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FIXED-POINTS AND UNIQUENESS OF ENTIRE AND MEROMORPHIC FUNCTIONS

Harina P. Waghamore[1], A. Tanuja[2 §], N. Shilpa[3]
[1,2]Department of Mathematics
Central College Campus
Bangalore University
Bangalore, 560 001, INDIA
[3]Department of Mathematics
M.E.S. Degree College
Malleswaram, Bangalore, 560 003, INDIA

Abstract: In this paper, we deal with the uniqueness problems on entire and meromorphic functions concerning differential polynomials that share fixed-points. These results improve and extend those given by Xiaojuan Li and C. Meng, see [2].

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1. Introduction

In this paper, we use the standard notations and terms in the value distribution theory [4]. For any nonconstant meromorphic function $f(z)$ on the complex plane C, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o\{T(r, f)\}$, as $r \to +\infty$, except possibly for a set of r of finite linear measures. A meromorphic function $a(z)$ is called a small function with respect to $f(z)$ if $T(r, a) = S(r, f)$. Let $S(f)$ be the set of meromorphic function in the complex plane C which are small functions with respect to f. Set $E(a(z), f) = \{z : f(z) - a(z) = 0\}$, $a(z) \in S(f)$, where a zero

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§Correspondence author

point with multiplicity m is counted m times in the set. If these zero points are only counted once, then we denote the set by $\overline{E}(a(z), f)$. Let k be a positive integer. Set $E_{k)}(a(z), f) = \{z : f(z) - a(z) = 0, \exists i, 1 \leq i \leq k$, such that, $f^{(i)}(z) - a^{(i)}(z) \neq 0\}$, where a zero point with multiplicity m is counted m times in the set.

Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, $a(z) \in S(f) \cap S(g)$. If $E(a(z), f) = E(a(z), g)$, then we say that $f(z)$ and $g(z)$ share the value $a(z)$ CM, especially, we say that $f(z)$ and $g(z)$ have the same fixed points when $a(z) = z$. If $\overline{E}(a(z), f) = \overline{E}(a(z), g)$, then we say that $f(z)$ and $g(z)$ share the function $a(z)$ IM. If $E_{k)}(a(z), f) = E_{k)}(a(z), g)$, we say that $f(z) - a$ and $g(z) - a$ have same zeros with the same multiplicities $\leq k$.

Moreover, we also use the following notations.

We denote by $N_{k)}(r, f)$ the counting function for poles of $f(z)$ with multiplicities $\leq k$, and by $\overline{N}_{k)}(r, f)$ the corresponding one for which the multiplicity is not counted. Let $N_{(k}(r, f)$ be the counting function for poles of $f(z)$ with multiplicities $\geq k$, and let $\overline{N}_{(k}(r, f)$ be the corresponding one for which the multiplicity is not counted. Set $N_k(r, f) = \overline{N}(r, f) + \overline{N}_{(2}(r, f) + ... + \overline{N}_{(k}(r, f)$.

Similarly, We have the notations

$$N_{k)}\left(r, \frac{1}{f}\right),\ \overline{N}_{k)}\left(r, \frac{1}{f}\right),\ N_{(k}\left(r, \frac{1}{f}\right),$$

$$\overline{N}_{(k}\left(r, \frac{1}{f}\right),\ N_k\left(r, \frac{1}{f}\right).$$

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and $\overline{E}(1, f) = \overline{E}(1, g)$. We denote by $\overline{N}_L\left(r, \frac{1}{(f-1)}\right)$ the counting function for 1-points of both $f(z)$ and $g(z)$ about which $f(z)$ has larger multiplicity than $g(z)$, with muliplicity not being counted, and denote by $N_{1]}\left(r, \frac{1}{(f-1)}\right)$ the counting function for common simple 1-points of both $f(z)$ and $g(z)$ where multiplicity is not counted. Similarly, we have the notation $\overline{N}_L\left(r, \frac{1}{(g-1)}\right)$.

In 2009, Xiaojuan Li and C. Meng[2] proved the following two theorems.

