overrightarrow{\mathcal{S}_0}^2 = 1.$$

Now, we present some examples related to golden split quaternions. Let's transform the variables using the transformation

$$T(q_1, q_2, q_3) = (yz, zx, xy)$$

where $x, y, z \in \mathbb{R}$. Under this transformation, we have

$$-q_1^2 + q_2^2 + q_3^2 = -y^2z^2 + z^2x^2 + x^2y^2 = \frac{5}{4}. \tag{16}$$

Besides the image of a Lorentz sphere of radius r centered at the origin under the projection

$$(x, y, z) \mapsto (yz, zx, xy)$$

is

$$-y^2z^2 + z^2x^2 + x^2y^2 - r^2xyz = 0. \tag{17}$$

This surface corresponds to the Steiner's Roman surface in the Lorentz space.

Then, Q is a golden split quaternion if

$$Q = \frac{1}{2} + (yz)e_1 + (zx)e_2 + (xy)e_3$$

has the property that the vector (x, y, z) spans the surface in (16).

Using equation (16), we can write equation (17). as

$$r^2xyz = \frac{5}{4}.$$

Furthermore, if we used the transform

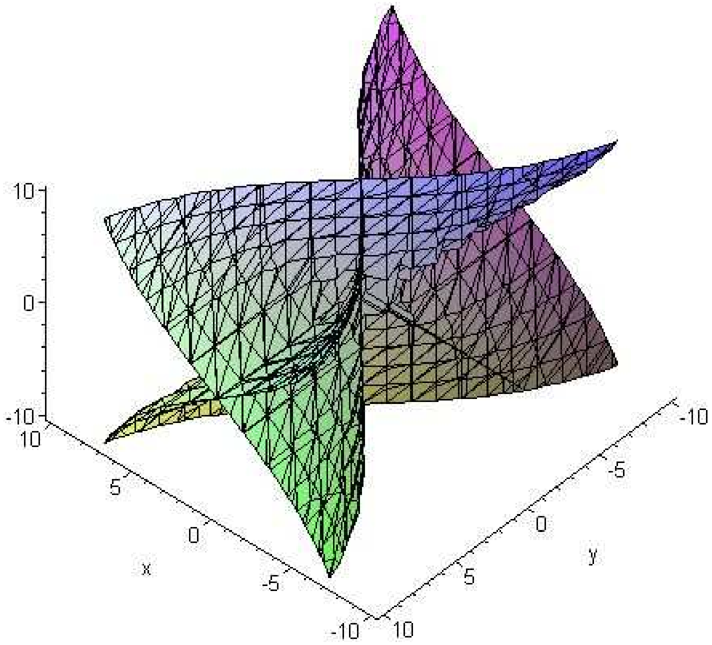
$$T(q_1, q_2, q_3) = (x, y, z),$$

then we would obtain the equality

$$-x^2 + y^2 + z^2 = \frac{5}{4},$$

which corresponds to a Lorentz sphere of radius $r = \frac{\sqrt{5}}{2}$.

International Electronic Journal of Pure and Applied Mathematics – IEJPAM, Volume 7, No. 3 (2014)



4.1. Golden Dual Split Quaternions

Now, we investigate the golden dual split quaternion. Let $Q = S_Q + \overrightarrow{V_Q}$. If we have a golden split quaternion with $\overrightarrow{V_Q} \neq 0$, then according to (12), we have

$$S_Q = \frac{1}{2} \text{ and } g(\overrightarrow{V_Q}, \overrightarrow{V_Q}) = \frac{5}{4},$$

where $Q = Q_0 + Q_1 e_1 + Q_2 e_2 + Q_3 e_3$ and $Q_j = q_j + \varepsilon q_j^* \in \mathbb{D}$, $j = 0, 1, 2, 3$. Then we have

$$Q_0 = \frac{1}{2} \text{ and } -Q_1^2 + Q_2^2 + Q_3^2 = \frac{5}{4}. \tag{18}$$