Theorem A. *Let f and g be two non-constant entire functions. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and a_i is the first nonzero coefficient from the right, and n, m, k be a positive integer with $n(> 2k+m+4)$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 CM, then:*

(1) If $0 \leq i < m$, then either $f(z) \equiv g(z)$ or f, g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

(2) If $i = m$, then either $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m} = 1$ or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k a_m^2 (c_1 c_2)^{n+m} [(n + m)c]^{2k} = 1.$$

Theorem B. *Let f and g be two non-constant transcendental entire functions. Let $P(f) = a_m f^m + a_{m-1} f^{m-1} + ... + a_1 f + a_0$, $(a_m \neq 0)$, and a_i is the first nonzero coefficient from the right, and n, m, k be a positive integer with $n + m > (5k + 7)(m + 1)$. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share the value 1 IM, then:*

(1) If $0 \leq i < m$, then either f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

(2) If $i = m$, then either $f(z) \equiv tg(z)$, where t is a constant satisfying $t^{n+m} = 1$ or $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, where c_1, c_2 and c are three constants satisfying

$$(-1)^k a_m^2 (c_1 c_2)^{n+m} [(n + m)c]^{2k} = 1.$$

One may ask the following question which is the motivation of the paper: Is it possible that value 1 can be replaced by a fixed-point z in the above theorem and if possible how far? We now state the following two theorems.

Theorem 1.1. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, where $a_0 \neq 0$, a_1,...,a_{m-1}, $a_m \neq 0$ are complex constants. If $E_l(z, [f^n P(f)]^{(k)}) = E_l(z, [g^n P(g)]^{(k)})$:*

i) If $l \geq 2$ and $n > 2k + 3m + 4$;

ii) If $l = 1$ and $n > 3k + 4m + 5$;

iii) If $l = 0$ and $n > 5k + 6m + 7$.

Then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n + m, ...n + m - i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1...m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

Theorem 1.2. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, where $a_0 \neq 0$, a_1,...,a_{m-1}, $a_m \neq 0$ are complex constants. If $E_l(z, [f^n P(f)]^{(k)}) = E_l(z, [g^n P(g)]^{(k)})$:*

i) If $l \geq 2$ and $n > 3k + 3m + 8$;

ii) If $l = 1$ and $n > 5k + 4m + 10$;

iii) If $l = 0$ and $n > 9k + 6m + 14$.

Then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n + m, ...n + m - i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1...m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

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2. Some Lemmas

In this section, we present some lemmas which are needed in the sequel.

Lemma 2.1. *(see [3]) Let $f(z)$ be a nonconstant meromorphic function and let a_0, a_1,...,a_n be finite complex numbers, $a_n \neq 0$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.2. (see [4]) *Let f be a nonconstant meromorphic function, let k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) \\ &\quad + S(r,f). \end{aligned}$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 2.3. (see [1]) *Let f and g be two nonconstant meromorphic functions, $k(\geq 1)$, $l(\geq 0)$ be integers. Suppose that $E_l(1, f^k(z)) = E_l(1, g^k(z))$. If one of the following conditions holds, then $f(z) \equiv g(z)$ or $f^{(k)}(z) g^{(k)}(z) \equiv 1$.*

i) $l \geq 2$ and

$$(k+2)\Theta(\infty, g) + 2\Theta(\infty, f) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > k + 7;$$

ii) $l = 1$ and

$$(k+2)\Theta(\infty, g) + (k+3)\Theta(\infty, f) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 2k + 9;$$

iii) $l = 0$ and

$$(2k+4)\Theta(\infty, f) + (2k+3)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 4k + 13.$$

Lemma 2.4. *Let f and g be two non-constant meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 1)$ be a integers. Then $[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \neq z^2$.*

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Proof. Let
$$[f^nP(f)]^{(k)}[g^nP(g)]^{(k)} \equiv z^2. \tag{2.1}$$
Let z_0 be a zero of f of order p_0. From (2.1) we get z_0 is a pole of g. Suppose that z_0 is a pole of g of order q_0. Again by (2.1), we obtain
$$np_0 - k = nq_0 + mq_0 + k,$$
i.e.
$$n(p_0 - q_0) = mq_0 + 2k.$$
The last equality implies that $q_0 \geq \frac{n-2k}{m}$ and so we have $p_0 \geq \frac{n+m-2k}{m}$.

Let z_1 be a zero of $f-1$ of order p_1, then z_1 is a zero of $[f^nP(f)]^{(k)}$ of order $p_1 - k$. Therefore from (2.1) we obtain
$$p_1 - k = nq_1 + mq_1 + k,$$
i.e.
$$p_1 \geq (n+m)s + 2k.$$

Let z_2 be a zero of f' of order p_2 that is not a zero of $fP(f)$, then from (2.1) z_2 is a pole of g of order q_2. Again by (2.1) we get
$$p_2 - (k-1) = nq_2 + mq_2 + k$$
i.e.
$$p_2 \geq (n+m)s + 2k - 1.$$

In the same manner as above, we have similar results for the zeros of $[g^nP(g)]^{(k)}$.

On other hand, suppose that z_3 is a pole of f. From (2.1), we get that z_3 is the zero of $[g^nP(g)]^{(k)}$.

Thus
$$\begin{aligned}\overline{N}(r,f) &\leq \overline{N}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g-1}\right) + \overline{N}\left(r,\frac{1}{g'}\right)\\ &\leq \frac{1}{p_0}N\left(r,\frac{1}{g}\right) + \frac{1}{p_1}N\left(r,\frac{1}{g-1}\right) + \frac{1}{p_2}N\left(r,\frac{1}{g'}\right)\\ &\leq \left[\frac{m}{n+m-2k} + \frac{1}{(n+m)s+2k} + \frac{2}{(n+m)s+2k-1}\right]T(r,g)\\ &+ S(r,g).\end{aligned} \tag{2.2}$$

By second fundamental theorem and equation (2.2), we have
$$T(r,f) \;\leq\; \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{f-1}\right) + \overline{N}(r,f)$$

$$\begin{aligned} &\leq \frac{m}{n+m-2k}N\left(r,\frac{1}{f}\right)+\frac{1}{(n+m)s+2k}N\left(r,\frac{1}{f-1}\right) \\ &+ \left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]T(r,g) \\ &+S(r,g)+S(r,f). \end{aligned}$$

$$\begin{aligned} T(r,f) \leq &\left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}\right]T(r,f) \\ &+\left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]T(r,g) \\ &+S(r,g)+S(r,f). \end{aligned} \tag{2.3}$$

Similarly, we have

$$\begin{aligned} T(r,g) \leq &\left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}\right]T(r,g) \\ &+\left[\frac{m}{n+m-2k}+\frac{1}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right]T(r,f) \\ &+S(r,g)+S(r,f). \end{aligned} \tag{2.4}$$

Adding (2.3) and (2.4) we get

$$\begin{aligned} T(r,f)+T(r,g) \leq &\left[\frac{2m}{n+m-2k}+\frac{2}{(n+m)s+2k}+\frac{2}{(n+m)s+2k-1}\right] \\ &\{T(r,f)+T(r,g)\}+S(r,g)+S(r,f). \end{aligned}$$

which is a contradiction. Thus Lemma proved.

3. Proof of the Theorem

In this section we present the proof of the main result.

Proof of Theorem 1.2. Let $F=f^nP(f)$, $G=g^nP(g)$.
By Lemma 2.1 we can easily we get

$$\Theta(0,F)=1-\limsup_{r\to\infty}\frac{\overline{N}\left(r,\frac{1}{F}\right)}{T(r,F)}\geq 1-\frac{m+1}{n+m}$$

Similarly, we have
$\Theta(0,G)\geq 1-\frac{m+1}{n+m}$, $\Theta(\infty,F)\geq 1-\frac{1}{n+m}$, $\Theta(\infty,G)\geq 1-\frac{1}{n+m}$
Moreover,

$$\delta_{k+1}(0,F)=1-\limsup_{r\to\infty}\frac{\overline{N}_{k+1}\left(r,\frac{1}{F}\right)}{T(r,F)}\geq 1-\frac{m+k+1}{n+m}$$

Similarly, we have,

$$\delta_{k+1}(0,G) \geq 1 - \frac{m+k+1}{n+m}.$$

Since $E_l(1,f^k) = E_l(1,g^k)$. We can discuss the following three cases:

i) $l \geq 2$. Because $n > 3k+3m+8$, we have

$$(k+2)\Theta(\infty,G) + 2\Theta(\infty,F) + \Theta(0,F) + \Theta(0,G) + \delta_{k+1}(0,F) + \delta_{k+1}(0,G)$$
$$\geq (k+4)\left(1 - \frac{1}{n+m}\right) + 2\left(1 - \frac{m+1}{n+m}\right) + 2\left(1 - \frac{m+k+1}{n+m}\right) > k+7.$$

ii) $l = 1$. Because $n > 5k+4m+10$, we have

$$(k+2)\Theta(\infty,G) + (k+3)\Theta(\infty,F) + \Theta(0,F) + \Theta(0,G) + 2\delta_{k+1}(0,F) + \delta_{k+1}(0,G)$$
$$\geq (2k+5)\left(1 - \frac{1}{n+m}\right) + 2\left(1 - \frac{m+1}{n+m}\right) + 3\left(1 - \frac{m+k+1}{n+m}\right) > 2k+9.$$