Thus we have

$$-q_1^2 + q_2^2 + q_3^2 = \frac{5}{4} \text{ and } -q_1 q_1^* + q_2 q_2^* + q_3 q_3^* = 0, \tag{19}$$

which implie that

$$I_Q = Q_0^2 + Q_1^2 - Q_2^2 - Q_3^2 = -1.$$

Hence we can write

$$Q = N_Q(\sinh\theta + \overrightarrow{S_0}\cosh\theta), \tag{20}$$

where $\sinh\theta = \frac{Q_1}{N_Q}$, $\cosh\theta = \frac{\sqrt{-Q_1^2+Q_2^2+Q_3^2}}{N_Q}$ and $\overrightarrow{\boldsymbol{S}_0} = \frac{Q_1e_1+Q_2e_2+Q_3e_3}{\sqrt{-Q_1^2+Q_2^2+Q_3^2}}$ is a space-like unit dual vector in E_1^3. Then, according to (18), (19) and (20), a golden split quaternion is in the form

$$Q = \frac{1}{2} + \overrightarrow{\boldsymbol{S}_0}\frac{\sqrt{5}}{2},\ g(\overrightarrow{\boldsymbol{S}_0}, \overrightarrow{\boldsymbol{S}_0}) = 1,\ \overrightarrow{\boldsymbol{S}_0}^2 = 1.$$

Now, we present some examples related to golden dual split quaternions. Let us transform the variables using

$$T(Q_1, Q_2, Q_3) = (\varepsilon yz, \varepsilon zx, \varepsilon xy)$$

where ε is the imaginary number of the algebra of dual numbers and $x, y, z \in \mathbb{R}$. Under this transformation, we have

$$-Q_1^2 + Q_2^2 + Q_3^2 = 0.$$

Hence, we can't use pure dual numbers in Q_1, Q_2, Q_3.

If we use the transforms

$$\begin{aligned}
T(Q_1, Q_2, Q_3) &= (\varepsilon x, y, z),\\
T(Q_1, Q_2, Q_3) &= (\varepsilon x, \varepsilon y, z),\\
T(Q_1, Q_2, Q_3) &= (\varepsilon x, y, \varepsilon z),\\
T(Q_1, Q_2, Q_3) &= (x, \varepsilon y, z),\\
T(Q_1, Q_2, Q_3) &= (x, \varepsilon y, \varepsilon z),\\
T(Q_1, Q_2, Q_3) &= (x, y, \varepsilon z),
\end{aligned}$$

then the solution surfaces will be cylinders and planes as given below:

$$\begin{aligned}
y^2 + z^2 &= \frac{5}{4},\\
z^2 &= \frac{5}{4},\\
y^2 &= \frac{5}{4},\\
x^2 + z^2 &= \frac{5}{4},\\
x^2 &= \frac{5}{4},\\
x^2 + y^2 &= \frac{5}{4}.
\end{aligned}$$

4.2. Golden Hyperbolic Split Quaternions

Finally, we investigate the golden hyperbolic split quaternion. Let $Q = S_Q + \overrightarrow{V_Q}$, If we have a golden split quaternion with $\overrightarrow{V_Q} \neq 0$, then according to (12), we have

$$S_Q = \frac{1}{2} \text{ and } g(\overrightarrow{V_Q}, \overrightarrow{V_Q}) = \frac{5}{4},$$

where $Q = Q_0 + Q_1e_1 + Q_2e_2 + Q_3e_3$ and $Q_j = q_i + jq_i' \in \mathcal{H}$, $i = 0, 1, 2, 3$. Then we have

$$Q_0 = \frac{1}{2} \text{ and } -Q_1^2 + Q_2^2 + Q_3^2 = \frac{5}{4}. \tag{21}$$

Thus we obtain

$$\begin{aligned} -q_1^2 + q_2^2 + q_3^2 - q_1'^2 + q_2'^2 + q_3'^2 &= \frac{5}{4} \text{ and} \\ -q_1q_1' + q_2q_2' + q_3q_3' &= 0 \end{aligned} \tag{22}$$

which implies that

$$I_Q = Q_0^2 + Q_1^2 - Q_2^2 - Q_3^2 = -1.$$

Since $I_Q = -1$, we can write

$$Q = N_Q(\sinh\theta + \overrightarrow{\mathbb{S}_0}\cosh\theta), \tag{23}$$

where $\sinh\theta = \frac{Q_1}{N_Q}$, $\cosh\theta = \frac{\sqrt{-Q_1^2+Q_2^2+Q_3^2}}{N_Q}$ and $\overrightarrow{\mathbb{S}_0} = \frac{Q_1e_1+Q_2e_2+Q_3e_3}{\sqrt{-Q_1^2+Q_2^2+Q_3^2}}$ is a spacelike unit hyperbolic vector in E_1^3. Then, according to (21), (22) and (23), a golden split quaternion is in the form

$$Q = \frac{1}{2} + \overrightarrow{\mathbb{S}_0}\frac{\sqrt{5}}{2}, \; g(\overrightarrow{\mathbb{S}_0}, \overrightarrow{\mathbb{S}_0}) = 1, \; \overrightarrow{\mathbb{S}_0}^2 = 1.$$