iii) $l = 0$. Because $n > 9k+6m+14$, we have

$$(2k+3)\Theta(\infty,G) + (2k+4)\Theta(\infty,F) + \Theta(0,F) + \Theta(0,G) + 3\delta_{k+1}(0,F) + 2\delta_{k+1}(0,G)$$
$$\geq (4k+7)\left(1 - \frac{1}{n+m}\right) + 2\left(1 - \frac{m+1}{n+m}\right) + 5\left(1 - \frac{m+k+1}{n+m}\right) > 4k+13.$$

Therefore, by Lemma 2.3, we deduce that either $F^{(k)}G^{(k)} \equiv z^2$ or $F \equiv G$.

If $F^{(k)}G^{(k)} \equiv z^2$, that is

$$[f^n(a_m f^m + a_{m-1}f^{m-1} + \dots + a_1 f + a_0)]^{(k)}[g^n(a_m g^m + a_{m-1}g^{m-1} + \dots + a_1 g + a_0)]^{(k)} \equiv z^2, \quad (3.1)$$

then by Lemma 2.4 we can get a contradiction.

Hence, we deduce that $F \equiv G$, that is

$$f^n(a_m f^m + a_{m-1}f^{m-1} + \dots + a_1 f + a_0) = g^n(a_m g^m + a_{m-1}g^{m-1} + \dots + a_1 g + a_0). \quad (3.2)$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (3.2) we obtain

$$a_m g^{n+m}(h^{n+m} - 1) + a_{m-1}g^{n+m-1}(h^{n+m-1} - 1) + \dots + a_0 g^n(h^n - 1) = 0,$$

which implies $h^d = 1$, where $d = (n+m, \dots, n+m-i, \dots n)$, $a_{m-1} \neq 0$ for some $i = 0,1,\dots m$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m, \dots, n+m-i, \dots n)$, $a_{m-i} \neq 0$ for some $i = 0,1,\dots m$.

If h is not a constant, then we know by (3.2) that f and g satisfy the algebraic equation $R(f,g) = 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

This completes the proof of Theorem 1.2.

Proof of Theorem 1.1. Since f and g are entire functions we have $\overline{N}(r,f) = \overline{N}(r,g) = 0$. Proceeding as in the proof of Theorem 1.2 we can easily prove Theorem 1.1.

4. Remark

It follows from the proof of Theorem 1.1 and Theorem 1.2 that if "z" is replaced by "$a(z)$" in Theorems 1.1 and Theorem 1.2, where $a(z)$ is a meromorphic function such that $a \neq 0, \infty$ and $T(r,a) = o\{T(r,f),\ T(r,g)\}$, then the conclusions of Theorems 1.1 and Theorem 1.2 still hold. So we obtain the following results.

Theorem 4.1. *Let $f(z)$ and $g(z)$ be two nonconstant entire functions and $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, where $a_0 \neq 0$, a_1,...,a_{m-1}, $a_m \neq 0$ are complex constants. If $E_l(a(z), [f^n P(f)]^{(k)}) = E_l(a(z), [g^n P(g)]^{(k)})$:*

i) If $l \geq 2$ and $n > 2k + 3m + 4$;

ii) If $l = 1$ and $n > 3k + 4m + 5$;

iii) If $l = 0$ and $n > 5k + 6m + 7$.

Then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m, ...n+m-i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1...m$, or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

Theorem 4.2. *Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions and $P(z) = a_m z^m + a_{m-1} z^{m-1} + ... + a_1 z + a_0$, where $a_0 \neq 0$, a_1,...,a_{m-1}, $a_m \neq 0$ are complex constants. If $E_l(a(z), [f^n P(f)]^{(k)}) = E_l(a(z), [g^n P(g)]^{(k)})$:*

i) If $l \geq 2$ and $n > 3k + 3m + 8$;

ii) If $l = 1$ and $n > 5k + 4m + 10$;

iii) If $l = 0$ and $n > 9k + 6m + 14$.

Then either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = (n+m, ...n+m-i, ...n)$, $a_{m-i} \neq 0$ for some $i = 0, 1...m$, or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n P(\omega_1) - \omega_2^n P(\omega_2)$.

Obviously, we can use the analog method of Theorems 1.1 and Theorem 1.2 to prove Theorems 4.1 and Theorem 4.2 easily. Here, we omit them.

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