Now, we give some examples for golden hyperbolic split quaternions. Let us transform the variables using the transformation

$$T(Q_1, Q_2, Q_3) = (jyz, jzx, jxy)$$

where $j^2 = 1$ is the imaginary number of the hyperbolic algebra and $x, y, z \in \mathbb{R}$. Under this transformation, we get

$$-Q_1^2 + Q_2^2 + Q_3^2 = -y^2z^2 + z^2x^2 + x^2y^2 = \frac{5}{4}. \tag{24}$$

Then, Q is a golden hyperbolic split quaternion if

$$Q = \frac{1}{2} + (jyz)e_1 + (jzx)e_2 + (jxy)e_3$$

has the property that the vector (x, y, z) span the Surface in (17).

Furthermore, if we use the transform

$$T(Q_1, Q_2, Q_3) = (jx, jy, jz),$$

then we obtain the equation

$$-x^2 + y^2 + z^2 = \frac{5}{4},$$

which corresponds to a Lorentz sphere of radius $r = \frac{\sqrt{5}}{2}$.

Finally, If we use the transforms

$$\begin{aligned}
T(Q_1, Q_2, Q_3) &= (jx, y, z), \\
T(Q_1, Q_2, Q_3) &= (jx, jy, z), \\
T(Q_1, Q_2, Q_3) &= (jx, y, jz), \\
T(Q_1, Q_2, Q_3) &= (x, jy, z), \\
T(Q_1, Q_2, Q_3) &= (x, jy, jz), \\
T(Q_1, Q_2, Q_3) &= (x, y, jz),
\end{aligned}$$

then the solution surfaces will be exactly a Lorentz sphere of radius $r = \frac{\sqrt{5}}{2}$.

References

[1] Antonuccio, F., Hyperbolic Numbers and the Dirac Spinor, Available from: arxiv:hep-th/9812036v1 3 Dec 1998.

[2] Crasmareanu M, Hretcanu CE. Golden differential geometry. Chaos, Solitons & Fractals 2008;38(5):1229-1238.

[3] Demir, S., Kompleks ve Dual Kuaterniyonların Fiziksel Uygulamaları, Doktora Tezi, Eskişehir, 2003.

[4] Emre, S., Golden Biquaternions, Generations and Spin, http://knol.google.com/k/golden-biquaternions-3-generations-and-spin#, 2009.

[5] Fischler RH. The shape of the Great Pyramid. Waterloo: Wilfrid Laurier University Press; 2000.

[6] Hacısalihoğlu, H.H., Hareket Geometrisi ve Kuaterniyonlar Teorisi, Gazi Üniversitesi Fen Edebiyat Fak. Yayınları, Math., 1983, No:2.

[7] Herz-Fischler R. A mathematical history of division in extreme and mean ratio. Waterloo, Ontario, Canada: Wilfrid Laurier University Press; 1987.

[8] Inoguchi, J., Timelike surfaces of Constant Mean Curvature in Minkowski 3-Space, Tokyo J. Math., (1998), 21(1), 141-152.

[9] İlarslan, K., Some special curves on non-euclidean manifolds, Ph.D. Thesis, Ankara University, Graduate School of Natural and Applied Sciences, 2002.

[10] Kantor, I.L., Lodovnikov, A.S., Hypercomplex numbers. An Elementary Introduction to Algebras, New York, Springer-Verlag, 1989.

[11] Kula, L., Yaylı, Y., Dual Split Quaternions and Screw Motıon in Minkowski 3-Space, Iranian Journal of Science & Technology, Transaction A, 2006; 30(3); 245-258.

[12] Livio M. The Golden Ratio: The Story of phi, the World's Most Astonishing Number, Broadway; 2002 [MR2003k:11025].

[13] O'neill, B., Semi-Riemannian Geometry with Applications to Relativity, London, Academic Press Inc., 1983.

[14] Özdemir, M., Ergin, A.A., Rotations with Unit Timelike Quaternions in Minkowski 3-Space, J. Geom. Phys.2006; 56: 322-336.

[15] Walser H. The Golden Section. The Mathematical Association of America; 2001.